2.1 1^{st} order Differential Equations

Example

$\frac{1}{t} y' + (\cos t)\, y = t^2$ linear ODE of 1^{st} order

$(\sin t)\, y' = t$ 1^{st} order linear ODE

$t y' + \frac{1}{t y} = 10$ 1^{st} order nonlinear ODE

$x y' + x y = 1 - y \Rightarrow x y' + (x+1) y = 1$ 1^{st} order linear ODE

$(x + e^{y})\, dy - dx = 0 \Rightarrow y' = \frac{1}{x + e^{y}}$ 1^{st} order nonlinear ODE

A linear 1^{st} order ODE has the general form

$$\frac{dy}{dt} = f(t, y) \quad , \; f \text{ is linear in } y$$

Examples ① Equations with costant coefficients

$$y' = -a y + b \qquad a, b \text{ constants}, \; a \neq 0$$

In this case, we use methods of calculus to find the solution:

$$\int \frac{\frac{dy}{dt}}{y - \frac{b}{a}} = \int -a \quad \Rightarrow \ln \left| y - \frac{b}{a} \right| = -at + C$$

$$\Rightarrow y = \frac{b}{a} + D e^{at} \quad , \; D = \pm e^{C}$$

② Equations with variable coefficients

$$y' + p(t)\, y = g(t)$$

In this case, we can't use methods of calculus because it does not work. We will use the method of integrating factor.

Method of Integrating Factors
(Variable Coefficient)

Consider a linear 1^{st} order ODE with variable coefficients $\frac{dy}{dt} + p(t)\,y = g(t)$

$\boxed{p(t) \text{ and } g(t) \text{ are continuous functions}}$

The method of integrating factors involves multiplying this equation by a function $\mu(t)$, chosen so that the resulting equation is easily integrated.

Example: take $p(t) = 2$ and $g(t) = e^{\frac{1}{2}t}$

$$y' + 2y = e^{t/2}$$

Multiply both sides by $\mu(t)$ $\Rightarrow$

$$\mu(t)\frac{dy}{dt} + \underline{\underline{2\mu(t)}}\,y = e^{\frac{1}{2}t}\mu(t) \quad \cdots\cdots *$$

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + \underline{\underline{\frac{d\mu(t)}{dt}}}\,y$$

$*$ becomes

$$\int e^{2t}\frac{dy}{dt} + 2e^{2t}y = \int e^{5t/2}$$

$$e^{2t}y = \frac{2}{5}e^{5t/2} + C$$

$$\boxed{y(t) = \frac{2}{5}e^{t/2} + Ce^{-2t}}$$

For us to be able to integrate both sides of $*$ we need to choose $\mu(t)$ such that

$$\frac{d\mu(t)}{dt} = 2\mu(t) \Rightarrow$$

$$\int \frac{\frac{d\mu(t)}{dt}}{\mu(t)} = \int 2 \Rightarrow$$

$$\ln|\mu(t)| = 2t$$

$$\mu(t) = ce^{2t} \quad \text{take } c=1$$

Example: Take $p(t) = a$

In this case we need to find the solution for the following linear 1^{st} order ODE

$$y' + ay = g(t)$$

Multiply by $\mu(t) \Rightarrow$

$$\mu(t)\frac{dy}{dt} + a\,\mu(t)\,y = \mu(t)\,g(t)$$

substitute $\mu(t) = e^{at} \Rightarrow$

$$\mu'(t) = a\,\mu(t)$$

$$\frac{\mu'(t)}{\mu(t)} = a$$

$$\boxed{\mu(t) = e^{at}}$$

$$\int e^{at}\frac{dy}{dt} + a\,e^{at}\,y = \int e^{at}\,g(t)$$

$$e^{at}\,y = \int e^{at}\,g(t)\,dt + c$$

$$\boxed{y(t) = \frac{1}{e^{at}}\int e^{at}\,g(t)\,dt + c\,e^{-at}} \rightarrow *$$

Example: Take $a = \frac{1}{5}$ and $g(t) = 5 - t$

$$y' + \frac{1}{5}y = 5 - t$$

using $* \Rightarrow y(t) = \frac{1}{e^{t/5}}\int e^{t/5}(5-t)\,dt + c\,e^{-t/5}$

$$y(t) = e^{-t/5}\Big[5\int e^{t/5}dt - \underbrace{\int t\,e^{t/5}dt}_{\text{by parts}}\Big] + c\,e^{-t/5}$$

$u = t$, $dv = e^{t/5}$

$du = dt$, $v = 5e^{t/5}$

$$\int u\,dv = uv - \int v\,du$$

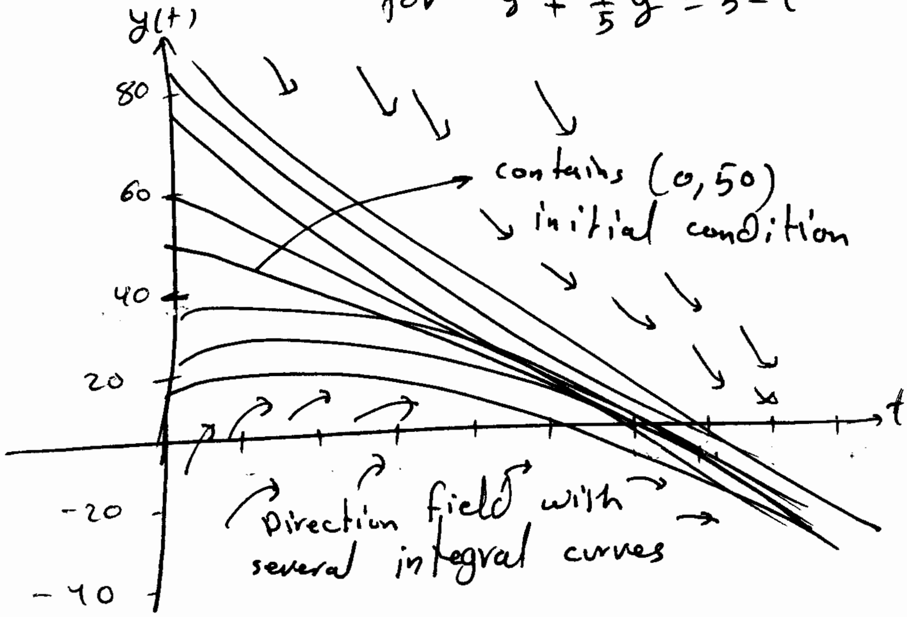
$$= 5e^{-t/5}\,5e^{t/5} - e^{-t/5}\Big[5t\,e^{t/5} - \int 5e^{t/5}dt\Big] + c\,e^{-t/5}$$

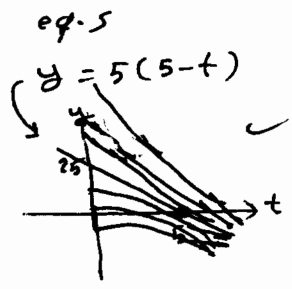
$$= 25 - 5t + 25 + c\,e^{-t/5}$$

$$\boxed{y(t) = 50 - 5t + c\,e^{-t/5}}$$

$$\Rightarrow y(t) = 50 - 5t + c\, e^{-t/5}$$

for $y' + \frac{1}{5} y = 5 - t$





convergence

Example: Take $p(t) = -\frac{1}{5}$ and $g(t) = 5 - t$

$$y' - \frac{1}{5} y = 5 - t$$

Using the previous formula $\Rightarrow y(t) = \frac{1}{e^{at}} \int e^{at} g(t)\, dt + c\, e^{-at}$

$$\Rightarrow y(t) = e^{t/5} \int e^{-t/5} (5 - t)\, dt + c\, e^{t/5}$$

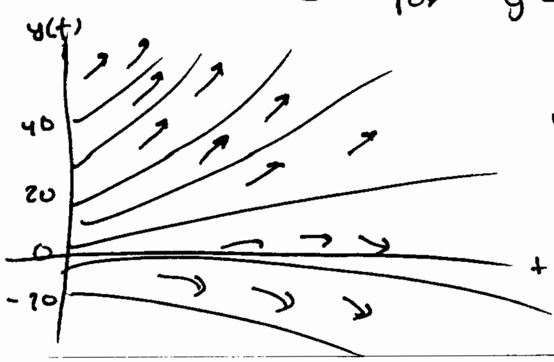
$\Rightarrow$ Integrating by parts $\Rightarrow \int e^{-t/5} (5-t)\, dt = \int 5 e^{-t/5} dt - \int t e^{-t/5} dt$

$$= -25 e^{-t/5} - \left[-5\, t e^{-t/5} + \int 5 e^{-t/5} dt\right]$$

$$= 5\, t\, e^{-t/5}$$

Thus $\Rightarrow y(t) = e^{t/5} \left[5 t e^{-t/5}\right] + c\, e^{t/5} = 5t + c\, e^{t/5}$

$y(t) = 5t + c\, e^{t/5}$ for $y' - \frac{1}{5} y = 5 - t$



Direction field with several integral curves

Diveragen

Method of Integrating Factors

For General First Order Linear Equation

Consider the general 1^{st} order linear ODE

$$y' + p(t)\,y = g(t)$$

p(t) and g(t) are continuous functions

The solution is

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\,g(t)\,dt + c\right]$$

where $\mu(t)$ is positive function given by

$$\mu(t) = e^{\int p(t)\,dt}$$

Proof: Multiply $y' + p(t)\,y = g(t)$ by $\mu(t)$ $\Rightarrow$

$$\mu(t)\frac{dy}{dt} + p(t)\,\mu(t)\,y = g(t)\,\mu(t) \quad \text{---} (*)$$

Now find $\mu(t)$ such that $\mu'(t) = p(t)\,\mu(t)$.

i.e $\int \frac{\mu'(t)}{\mu(t)} = \int p(t)$

$\ln \mu(t) = \int p(t)\,dt + D$

assume $D=0$

$$\boxed{\mu(t) = e^{\int p(t)\,dt}}$$

$\rightarrow$ substitute $\mu(t)$ in $*$, we get

$$\int e^{\int p(t)dt}\frac{dy}{dt} + p(t)\,e^{\int p(t)dt}\,y = \int g(t)\,e^{\int p(t)dt}\,dt$$

$$C + \mu(t)\,y = \int g(t)\,\mu(t)\,dt$$

$$y = \frac{1}{\mu(t)}\left[\int g(t)\,\mu(t)\,dt + C\right]$$

Example: Solve the IVP $ty' - 2y = 5t^2$, $y(1) = 2$, $t > 0$ (21)

First $\Rightarrow$ $y' - \frac{2}{t} y = 5t$, $t > 0$

Second $\Rightarrow$ Find the integrating factor

$$\mu(t) = e^{\int p(t)\,dt} = e^{-\int \frac{2}{t}\,dt} = e^{-2\ln t} = e^{\ln \frac{1}{t^2}} = \frac{1}{t^2}$$

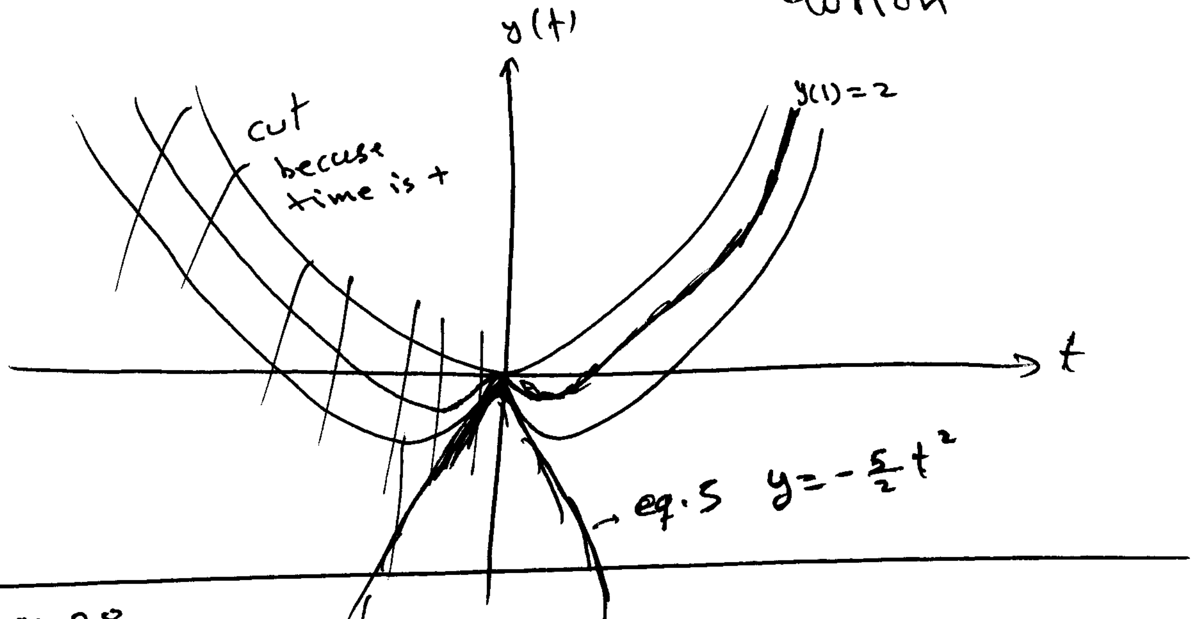
Hence, the solution $y(t) = \frac{1}{\mu(t)} \int \mu(t) g(t)\,dt + c$

$$y(t) = t^2 \left[\int \frac{1}{t^2}(5t)\,dt + c \right]$$

$$y(t) = 5t^2 \ln t + c t^2 \rightarrow \text{general solution}$$

$$y(1) = 5 \ln 1 + c(1)^2 = 2 \Rightarrow \boxed{c = 2}$$

$$\therefore \; y(t) = 5t^2 \ln t + 2t^2 \rightarrow \text{particular solution}$$



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