$\boxed{2.2}$ Separable Equations

* Recall the general 1^{st} order DE: $\frac{dy}{dx} = f(x, y)$

$\Rightarrow$ This equation can be written as $M(x,y) + N(x,y)\frac{dy}{dx} = 0$

$\Rightarrow$ If M is a function of x and N is a function of y

i.e $M(x)\,dx + N(y)\,dy = 0$, then the equation is called separable.

Example: Solve the following D.E $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

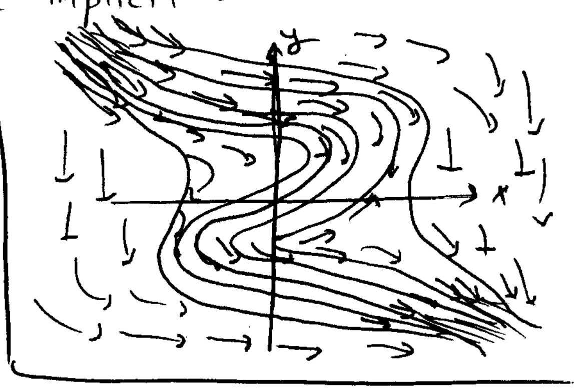
$$(1-y^2)\,dy = x^2\,dx$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + C \Rightarrow y^3 - 3y = -x^3 + C$$

"implicit solution"

Example: Solve the IVP:

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1$$



separating variables

$$2(y-1)\,dy = (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$y(0) = -1 \Rightarrow$

$(-1)^2 - 2(-1) = C \Rightarrow \boxed{C = 3}$

$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + 3}$* is the implicit solution.

To find the explicit solution we write * as

$$y^2 - 2y - (x^3+2x^2+2x+3) = 0$$

$$\Rightarrow y = \frac{2 \pm \sqrt{4+4(x^3+2x^2+2x+3)}}{2}$$

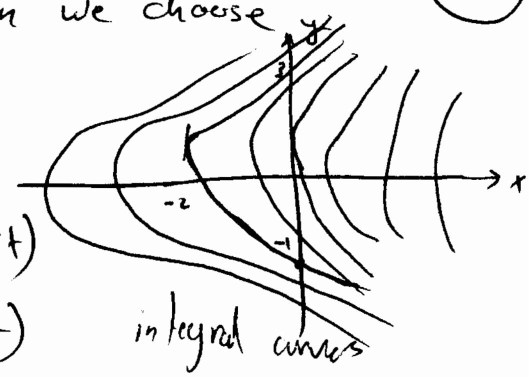
$$y = 1 \pm \frac{\sqrt{4+4(x^3+2x^2+2x+3)}}{2}$$

$$y = 1 \pm \sqrt{x^3+2x^2+2x+4}$$

$\boxed{y = 1 - \sqrt{x^3+2x^2+2x+4}}$ because $y(0) = -1$

$\Rightarrow$ If the initial condition $y(0)=3$ then we choose

$$y = 1+\sqrt{x^3+2x^2+2x+4}$$



integral curves

Hence, our solution is

$$y^2-2y = x^3+2x^2+2x+3 \quad \text{(implicit)}$$

$$y = 1-\sqrt{x^3+2x^2+2x+4} \quad \text{(explicit)}$$

$$= 1-\sqrt{x^2(x+2)+2(x+2)}$$

$$= 1-\sqrt{(x+2)(x^2+2)}$$

The domain of y is $(-2,\infty)$

Note when $x=-2 \Rightarrow y=1 \Rightarrow$ the denominator of $\frac{dy}{dx}$ is zero

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<u>Example</u> Solve the IVP $\frac{dy}{dx} = \frac{y\cos x}{1+3y^3}$, $y(0)=1$

separating variables $(1+3y^3)\,dy = y\cos x\,dx$

$$\left(\frac{1}{y}+3y^2\right)dy = \cos x\,dx$$

$$\ln|y| + y^3 = \sin x + c \qquad \text{since } y(0)=1 \qquad \begin{matrix} x=0 \\ y=1 \end{matrix}$$

$$\ln 1 + (1)^3 = \sin 0 + c \Rightarrow c = 1$$

$$\boxed{\ln|y| + y^3 = \sin x + 1}$$

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Example: Solve $\frac{dy}{dx} = \frac{4y-3x}{2x-y} = \frac{4v-3}{2-v}$ $\quad v=\frac{y}{x} \Rightarrow y = xv$

$\Downarrow$ solution $\quad \Rightarrow y' = xv' + v$

$$c = -5\ln|y-3x| + \ln|y-x|$$

$\Downarrow$

$$\ln|x| = \int \frac{(2-v)}{(v+3)(v-1)}\,dv$$

$$= -\frac{5}{4}\int\frac{dv}{v+3} + \frac{1}{4}\int\frac{dv}{v-1}$$