2.3 Modeling with 1st Order Equations

The idea is to construct Mathematical models that characterize problems in physical, biological and social sciences using differential equations. The solution of these mathematical models lead to equations relating the variables and parameters in the problem.

* Three steps are present in the process of Mathematical models:

① Model Construction: Translating the physical situation into mathematical terms. State clearly the physical principles that are believed to govern the process. DE is a mathematical model of the process, typically an approximation.

② Analysis of the Model: solving DEs or obtaining understanding of solution. simplify the model.

③ Comparsion with Experiment or observation: Verifies solution or suggest refinement of the model.

1 lb (pound) $\simeq$ 453.6 grams

Example: A time $t=0$, a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing $\frac{1}{4}$ lb of salt/gal is entering tank at rate of r gal/min and leaves at same rate.

(a) Set up the IVP that discribes this salt solution flow process

(b) Find amount of salt $Q(t)$ in tank at any given time t.

(c) Find the limiting amount Q_L of salt $Q(t)$ in the tank after a very long time.

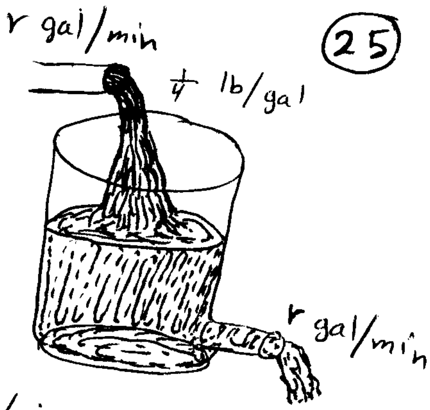
(d) If $r=3$ and $Q_0 = 2Q_L$, find time T after which salt is within 2% of Q_L.

(e) Find flow rate r required if T is not to exceed 45 min.

The rate of change of salt in the take is

(a)

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$



- rate in = the concentration of the entering salt X the flow in rate

$= \frac{1}{4}$ lb/gal X r gal/min $= \frac{r}{4}$ lb/min

- rate out = the concentration of the salt in the tank X the flow out rate

= (Let $Q(t)$ be the quantity of salt lb in the tank at time t $\Rightarrow$ the concentration $= \frac{Q(t)}{100}$) X r gal/min

$$= r\,\frac{Q(t)}{100}$$

$\Rightarrow$ The IVP becomes $\boxed{\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}, \quad Q(0) = Q_0}$ *

(b) To solve the IVP $\frac{dQ}{dt} + \frac{rQ}{100} = \frac{r}{4}$, $Q(0) = Q_0$ we use the method of integrating factor: or ($Q(t) = \frac{b}{a} + (Q_0 - \frac{b}{a})\,e^{at}$, $Q' = \frac{-r}{100}Q + \frac{r}{4}$

$$\mu(t) = e^{\int p(x)dx} = e^{\int \frac{r}{100}dt} = e^{\frac{rt}{100}}$$

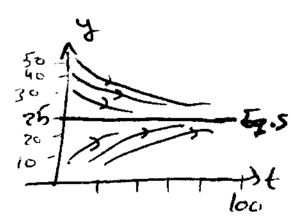
$$Q(t) = \frac{1}{e^{rt/100}}\left[\int e^{\frac{rt}{100}}\,\frac{r}{4}\,dt + C\right] = e^{-rt/100}\left[\frac{\cancel{100}^{25}}{\cancel{r}}\,e^{\frac{rt}{100}}\,\frac{\cancel{r}}{\cancel{4}} + C\right] = 25 + C\,e^{\frac{-rt}{100}}$$

$$Q(0) = Q_0 = 25 + C\,e^{0} \Rightarrow Q_0 = 25 + C \Rightarrow C = Q_0 - 25$$

$$\boxed{Q(t) = 25 + (Q_0 - 25)\,e^{\frac{-rt}{100}}}$$

(c) $$Q_L = \lim_{t\to\infty} Q(t) = \lim_{t\to\infty} 25 + (Q_0 - 25)\,e^{\frac{-rt}{100}} = 25 \text{ lb}$$

The limiting amount of salt $Q_L = 25$ lb



(d) $Q_0 = 2Q_L = 2(25) = 50$ lb and note that $r = 3$

$$\Rightarrow Q(t) = 25 + (50 - 25)\,e^{\frac{-rt}{100}}$$

$$Q(t) = 25 + 25\,e^{-0.03t}$$

$\Rightarrow$ we need to find the time T at which $Q(t)$ is within 2% of Q_L i.e

$$Q(T) = Q_L + \frac{2}{100} Q_L$$
$$= 25 + \frac{2}{100} \times 25$$
$$= 25 + 0.5 = 25.5$$

$$Q(T) = 25 + 25\, e^{-0.03T} = 25.5 \Rightarrow \frac{\frac{1}{2}}{25} = e^{-0.03T}$$

$$\Rightarrow 0.02 = e^{-0.03T} \Rightarrow -0.03T = \ln(0.02) \Rightarrow T \simeq 130.4 \text{ min}$$

(e) To find r so that $T = 45$: $\quad Q_0 = 2Q_L = 50$ lb

$$Q(t) = 25 + 25\, e^{\frac{-rt}{100}}$$
$$Q(T) = 25 + 25\, e^{\frac{-r}{100}T}$$
$$25.5 = 25 + 25\, e^{\frac{-r}{100} \times 45} \Rightarrow 0.02 = e^{-0.45r}$$
$$\Rightarrow \ln(0.02) = -0.45r \Rightarrow r = \frac{\ln(0.02)}{-0.45} \simeq 8.69 \text{ gal/min}$$

Example: Consider a pond that initially contains 10 million gallons of fresh water. Water containing toxic waste (نفايات سامة) flows into the pond at the rate of 5 million gal/year and exists at same rate. The concentration $c(t)$ of toxic waste in the incoming water varies periodically with time accordingly to $c(t) = 2 + \sin 2t$ g/gal

(a) Construct a mathematical model of this flow process.

(b) Find the amount of toxic waste in pond at any time.

(a) The rate of change of toxic waste in the pond is $\frac{dQ}{dt}$ given by $\frac{dQ}{dt}$ = rate in − rate out

$$= \underbrace{(2 + \sin 2t)}_{\frac{g}{gal}} \times \underbrace{5 \times 10^6 \frac{gal}{year}}_{\frac{gal}{year}} - \frac{5 \times 10^6}{10 \times 10^6} Q(t) \frac{g}{year}$$

the amount of waste in the pond at time t

$$\boxed{\frac{dQ}{dt} = (2 + \sin 2t)(5 \times 10^6) - \frac{Q(t)}{2}, \quad Q(0) = 0} \ *$$

(b) To solve the IVP, we simplify notations using change of variables: divide $*$ by 10^6 and

Let $q(t) = \dfrac{Q(t)}{10^6} \Rightarrow \dfrac{dq}{dt} = \dfrac{1}{10^6}\dfrac{dQ(t)}{dt}$

Then $*$ becomes:

$$\frac{dq}{dt} = (2+\sin 2t)\times 5 - \frac{q(t)}{2}, \quad q(0) = \frac{Q(0)}{10^6} = 0$$

$$\boxed{\frac{dq}{dt} + \frac{q(t)}{2} = 10 + 5\sin 2t, \quad q(0) = 0}$$

Using the method of integrating factor:-

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{2}dt} = e^{\frac{t}{2}}$$

$$q(t) = \frac{1}{e^{\frac{t}{2}}}\left[\underbrace{\int e^{\frac{t}{2}}(10+5\sin 2t)\,dt}_{\text{(by parts 3 or 4 times see below)}} + C\right] \longrightarrow \text{Exercises 1}$$

$$q(t) = e^{-\frac{t}{2}}\left[20e^{\frac{t}{2}} - \frac{40}{17}e^{t/2}\cos 2t + \frac{10}{17}e^{t/2}\sin 2t + C\right]$$

$q(0)=0 \Rightarrow C = 20 \Rightarrow$ $q(t) = 20 - \dfrac{40}{17}\cos 2t + \dfrac{10}{17}\sin 2t - \dfrac{300}{17}e^{-\frac{t}{2}}$

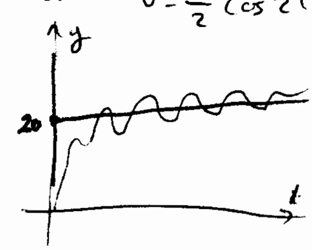
$u = e^{\frac{t}{2}}$ $\quad dv = \sin 2t\,dt$

$du = \frac{1}{2}e^{\frac{t}{2}}dt$ $\quad v = \frac{-1}{2}\cos 2t$

$$\int e^{\frac{t}{2}}\sin 2t\,dt = \left[-\frac{1}{2}e^{\frac{t}{2}}\cos 2t + \frac{1}{4}\int e^{\frac{t}{2}}\cos 2t\,dt\right]$$

$$= \left[\frac{-1}{2}e^{\frac{t}{2}}\cos 2t + \frac{1}{4}\left(\frac{1}{2}e^{\frac{t}{2}}\sin 2t - \frac{1}{4}\int e^{\frac{t}{2}}\sin 2t\,dt\right)\right]$$

$$\int e^{\frac{t}{2}}\sin 2t\,dt = -\frac{1}{2}e^{\frac{t}{2}}\cos 2t + \frac{1}{8}e^{t/2}\sin 2t - \frac{1}{16}\int e^{t/2}\sin 2t\,dt$$



$$\frac{17}{16}\int e^{t/2}\sin 2t\,dt = -\frac{1}{2}e^{t/2}\cos 2t + \frac{1}{8}e^{t/2}\sin 2t$$

$$\int e^{t/2}\sin 2t\,dt = -\frac{8}{17}e^{t/2}\cos 2t + \frac{2}{17}e^{t/2}\sin 2t$$

$$\therefore 5\int e^{t/2}\sin 2t\,dt = -\frac{40}{17}e^{t/2}\cos 2t + \frac{10}{17}e^{t/2}\sin 2t$$