2.4 Differences Between linear and Nonlinear Eq's

Recall that a 1^{st} order ODE has the form $\boxed{y' = f(t,y)}$ — $*$.

→ $*$ is linear if f is linear in y

→ $*$ is nonlinear if f is nonlinear in y.

Theorem 2.4.1 Consider the linear 1^{st} order IVP

$$\frac{dy}{dt} + p(t)\,y = g(t) \quad , \quad y(t_0) = y_0 \quad \text{------} \ \boxed{1}$$

If the functions p and g are continous on an open interval $I: \alpha < t < \beta$ containing t_0, then there exist a unique function $y = \phi(t)$ that satisfies $y' + p(t)\,y = g(t)$ $\forall t \in I$ and satisfies the initial condition $y(t_0) = y_0$

From Th. 2.4.1 we observe that for a given IVP, it has a solution and it is only one solution (existance and uniqueness). The solution exists for every $t \in (\alpha, \beta)$; including the initial point $t = t_0$, in which p and g are continous.

Proof: Existence $\Rightarrow$ $y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\,dt + c\right]$, $\mu(t) = e^{\int p(s)\,ds}$

this is the general solution ⇒ explicitly defined.

Uniqueness $\Rightarrow y(t_0) = y_0 \Rightarrow y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(t)\, g(t)\,dt + y_0\right]$, $\mu(t) = e^{\int_{t_0}^{t} p(s)\,ds}$

The continuity of p implies that μ is nonzero continous function and moreover $\mu(t) = e^{\int p(s)\,ds}$ is differentiable. The continuity of μ and g implies that μg is integrable.

The fundemental Existence and Uniqueness Theorem for 1^{st} order IVP:

Theorem 2.4.2: Consider the IVP

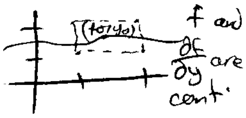
$$\frac{dy}{dt} = f(t,y) \quad , \quad y(t_0) = y_0 \quad \text{- - - -} \ \boxed{2}$$

linear or Nonlinear

If the functions f and $\frac{\partial f}{\partial y}$ are continous in some open rectangle $(t,y) \in (\alpha,\beta) \times (\gamma,\delta)$ containing the point (t_0, y_0), then in some interval $(t_0 - h, t_0 + h) \subset (\alpha, \beta)$ there is a unique solution $y = \phi(t)$ satisfies $\boxed{2}$.

*Note that the continuity of f and $\frac{\partial f}{\partial y}$ in Th 2.4.2 is equivalent to continuity of p and g in Th 2.4.1

$f(t,y) = -p(t)y + g(t)$ and $\frac{\partial f}{\partial y} = -p(t)$



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* The proof of Th 2.4.2 is more difficult because there is no expression for the general solution.

Remark 2.4.2

* The conditions stated in Th 2.4.2 are sufficient to guarantee the existance of a unique solution of the IVP ② in some interval $t_0-h<t<t_0+h$. but they are not necessary. That is the continuity of f ensures the existence (but not uniquencess) of ϕ.

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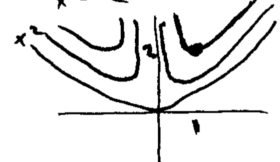
quiz 1

Example ① Solve the IVP $xy' + 2y = 4x^2$, $y(1) = 2$ linear

$y' + \frac{2}{x}y = 4x$ $\qquad \mu(x) = e^{\int^x \frac{2}{s}ds} = e^{2\ln s\big|_1^x} = e^{2\ln x} = e^{\ln x^2} = x^2$

$$\Rightarrow y(x) = \frac{1}{x^2}\left[\int_1^x s^2(4s)ds + 2\right] = \frac{1}{x^2}\left[s^4\Big|_1^x + 2\right] = \frac{1}{x^2}\left[x^4 + 1\right]$$

$y(x) = x^2 + \frac{1}{x^2}$ this is explicit solution



② Find an interval in which the IVP above has a unique solution.
We use Th 2.4.1 : $p(x) = \frac{2}{x}$ and $g(x) = 4x$

$\Rightarrow g(x)$ is continous for all x but

$\Rightarrow p(x)$ is continous only for $x<0$ or $x>0$. The interval $x>0$ contains the initial point $\Rightarrow$ Th 2.4.1 guarantees the IVP above has a unique solution on the interval $(0,\infty)$ given by $y(x) = x^2 + \frac{1}{x^2}$, $x>0$

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separable

Example Apply Th 2.4.2 to the IVP $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$ (a) $y(0) = -1$ (b) $y(0) = 1$

non linear $\Downarrow$ Th 2.4.1 does not work

(a) $f(x,y) = \frac{3x^2+4x+2}{2(y-1)}$ $\Rightarrow \frac{\partial f}{\partial y} = -\frac{3x^2+4x+2}{2(y-1)^2}$

f and $\frac{\partial f}{\partial y}$ are continous everywhere except on the line $y=1$

We can draw an open rectangle about $(0,-1)$ on which f and $\frac{\partial f}{\partial y}$ are continvous as long as it does not cover $y=1$ $\Rightarrow$ Th 2.4.2 guarantees the IVP has a unique solution in some interval about $x=0$

Recall the solution $y(x) = 1 - \sqrt{x^3+2x^2+2x+4}$, $x > -2$

(b)
* Suppose now $y(0) = 1$ $\Rightarrow$ the initial point now lies on the line $y = 1$ where f and $\frac{\partial f}{\partial y}$ are not continuous $\Rightarrow$ no rectangle can be drawn $\Rightarrow$ Th 2.4.2 is not satisfied.

$y^2 - 2y = x^3 + 2x^2 + 2x + c$ $\quad$ $y(0) = 1 \Rightarrow c = -1$

$\Rightarrow$ The solution exists $y = 1 \pm \sqrt{x^3+2x^2+2x}$, $x > 0$ but it is not unique.

Example: Consider the IVP $y' = y^{\frac{1}{3}}$, $y(0) = 0$, $t \geq 0$.

Apply Th 2.4.2 to this IVP and solve the problem.

$f(t, y) = y^{\frac{1}{3}}$ is continuous every where

$\frac{\partial f}{\partial y} = \frac{1}{3} y^{\frac{-2}{3}}$ is not continuous when $y = 0$ "singularity point"

$\Rightarrow$ Th 2.4.2 does not apply for this problem.

* Using Remark 2.4.2 $\Rightarrow$ The continuity of f does ensure the existence of solutions but not their uniqueness:

$y^{-\frac{1}{3}} dy = dt \Rightarrow \frac{3}{2} y^{\frac{2}{3}} = t + c$ $\Rightarrow y(0) = 0 \Rightarrow c = 0$

$\Rightarrow y = \pm \left(\frac{2}{3} t\right)^{\frac{3}{2}}$ $\Rightarrow y_1 = \left(\frac{2}{3} t\right)^{\frac{3}{2}}$, $y_2 = -\left(\frac{2}{3} t\right)^{\frac{3}{2}}$, $y_3 = 0$ are all solutions.

can not be obtained by solving IVP

Example: Solve the IVP $y' = y^2$, $y(0) = 1$ and find the interval in which the solution exists.

$f(t, y) = y^2$ continuous everywhere

$\frac{\partial f}{\partial y} = 2y$ Continuous everywhere

Thus they are continuous at $t = 0$, so Th 2.4.2 guarantees that solution exists and is unique

$\Rightarrow$ seprable:

$y^{-2} dy = dt \Rightarrow -y^{-1} = t + c \Rightarrow y = \frac{-1}{t+c}$, $y(0) = 1$

$\Rightarrow y = \frac{-1}{t-1} = \frac{1}{1-t}$

$\Rightarrow y = \dfrac{1}{1-t}$ The solution is unbounded as $t \to 1$

$\Rightarrow$ The solution exists only on the interval $(-\infty, 1)$ $\Rightarrow$ because $y(0) = 1$ $\Rightarrow t_0 = 0 \in (-\infty, 1)$

	1^{st} order linear $y' + p(t)y = g(t)$, $y(0) = y_0$	1^{st} order nonlinear $y' = f(t,y)$ $y(0) = y_0$
Interval of definitions	The solution exists through any interval about $t = t_0$ on which p and g are continuous	The interval on which a solution exists may be difficult to find.
General Solution	It is possible to obtain a solution containing one arbitrary constant, from which all solutions follow by specifying values for this constant.	No general Solution. Even though a solution containing an arbitrary constant, may be found, there may be other solutions that can't be obtained by specifying values for this constant.
Explicit Solutions	$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s) g(s) ds + y_0\right]$ $\mu(t) = e^{\int_{t_0}^{t} p(s)\, ds}$ as long as the integrals can be solved.	May not exist. Unless the equation is simple enough.
Singularities "points of discontinuity" of Solution	Can be found without solving the problem	Usually can not be found without solving the problem

In section 2.8 : we will discuss the Existance and Uniqueness Theorem.