2.5 Autonomous Equations and Population Dynamics

* A 1st order equations whose independent variable does not appear explicitly are called autonomous and have the form
$$\frac{dy}{dt} = f(y)$$

* the idea here is to learn geometric methods can be used to obtain qualitative information directly from differential equation without solving it.

Example " Exponential Growth or decline : $\frac{dy}{dt} = ry$, $y(0) = y_0$

→ y is the population of a given species at time t.

→ r is the rate of growth "when it is positive" or the rate of decline "when it is negative"

r is constant

Solution ⟹ $y = y_0 e^{rt}$ assuming $r > 0$

Example " logistic Growth"

$$\frac{dy}{dt} = (r - ay)y \;,\; r, a > 0 \text{ : Verhulst or logistic equation}$$

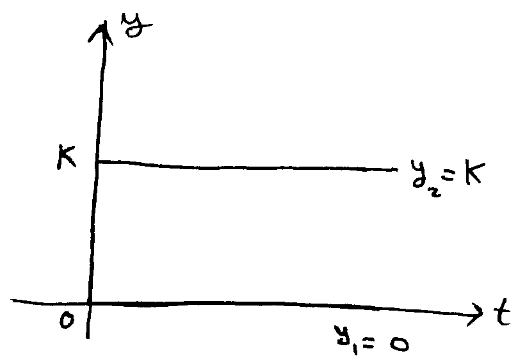
$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y \;,\quad K = \frac{r}{a} \text{ which is the carrying capacity of the population}$$

⟹ The eq. solutions are $y_1 = 0$ and $y_2 = K$

They are the roots of $f(y) = 0$.

The roots are called critical points.

⟹ In the Figure, how the direction fields look like?



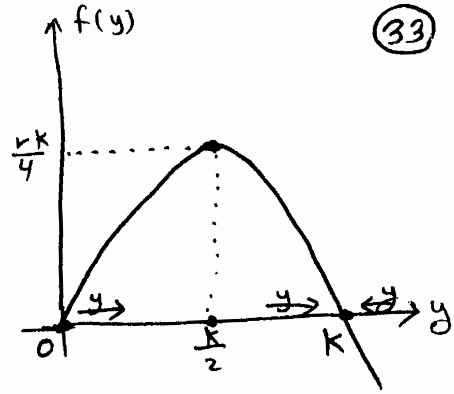
we start graphing $f(y)$ vs. y

$$f(y) = r\left(1 - \frac{y}{K}\right)y$$

The intercepts are $(0,0)$ and $(K,0)$

The graph is parabola.

The vertex of the parabola is $\left(\frac{K}{2}, \frac{rK}{4}\right)$



Because $f'(y) = r\left[y\left(\frac{-1}{K}\right) + \left(1 - \frac{y}{K}\right)\right]$

$$f'(y) = \frac{-r}{K}(2y - K)$$

$$f'(y) = 0 \Rightarrow y = \frac{K}{2}$$

$$f\left(\frac{K}{2}\right) = r\left(1 - \frac{K}{2K}\right)\left(\frac{K}{2}\right) = \frac{rK}{4}$$

$\Rightarrow$ $f(y) > 0$ (i.e $\frac{dy}{dt} > 0$) for $0 < y < K$

$\Rightarrow$ y is an increasing function of t ($\rightarrow$)

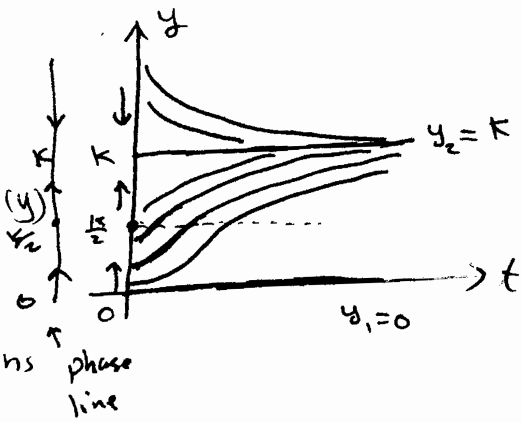
$\Rightarrow$ $f(y) < 0$ (i.e $\frac{dy}{dt} < 0$) for $y > K$

$\Rightarrow$ y is a decreasing function of t ($\leftarrow$)

$\Rightarrow$ Concavity / Convexity

For that we need y''

$$\frac{dy}{dt} = f(y) \Rightarrow \frac{d^2y}{dt^2} = f'(y)\frac{dy}{dt} = f'(y)\, f(y)$$



$\Rightarrow$ y is concave if $y'' < 0$
i.e f and f' have opposite signs

$\Rightarrow$ y is convex if $y'' > 0$
i.e f and f' have same signs.

$\Rightarrow$ The inflection points may occure when $f'(y) = 0$ i.e $y = \frac{K}{2}$

$\Rightarrow$ The solution is convex $\forall\, y \in (0, \frac{K}{2})$ where f is increasing ↗ +

The solution is concave $\forall\, y \in (\frac{K}{2}, K)$ where f is increasing ↘ +

$\Rightarrow$ The solution is convex $\forall\, y \in (K, \infty)$ where f is decreasing ↘ −

* The slope (when $y = 0$ or $y = K$) is zero

$\lim_{t \to \infty} y = K \Rightarrow K$ is asymptotically stable

0 is unstable

* K is called saturation level or carrying capacity (it's like an upper bound).

* If we wish to know the value of the population at some particular time, then we need to solve the problem, i.e 34

$$\frac{dy}{dt} = r\left(1-\frac{y}{K}\right)y \qquad y(0)=y_0 \qquad y\neq 0 \text{ and } y\neq K$$

$\Rightarrow \dfrac{dy}{\left(1-\frac{y}{K}\right)y} = r\,dt$ using partial fraction expansion $\Rightarrow$

$$\frac{1}{\left(1-\frac{y}{K}\right)y} = \frac{A}{1-\frac{y}{K}} + \frac{B}{y} \Rightarrow 1 = Ay + B\left(1-\frac{y}{K}\right) \Rightarrow B=1 \text{ and } A=\frac{1}{K}$$

$\Rightarrow$ The logistic equation can be written as

$$\left[\frac{1}{y} + \frac{\frac{1}{K}}{1-\frac{y}{K}}\right]dy = r\,dt \Rightarrow \ln|y| - \ln\left|1-\frac{y}{K}\right| = rt + c$$

$$\ln y_0 - \ln\left|1-\frac{y_0}{K}\right| = 0 + c$$

$\Rightarrow$ if $y_0 \in (0,K)$ then $y \in (0,K)$, so we remove the absolute value

$$\ln\left(\frac{y}{1-\frac{y}{K}}\right) = rt + c \Rightarrow \frac{y}{1-\frac{y}{K}} = e^{rt+c}$$

$$\ln\frac{y_0}{1-\frac{y_0}{K}} = c \qquad e^{c} = \frac{y_0}{1-\frac{y_0}{K}}$$

مرتب بعرص

$$\Rightarrow \boxed{y = \frac{y_0 K}{y_0 + (K-y_0)e^{-rt}}, \quad y_0 = y(0) \in (0,K)} \;\text{—}\; *$$

$\Rightarrow$ if $y_0 > K$, then $*$ still valid.

$\Rightarrow$ from $*$, we can see that $y_1 = 0$ and $y_2 = K$ are solutions.

$\Rightarrow \lim\limits_{t\to\infty} * = K$ (assymptotically stable)

$\Rightarrow$ The only way to gaurantee solution remains near zero is to make $y_0 = 0$

Example: (Halibut in the Pacific Ocean) Let y be the biomass in Kg of halibut population at time t, with $r = 0.71$ for every year and $K = 80.5 \times 10^6$ Kg. If $y_0 = 0.25\,K$, then find:

(a) biomass 2 years later

(b) the time τ such that $y(\tau) = 0.75\,K$.

(a) From $*$, we have $\boxed{\dfrac{y}{K} = \dfrac{\frac{y_0}{K}}{\frac{y_0}{K} + \left(1-\frac{y_0}{K}\right)e^{-rt}}}$ ① $\Rightarrow$

$$\frac{y(2)}{K} = \frac{0.25}{0.25 + 0.75\, e^{-0.71(2)}} \Rightarrow y(2) = 0.5797\ K$$
$$\approx 46.7 \times 10^6 \text{ kg}$$

(b) From ①, we have $\dfrac{y(\tau)}{K} = \dfrac{0.25}{0.25 + 0.75\, e^{-0.71\tau}}$

$$0.75 = \frac{0.25}{0.25 + 0.75\, e^{-0.71\tau}} \Rightarrow$$

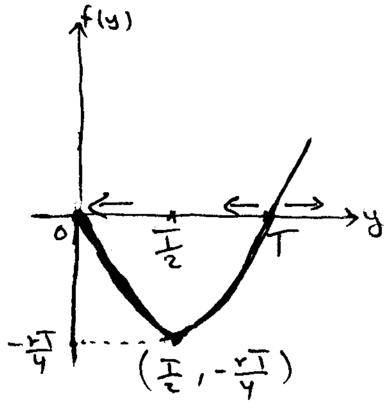
$$0.75\left(0.25 + 0.75\, e^{-0.71\tau}\right) = 0.25 \Rightarrow \frac{1}{3} = \frac{1}{4} + \frac{3}{4} e^{-0.71\tau}$$

$$\Rightarrow \frac{4}{3} - 1 = 3e^{-0.71\tau} \Rightarrow \frac{1}{9} = e^{-0.71\tau} \Rightarrow -0.71\tau = -\ln 9 \Rightarrow$$
$$\tau \approx 3.095 \text{ years}$$

Critical Threshold Equation

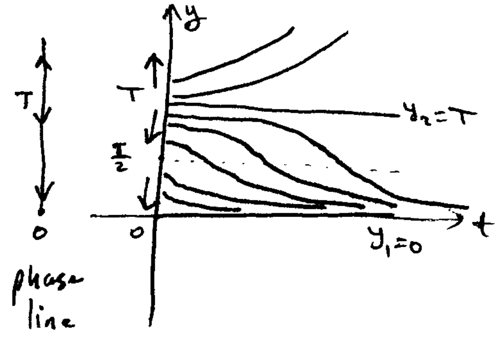
$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y, \quad r > 0 \quad \text{-----} \ ③$$

* Two modifications were made in the logistic ODE:
  1) minus sign ② K replaced by T "Threshold value for y_0"
* Note that the solution of ③ is very different from the logistic equation.



* The critical points are $y=0$ and $y=T$ $\Rightarrow$ the equilibrium solutions are $y_1 = 0$ and $y_2 = T$
* If $y_0 > T$, then the population grows unbounded
  If $y_0 < T$, then " " dies.
* $y_1(t) = 0$ is an asymptotically stable eq. solution.
  $y_2(t) = T$ is unstable one.
* Solving ③, we get $y = \dfrac{y_0 T}{y_0 + (T - y_0)e^{rt}}$

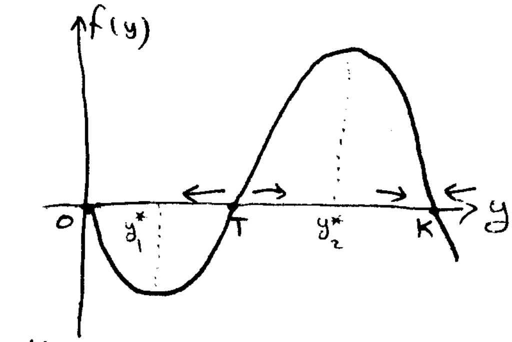
  clearly $\lim\limits_{t\to\infty} y = 0$



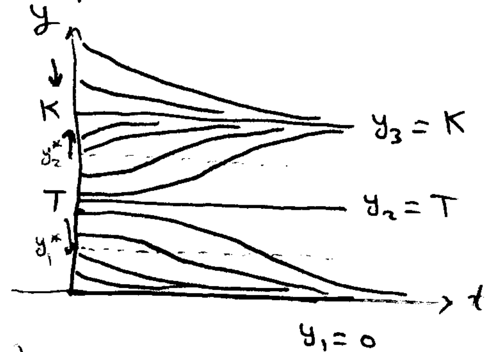
Logistic Growth with a Threshold

To avoid unbounded growth when $y > T$, we consider the following modification to the logistic equation

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y \quad , \; r > 0 \text{ and } 0 < T < K$$

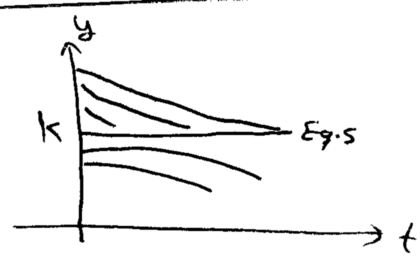


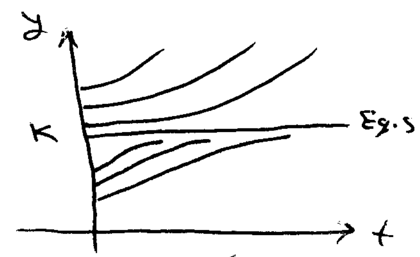
- K is the carrying capacity level
  T is the threshold value for y_0
- $y_1(t) = 0$ and $y_3(t) = K$ are stable equilibrium solutions and
  $y_2(t) = T$ is an unstable eq. solution.



- The critical points are the roots of $f(y) = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$
  i.e. $y = 0, T, K$
- The inflection points are y_1^* and y_2^*
  $f'(y) = 0 \Rightarrow y_{1,2}^* = \left(K + T \pm \sqrt{K^2 - KT + T^2}\right)/3$

Note: **Semistable Equilibrium Solutions:** is a solution has the property that solutions lying on one side of the eq. solution tend to approach it, whereas solutions lying on the other side depart from it.





If $\frac{dy}{dt} = f(y)$ and y_1 is an critical point $(f(y_1) = 0)$, then y_1 is asymptotically stable if $f'(y_1) < 0$ and y_1 is unstable if $f'(y_1) > 0$.