2.6 Exact Equations and Integrating Factors

Example 1: Solve the DE $2x + y^2 + 2xy\,y' = 0$

This equation is niether linear nor separable. How?!

Th 2.6.1: Suppose an ODE has the form $\boxed{M(x,y) + N(x,y)\,y' = 0}$ ①

where the functions M, N, M_y, N_x are all continous in the region $R: (x,y) \in (\alpha, \beta) \times (\gamma, \delta)$. Then equation ① is exact differential equation iff $\boxed{M_y(x,y) = N_x(x,y) \quad \forall (x,y) \in R}$ ②

That is, $\exists$ a function ψ satisfying the conditions

$$\boxed{\psi_x(x,y) = M(x,y) \quad \text{and} \quad \psi_y(x,y) = N(x,y)} \text{ ③}$$

iff M and N satisfy equation ②.

Note that when $M(x,y) + N(x,y)\,y' = 0 \Rightarrow$

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0 \Rightarrow$$

$$\frac{d}{dx}\psi(x,y) = 0 \Rightarrow$$

$\psi(x,y) = c$ defines a solution implicity $y = \phi(x)$

Example 1 Solve the DE $2x + y^2 + 2xy\,y' = 0$

$M(x,y) = 2x + y^2 \Rightarrow M_y = 2y$
$N(x,y) = 2xy \Rightarrow N_x = 2y$
$\rightarrow$ ODE is exact

Thus, $\exists$ a function ψ s.t $\psi_x(x,y) = 2x + y^2$ and $\psi_y(x,y) = 2xy$

$$\psi(x,y) = \int \psi_x(x,y)\,dx = \int M(x,y)\,dx = \int (2x + y^2)\,dx$$

$$\boxed{\psi(x,y) = x^2 + xy^2 + h(y)}$$

$$\Rightarrow \psi_y(x,y) = 2xy + h'(y) = N(x,y) = 2xy \Rightarrow h'(y) = 0 \Rightarrow h(y) = c$$

$\Rightarrow \psi(x,y) = x^2 + xy^2 + c$ and the solution is given implicitly by $x^2 + xy^2 = c$

<u>Proof</u>: <u>1st part</u>: suppose $\exists$ a function ψ s.t ③ holds, we need to show that equation ② holds.

From equation ③, we have:

$M_y(x,y) = \psi_{xy}(x,y)$ and $N_x(x,y) = \psi_{yx}(x,y)$.

Since M_y and N_x are continuous, we have ψ_{xy} and ψ_{yx} are continuous. Thus are equal and ② holds.

<u>2nd part</u>: suppose M and N satisfy ②, we need to show that ① is exact. That is, we need to construct a function ψ satisfies ③.

Recall that ③ is given by

$\psi_x(x,y) = M(x,y)$ and $\boxed{\psi_y(x,y) = N(x,y)}$ *

To find ψ, we integrate, for example, the first equality

$$\boxed{\psi(x,y) = \int M(x,y)\,dx + h(y)} \; * \quad \Rightarrow \text{differentiate w.r.t } y$$

$$\psi_y(x,y) = \frac{\partial}{\partial y}\int M(x,y)\,dx + h'(y) \quad \Rightarrow \text{use } *$$

$$N(x,y) = \int M_y(x,y)\,dx + h'(y) \quad \Rightarrow \text{Solve for } h'(y)$$

$$\boxed{h'(y) = N(x,y) - \int M_y(x,y)\,dx} \; Ⓐ \quad \Rightarrow \text{integrate w.r.t } y$$

$$h(y) = \int N(x,y)\,dy - \iint M_y(x,y)\,dx\,dy \quad \Rightarrow \text{substitute in } *$$

$$\boxed{\psi(x,y) = \int M(x,y)\,dx + \int N(x,y)\,dy - \iint M_y(x,y)\,dx\,dy} \; Ⓑ$$

Note that the right hand side of Ⓐ is only function of y because if we differentiate w.r.t x, we get $N_x(x,y) - M_y(x,y)$ which is zero by ②

Hence from Ⓑ $\Rightarrow \psi_x(x,y) = M(x,y)$ and $\psi_y(x,y) = N(x,y)$. Thus, ③ holds.

Example : Solve the DE $\frac{dy}{dx} = -\frac{x+4y}{4x-y}$

$$(4x-y)\frac{dy}{dx} = -(x+4y) \iff (x+4y) + (4x-y)y' = 0$$

- $M(x,y) = x+4y$ and $N(x,y) = 4x-y$
- $M_y = 4 = N_x \Rightarrow$ ODE is exact

Thus, $\exists$ a function ψ s.t $\psi_x = M(x,y) = x+4y$ and

$\psi_y = N(x,y) = 4x-y$

$$\psi(x,y) = \int \psi_x(x,y)\,dx = \int (x+4y)\,dx = \tfrac{1}{2}x^2 + 4xy + h(y) \quad *$$

$$\Rightarrow \psi(x,y) = \tfrac{1}{2}x^2 + 4xy + h(y)$$

$$\psi_y(x,y) = 4x + h'(y) = 4x - y$$

$$h'(y) = -y \Rightarrow h(y) = -\tfrac{1}{2}y^2 + c$$

Thus $*$ becomes

$\psi(x,y) = \tfrac{1}{2}x^2 + 4xy - \tfrac{1}{2}y^2 + c$ and the solution is given implicitly by $x^2 + 8xy - y^2 = c$

Example: Solve the DE $(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$

- $M(x,y) = y\cos x + 2xe^y$ and $N(x,y) = \sin x + x^2e^y - 1$
- $M_y(x,y) = \cos x + 2xe^y = N_x(x,y) \Rightarrow$ ODE is exact.

Thus, $\exists$ a function ψ such that

$\psi_x(x,y) = M(x,y) = y\cos x + 2xe^y$ and $\psi_y(x,y) = N(x,y) = \sin x + x^2e^y - 1$

$$\Rightarrow \psi(x,y) = \int \psi_x(x,y)\,dx = \int (y\cos x + 2xe^y)\,dx = y\sin x + x^2e^y + h(y) \quad *$$

$$\psi_y(x,y) = \sin x + x^2e^y + h'(y) = \sin x + x^2e^y - 1$$

$\Rightarrow h'(y) = -1 \Rightarrow h(y) = -y + c$

substituting $h(y)$ in $*$ gives

$\psi(x,y) = y\sin x + x^2 e^y - y + c$ and the solution is given implicitly by $y\sin x + x^2 e^y - y = c$

Example Solve the DE $\boxed{(3xy + y^2) + (xy + x^2)y' = 0}$ — ①

$M(x,y) = 3xy + y^2$ and $N(x,y) = xy + x^2$

$M_y = 3x + 2y \neq y + 2x = N_x$ $\Rightarrow$ ODE is not exact

Our DE cannot be solved by this method. To see that

$\psi_x = M = 3xy + y^2$ and $\psi_y = N = xy + x^2$ $\quad *$

$\Rightarrow \psi(x,y) = \int \psi_x \, dx = \int (3xy + y^2)\, dx = \frac{3}{2}x^2 y + xy^2 + h(y)$

$\psi_y(x,y) = \cancel{xy} + x^2 = \frac{3}{2}x^2 + 2\cancel{xy} + h'(y) \Rightarrow$

$h'(y) = -xy - \frac{1}{2}x^2 \Rightarrow h(y) \overset{??}{=} -\frac{x}{2}y^2 - \frac{1}{2}x^2 y$ depends on x and y as well

$\Rightarrow$ it is impossible to slove $h(y)$ for y. Hence $\nexists\ \psi$ satisfies $*$

How to solve ① then?!

We solve ① using the Integrating Factors: It is sometimes possible to convert a DE that is not exact to an exact one by multiplying the DE by a suitable integrating factor $I(x,y)$:

$$M(x,y) + N(x,y)\,y' = 0$$

$$I(x,y)\,M(x,y) + I(x,y)\,N(x,y)\,y' = 0 \quad \text{......} \quad Ⓐ$$

For Ⓐ to be exact, we need

$$(IM)_y = (IN)_x$$

يشبه سؤال (24) في الوظيفة

$$\Leftrightarrow \quad I_y M + I M_y = I_x N + I N_x$$

$$\Leftrightarrow \quad I_y M - I_x N + (M_y - N_x) I = 0 \quad \text{.... (B)}$$

Note that (B) is a 1^{st} order partial differential equation

function of x, $I(x)$, or function of y, $I(y)$, or function of xy, $I(xy)$

(1) If $\dfrac{M_y - N_x}{N} = f(x)$, then $I(x) = e^{\int f(x)\,dx}$

This means that $I(x)$ is only function of x and so $I_y = 0$

so from (B) we have $I_x N = (M_y - N_x) I \Rightarrow \dfrac{I_x}{I} = \dfrac{M_y - N_x}{N}$

$\Rightarrow \int \dfrac{I_x}{I}\,dx = \int \dfrac{M_y - N_x}{N}\,dx \Rightarrow \ln I = \int f(x)\,dx \Rightarrow I(x) = e^{\int f(x)\,dx}$

(2) If $\dfrac{M_y - N_x}{M} = g(y)$, then $I(y) = e^{-\int g(y)\,dy}$

This means that $I(y)$ is only function of y and so $I_x = 0$

so from (B) we have $I_y M = -(M_y - N_x) I \Rightarrow \dfrac{I_y}{I} = -\dfrac{M_y - N_x}{M}$

$\Rightarrow \int \dfrac{I_y}{I}\,dy = -\int \dfrac{M_y - N_x}{M}\,dy \Rightarrow \ln I = -\int g(y)\,dy \Rightarrow I(y) = e^{-\int g(y)\,dy}$

(3) If $\dfrac{N_x - M_y}{xM - yN} = h(v)$, then $I(v) = e^{\int h(v)\,dv}$ where $v = xy$

This means that $I = xy \Rightarrow I_x = y$ and $I_y = x$

$\Rightarrow$ so from (B) we have $xM - yN = (N_x - M_y)\,xy \Rightarrow \dfrac{1}{xy} = \dfrac{N_x - M_y}{xM - yN}$

$\Rightarrow \dfrac{1}{v} = h(v) \Rightarrow \ln v = \int h(v)\,dv \Rightarrow I(v) = e^{\int h(v)\,dv}$.

Now let us solve the previous example : $(3xy + y^2) + (xy + x^2)y' = 0$

We know that $M_y \neq N_x$ so it is not exact.

$M = 3xy + y^2$

$M_y = 3x + 2y$

$N = xy + x^2$

$N_x = y + 2x$

(1) $\dfrac{M_y - N_x}{N} = \dfrac{3x + 2y - 2x - y}{xy + x^2} = \dfrac{x + y}{x(x+y)} = \dfrac{1}{x}$

$I(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$

$\Rightarrow$ Multiply the DE by $I(x) = x$ "integrating factor"

$$\Rightarrow \boxed{(3x^2y + xy^2) + (x^2y + x^3)\, y' = 0} \quad (C)$$

$M(x,y) = 3x^2y + xy^2$ and $N(x,y) = x^2y + x^3$

$M_y(x,y) = 3x^2 + 2xy = N_x(x,y)$ $\Rightarrow$ C is exact ODE

Thus, $\exists$ a function ψ s.t

$$\psi_x = M = 3x^2y + xy^2 \quad \text{and} \quad \psi_y = N = x^2y + x^3$$

$$\psi(x,y) = \int \psi_x(x,y)\,dx = \int (3x^2y + xy^2)\,dx = x^3y + \tfrac{1}{2}x^2y^2 + h(y)$$

$$\psi_y = \cancel{x^2y} + \cancel{x^3} = \cancel{x^3} + \cancel{x^2y} + h'(y) \quad \Rightarrow h'(y) = 0 \Rightarrow h(y) = c$$

Hence, $\psi(x,y) = x^3y + \frac{1}{2}x^2y^2 + c$ and the solution is given implicitly by $x^3y + \frac{1}{2}x^2y^2 = c$