2.7 Numerical Approximations: Euler's Method

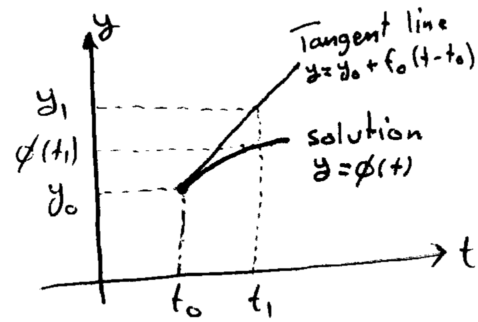
Recall that a 1^{st} order IVP has the form $\boxed{\frac{dy}{dt} = f(t,y) \ , \ y(t_0) = y_0}$ — ①

- If f and $\frac{\partial f}{\partial y}$ are continuous, then ① has a unique solution $y = \phi(t)$ in some interval about t_0

- If ① is linear or seprable or exact, then we can find the solution. Otherwise, it not possible to find analytical solution for equations of the form①. Therefore, it is important to be able to approach the problem in other ways. For example: Direction field: to study the behavior of the solution without solving the differential equation. However, direction field does not lend itself to quantitative computation or comparisons.

- An alternative way to approximate solution is the tangent line method or Euler method.

  - We know the solution $y = \phi(t)$ passes through the initial point (t_0, y_0) and from ① we know also the slope at this point is $f(t_0, y_0)$. Thus, the tangent line to the solution at (t_0, y_0) is



$$y = y_0 + f(t_0, y_0)(t - t_0)$$

- The tangent line is a good approximation to the solution curve on an interval short enough (the slope does not change).

- Thus, if t_1 is close enough to t_0, we approximate $y_1 = \phi(t_1)$ by

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$

- If t_2 is close enough to t_1, we approximate $y_2 = \phi(t_2)$ using the line passes through (t_1, y_1) with slope $f(t_1, y_1)$ by

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

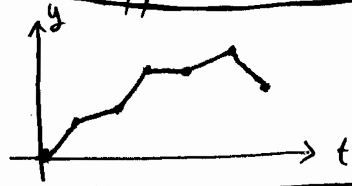
⇒ Hence, we can create a sequence y_n of approximations to $\phi(t_n)$ by:

$$y_1 = y_0 + f_0\,(t_1 - t_0)$$
$$y_2 = y_1 + f_1\,(t_2 - t_1)$$
$$\vdots$$
$$y_{n+1} = y_n + f_n\,(t_{n+1} - t_n) \qquad \text{where } f_n = f(t_n, y_n)$$

- For a uniform step size $h = t_{n+1} - t_n$, Euler's formula becomes

$$\boxed{y_{n+1} = y_n + f_n h \qquad n = 0, 1, 2, \dots} \quad (*)$$

- Euler Approximation: To graph an Euler approximation, we plot the points $(t_0, y_0), (t_1, y_1), \dots, (t_n, y_n)$ and then connect these points with line segments



Example 1 Consider the IVP $\boxed{y' = 9.8 - 0.2y \;,\; y(0) = 0}$ ②

(a) Use Euler's method with step size $h = 0.1$ to approximate values of the solution of ② at $t = 0.1, 0.2, 0.3, 0.4$.

(b) Compare the values in (a) with the corresponding values of the actual solution of ②.

(a) Using $(*)$ we have $y_1 = y_0 + f_0\,h = 0 + 9.8\,(0.1) = 0.98$

$$y_2 = y_1 + f_1 h = 0.98 + [9.8 - 0.2(0.98)](0.1) \approx 1.94$$
$$y_3 = y_2 + f_2 h = 1.94 + [9.8 - 0.2(1.94)](0.1) \approx 2.88$$
$$y_4 = y_3 + f_3 h = 2.88 + [9.8 - 0.2(2.88)](0.1) \approx 3.80$$

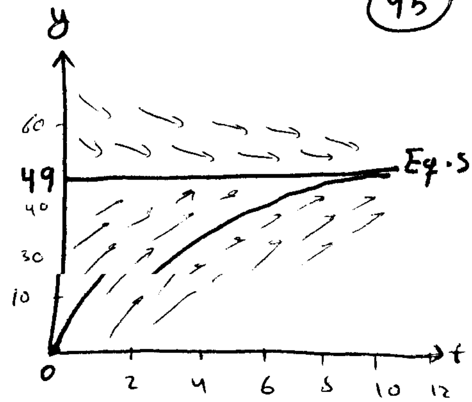
(b) We find the exact solution of ②: $y' = -0.2(y - 49)$

$$\frac{dy}{y-49} = -0.2\,dt \Rightarrow \ln|y-49| = -0.2t + C \Rightarrow y = 49 + k e^{-0.2t}$$
$$k = \pm e^{C}$$

$$y(0) = 0 \Rightarrow k = -49 \Rightarrow \boxed{y = 49\left(1 - e^{-0.2t}\right)}$$

Euler's Method with h = 0.1

t	Exact y	Approximat y	Error
0.00	0.00	0.00	0.00
0.10	0.97	0.98	-1.03
0.20	1.93	1.94	-1.04
0.40	3.77	3.80	-0.80



Example 2 Consider the IVP $\frac{dy}{dt} = 3 + e^{-t} - \frac{1}{2}y$, $y(0) = 1$

(a) Use Euler's Method with step sizes $h = 0.1, 0.05, 0.025, 0.01$ to approximate values of the solution for $t = 0, 1, 2, 3, 4, 5$.

(b) Compare the calculated results with the crrosponding values of the exact solution.

The exact solution "Integrating factor" is $y = \phi(t) = 6 - 2e^{-t} - 3e^{-\frac{t}{2}}$

t	Exact	h = 0.1	h = 0.05	h = 0.025	h = 0.01
0.00	1.00	1.00	1.00	1.00	1.00
1.00	3.44	3.52	3.48	3.46	3.45
2.00	4.63	4.70	4.66	4.64	4.63
3.00	5.23	5.29	5.26	5.25	5.24
4.00	5.56	5.60	5.58	5.57	5.56
5.00	5.74	5.77	5.76	5.75	5.74

Note that we are making some cut in this table, i.e

To go from $t=0$ to $t=5$

- using $h = 0.1$, we need 50 step
- using $h = 0.05$, —— ..100 steps
- using $h = 0.025$, ... 200 steps
- using $h = 0.01$, ... 500 steps

From the table, we observ the following:

1) For a fixed value of t, the approximate values become more accurate as h decreases.

2) For a fixed step size h, the approximate values become more accurate as t increases.

Note that $\lim_{t \to \infty} y = 6$, so we can see from the table that the values approach the limit 6. Thus we have convergence for the computed values.

Example $\boxed{3}$ Consider the IVP $y' = 4 - t + 2y$, $y(0) = 1$

Use the Euler's Method with $h = 0.1$ to approximate the solution at $t = 1, 2, 3, 4$ and compare the result with the exact values.

$t_{n+1} = t_n + h$

$f_n = f(t_n, y_n)$

$$y_1 = y_0 + f_0 h = 1 + [4 - 0 + 2(1)](0.1) = 1.6$$

$$y_2 = y_1 + f_1 h = 1.6 + [4 - 0.1 + 2(1.6)](0.1) = 2.31$$

$$y_3 = y_2 + f_2 h = 2.31 + [4 - 0.2 + 2(2.31)](0.1) \simeq 3.15$$

$\vdots$

Note that the exact solution (integrating factor) is

$$y = \phi(t) = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

t	Exact	Approximaty	Error
0.00	1.00	1.00	0.00
1.00	19.07	15.78	17.27
2.00	149.39	104.68	29.93
3.00	1109.18	652.53	41.17
4.00	8197.88	4042.12	50.69

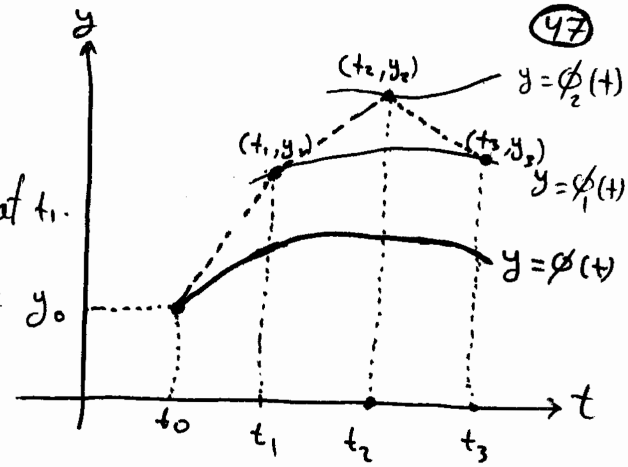
Note that we are making some cut in this table

To go from $t=0$ to $t=4$ we need 40 steps

Note that $\lim_{t \to \infty} y = \infty$, so we can see from the table the values increase exponentially without upper bound. Thus we have diveragen (to ∞) for the computed values.

Question: Why the error in Example $\boxed{3}$ is much Larger than those in Examples $\boxed{1}$ and $\boxed{2}$?

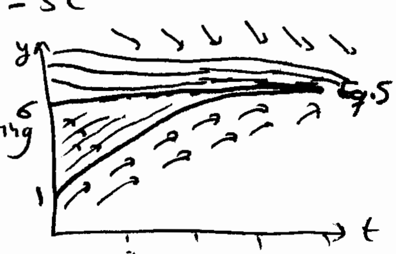
- Recall the general IVP $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$ with solution denoted by $y = \phi(t)$.
- Recall that a 1^{st} order DE has an infinite family of solutions, indexed by an arbitrary constan c. The initial condition picks out one member of this infinite family by determining the value of c.
- Thus $\phi(t)$ is the member of the infinite family of solutions that satisfies the initial condition $\phi(t_0) = y_0$.



- The 1^{st} step of Euler's Method uses the tangent line of $y = \phi(t)$ at the point (t_0, y_0) to estimate y_1 at t_1.
- The point (t_1, y_1) is typically not on the graph of ϕ i.e $y_1 \neq \phi(t_1)$ because y_1 is an approximation of $\phi(t_1)$.
- Hence, the next iteration of Euler's method does not use a tangent line approximation to $y = \phi(t)$, but rather to nearby solution $y_1 = \phi_1(t)$ that passes through the point (t_1, y_1).
- Thus, Euler's method uses a succession of tangent lines to a sequence of different solutions $\phi(t)$, $\phi_1(t)$, $\phi_2(t)$, $\cdots$ to the differential equation. Hence, the accuracy after many steps depends on the behavior of solutions passing through (t_n, y_n), $n = 1, 2, 3, \cdots$
- Convergent Family of Solutions: Consider the IVP in Example $\boxed{2}$:

$$y' = 3 + e^{-t} - \tfrac{1}{2} y \;,\; y(0) = 1 \;\Rightarrow\; y = \phi(t) = 6 - 2e^{-t} - 3e^{-\frac{t}{2}}$$

- The direction field tells us the behavior of solution.
- It does not matter which solutions we are approximating with tangent lines, because all solutions get closer to each other as t increases.



Converging family of solutions.

- Divergent Family of solutions: Consider the IVP in Example $\boxed{3}$: $y' = 4 - t + 2y$, $y(0) = 1$

$$\Rightarrow y = \phi(t) = -\frac{7}{4} + \frac{t}{2} + \frac{11}{4} e^{2t}$$

- Since the family of solutions is divergent, at each step of Euler's method, we are following a different solution separating from the desired one more and more as t increases.