2.8 The Existence and Uniqueness Theorem

In this section, we discuss the proof of Th 2.4.2 "The fundemental Existence and Uniqueness Theorem" which is equivellent to

Th 2.8.1: If f and $\frac{\partial f}{\partial y}$ are continuous in a rectangle $R: |t| \leq a, |y| \leq b$ then there is some interval $|t| \leq h \leq a$ in which there exist a unique solution $y = \phi(t)$ of the IVP

$$\boxed{y' = f(t, y) \quad , \quad y(0) = 0} \quad (1)$$

Note that in Th 2.4.2 the initial condition is $y(t_0) = y_0$ and here the initial condition is $y(0) = 0$. Actually we can always use change of variables, translation of the coordinate axes, and take a given point (t_0, y_0) to the origin.

Example Transform the IVP $y' = y^3 - t^2$, $y(2) = -5$ to an equivalent IVP with the initial condition at the origin.

$s = t - 2$ so when $t_0 = 2 \Rightarrow s_0 = 2 - 2 = 0 \Rightarrow \boxed{t = s + 2}$ (a)

$z = y + 5$ so when $y_0 = -5 \Rightarrow z_0 = -5 + 5 = 0 \Rightarrow \boxed{y = z - 5}$ (b)

substitute (a) and (b) in the IVP, we get

$$z' = (z - 5)^3 - (s + 2)^2 \quad , \quad z(0) = 0$$

Note that we can solve the IVP (1) in some cases (if it is linear, exact or separable), but in general there is no method of solving (1) that applies in all cases.

Proof Th 2.8.1: The existence: suppose $\exists$ a function $y = \phi(t)$ that satisfies the IVP (1). Then $f(t, \phi(t))$ is a continuous function of t only. Integrate (1) from $t = 0$ to an arbitrary value of t:

$$\boxed{\phi(t) = \int_0^t f(s, \phi(s))\, ds} \quad (2)$$

Note that equation (2) satisfies the initial condition $\phi(0)=0$. Moreover, (2) contains an integral of unknow function ϕ, so it is called an <u>integral equation</u>.

* The IVP (1) and the integral equation (2) are equivalent in the sense that any solution of one is also a solution of the other. That is ($\Leftarrow$) suppose $\exists$ a continuous function $y=\phi(t)$ satisfies (2), then $\phi(0)=0$. Since f in (2) is continuous, it follows from the <u>Fundemental theorem of calculus</u>[1] that $\phi'(t)=f(t,\phi(t))=y$. Thus $y=\phi(t)$ is a solution to the IVP (1).

* Thus, it is more convenient to show that $\exists$ a unique solution of the integral equation (3) (and so $\exists$ a unique solution of the IVP (1)).

* One method to show that (2) has a unique solution is known as:

(1): Let f be a continuous real valued function on $[a,b]$. Let F be the function defined by $F(x)=\int_a^x f(t)\,dt \quad \forall\, x\in[a,b]$.

Then F is continuous on $[a,b]$, differentiable on (a,b) and $F'(x)=f(x) \quad \forall\, x\in(a,b)$.

Picard's iteration method or
Method of successive approximation:

→ Choose an initial function ϕ_0: The simplest choice is $\phi_0(t)=0$.

→ Construct:

$$\phi_1(t)=\int_0^t f(s,\phi_0(s))\,ds$$

$$\phi_2(t)=\int_0^t f(s,\phi_1(s))\,ds$$

$$\phi_3(t)=\int_0^t f(s,\phi_2(s))\,ds$$

$$\vdots$$

$$\phi_{n+1}(t)=\int_0^t f(s,\phi_n(s))\,ds$$

$$\vdots$$

- Thus, we generate the sequence of functions $\{\phi_n\}=\phi_0,\phi_1,\dots,\phi_n,\dots$ Each satisfies the initial condition, but in general none satisfies the DE.
- If $\phi_{n+1}(t)=\phi_n(t)$, then ϕ_n is a solution of (2), and so it is a solution of (1) and we stope at n.

$\Rightarrow$ In general, this does not occur, and we need to consider the entire infinite sequence.

<u>Example</u>: Use the Picard's iteration method to solve the IVP

$$y' = 2t(1+y) \quad , \quad y(0) = 0 \quad \text{------} \; A$$

$y' - 2ty = 2t$
$\mu(t) = e^{-\int 2t} = e^{-t^2}$
$y(t) = e^{t^2}[\ldots]$
$= e^{t^2} - 1$

$\rightarrow$ Take the initial approximation $\phi_0(t) = 0$

$$\rightarrow \phi_1(t) = \int_0^t f(s, \phi_0(s))\,ds = \int_0^t f(s,0)\,ds = \int_0^t 2s(1+0)\,ds = t^2$$

$$\rightarrow \phi_2(t) = \int_0^t f(s, \phi_1(s))\,ds = \int_0^t f(s, s^2)\,ds = \int_0^t 2s(1+s^2)\,ds = t^2 + \frac{t^4}{2}$$

$$\rightarrow \phi_3(t) = \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2}\right)ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}$$

$$\rightarrow \phi_4(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \frac{t^8}{24}$$

$\vdots$

$$\boxed{\phi_n(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{3!} + \cdots + \frac{t^{2n}}{n!} \qquad n \geq 1} \quad \text{------} \; (*)$$

$(*)$ is true and we can prove it by induction:

$\rightarrow$ $(*)$ is true for $n=1$ $\Rightarrow \phi_1(t) = t^2$ ✓

$\rightarrow$ Assume $(*)$ is true for $n=k$, we need to show it is true for $n = k+1$

$$\phi_{k+1}(t) = \int_0^t 2s\left[1 + \phi_k(s)\right]ds$$

$$= \int_0^t 2s\left[1 + s^2 + \frac{s^4}{2} + \frac{s^6}{6} + \cdots + \frac{s^{2k}}{k!}\right]ds$$

$$= t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2k+2}}{(k+1)!} \quad ✓$$

Note that $\phi_n(t)$ in $(*)$ is the n^{th} partial sum of the infinite series.

$$\boxed{\sum_{k=1}^{\infty} \frac{t^{2k}}{k!}} \text{ --- } (a)$$

Hence $\lim_{n \to \infty} \phi_n(t)$ exists iff (a) converges.

$\Rightarrow$ To check if ⓐ converges or not we apply the ratio test[2]: For each t, we have

$$\left| \frac{t^{2K+2}}{(K+1)!} \, \frac{K!}{t^{2K}} \right| = \frac{t^2}{K+1} \longrightarrow 0 \quad \text{as } K \to \infty$$

(2): suppose we have the series $\sum a_n$. Define $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$.

$\to$ if $L > 1$, then the series is absolutely convergent (and hence convergent).

$\to$ if $L < 1$, then the series is divergent.

$\to$ if $L = 1$, then the ratio test fails.

Thus the series ⓐ converges $\forall t$, and its sum $\phi(t)$ is the limit of the sequence $\{\phi_n(t)\}$

$$\phi(t) = \lim_{n\to\infty} \phi_n(t)$$

* Note that: $$\phi(t) = \lim_{n\to\infty} \phi_n(t) = e^{t^2} - 1$$

Taylor series

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

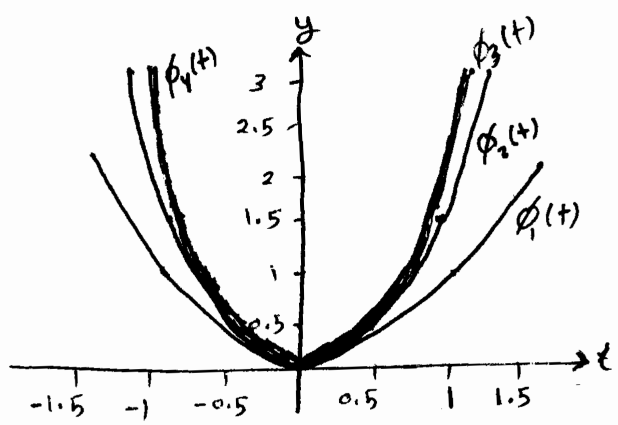
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \Rightarrow$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots \Rightarrow$$

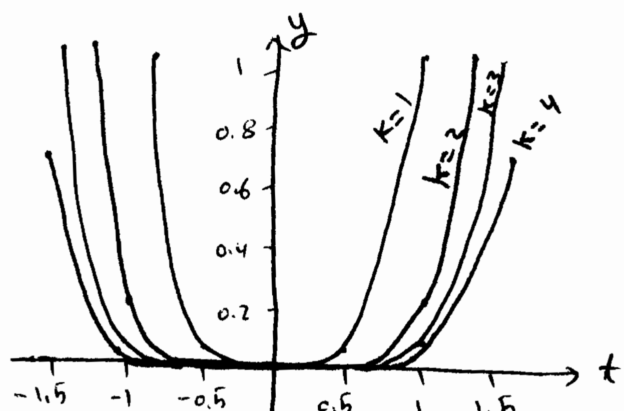
$$e^{x^2} - 1 = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots = \sum_{K=1}^{\infty} \frac{x^{2K}}{K!}$$

* Note that $\phi(t) = e^{t^2} - 1$ is a solution of the integral equation ②. Thus, it is also a solution of the IVP ①.

*



Plots of $\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)$



Plots of $\phi(t) - \phi_K(t)$ $K = 1,2,3,4$ to visualize the convergence.

Proof Th 2.8.1 : The uniqueness

Assume that ϕ and ψ are solutions of the IVP (A). Thus, ϕ and ψ are solutions of it's integral equation:

$$\phi(t) = \int_0^t 2s\left(1+\phi(s)\right) ds \quad \text{and} \quad \psi(t) = \int_0^t 2s\left(1+\psi(s)\right) ds$$

We are Considering the IVP in the previous example (as a special case) in which the integral equation $h(t) = \int_0^t 2s\,[1+h(s)]\, ds$.

$$\phi(t) - \psi(t) = \int_0^t 2s\,[\phi(s) - \psi(s)]\, ds \quad \Rightarrow$$

$$|\phi(t) - \psi(t)| = \left|\int_0^t 2s\,[\phi(s) - \psi(s)]\, ds\right| \overset{(1)}{\leq} \int_0^t 2s\,|\phi(s) - \psi(s)|\, ds$$

Let t be restricted $0 \leq t \leq \frac{A}{2}$ so that

$$0 \leq 2t \leq A$$

$$\Rightarrow \boxed{|\phi(t) - \psi(t)| \leq A \int_0^t |\phi(s) - \psi(s)|\, ds} \quad (5)$$

(1): property of integral:
$\int -|f(x)| \leq \int f(x) \leq \int |f(x)|$
$-\int |f(x)|dx \leq \int f(x)dx \leq \int |f(x)|dx$
$\left|\int f(x)dx\right| \leq \int |f(x)|dx$

Let $U(t)$ be the function given by

$$U(t) = \int_0^t |\phi(s) - \psi(s)|\, ds \quad \Rightarrow U(0) = 0 \quad \text{and} \;\cdots\cdots (6)$$

$$U(t) \geq 0 \quad \forall\; t \geq 0 \cdots (7)$$

$$U'(t) = |\phi(t) - \psi(t)| \cdots (8)$$

Thus (5) becomes $U'(t) - A\,U(t) \leq 0$ multiply by positive quantity e^{-At}

$$\Rightarrow e^{-At}\,U'(t) - A\,e^{-At}\,U(t) \leq 0$$

$$\Rightarrow \left[e^{-At}\,U(t)\right]' \leq 0 \qquad \text{integrating from } 0 \text{ to } t$$

$$\Rightarrow e^{-At}\,U(t) \leq c \qquad \text{using (6)}$$

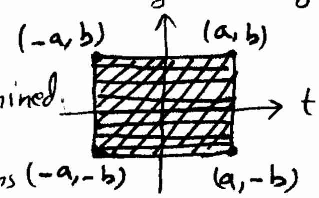
$$\Rightarrow e^{-At}\,U(t) \leq 0 \quad \text{for } t \geq 0$$

$\Rightarrow U(t) \le 0$ for $t \ge 0$, and in conjuction with ⑦, we must have $U(t) = 0 \;\; \forall \; t \ge 0$.

Thus, $U'(t) = 0 \;\; \forall \; t \ge 0$. From ⑧ we conclude that $\phi(t) = \psi(t)$, which contradicts the original hypothesis.

Notes

1. Do all members of the sequence $\{\phi_n\}$ exist? In the previous example f and $\frac{\partial f}{\partial y}$ were continuous in the whole ty-plane, so each member of the sequence could be explicitly calculated. But in general f and $\frac{\partial f}{\partial y}$ are continuous in $R: |t| \le a, |y| \le b$. Hence, the member of the sequence can not be explicitely determined.



→ For $n = k$, the graph of $y = \phi_k(t)$ may contains points lie outside R, so to compute $\phi_{k+1}(t)$, we need to evaluate $f(t, y)$ at points where it is not known to be continous or even exist!!

→ To avoid this, we restrict t to a smaller interval than $|t| \le a$.

<u>To find such an interval</u>: Recall that a continous function on a closed interval is bounded. Hence, f is bounded on R. Thus, $\exists \; M > 0$ s.t $|f(t, y)| \le M \;\; \forall \; (t, y) \in R$. Since $f(t, \phi_k(t)) = \phi'_{k+1}(t) = M$, it follows that M represents the maximum absolute slope of the graph of $y = \phi_{k+1}(t)$.

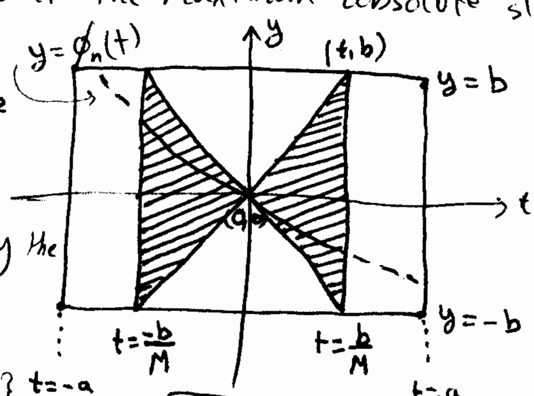
$M = \frac{\Delta y}{\Delta x} = \frac{b - 0}{t - 0} = \frac{b}{t}$

$\Rightarrow t = \frac{b}{M}$

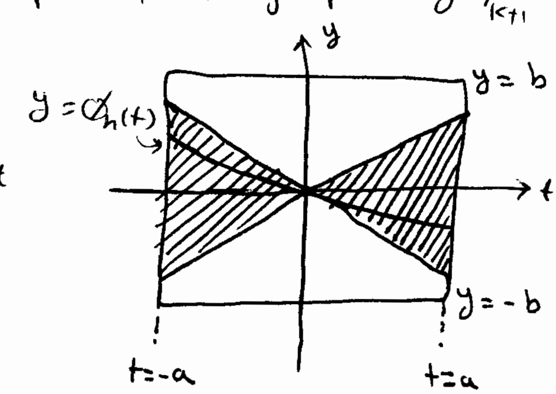
We consider only the rectangle

$D: |t| \le h, |y| \le b$

where $h = \min\{a, \frac{b}{M}\}$



$\boxed{\frac{b}{M} < a}$



$\boxed{\frac{b}{M} > a}$

Regions in which successive iterates lie.

[2] Does the sequence $\{\phi_n(t)\}$ converge?
We can write $\phi_n(t)$ as follows:

$$\phi_n(t) = \phi_1(t) + [\phi_2(t) - \phi_1(t)] + [\phi_3(t) - \phi_2(t)] + \cdots + [\phi_n(t) - \phi_{n-1}(t)]$$

$$= \phi_1(t) + \sum_{k=1}^{\infty} [\phi_{k+1}(t) - \phi_k(t)] \quad \text{-----} \quad (9)$$

If (9) converges, then the sequence $\{\phi_n(t)\}$ converges, and we write $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

[3] What are the properities of the limit function ϕ?
Is ϕ ~~continuous~~ No? Note that sometimes a sequence of continuous functions converges to a limit function that is discontinous (see problem 13 page 113). To ensure that ϕ is continuous, not only we need the sequence $\{\phi_n\}$ converges, but also it converges uniformly, so we ensure the continuity of the limit function ϕ in the interval $|t| \le h$. Note that

$$\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds \qquad \text{make } n \to \infty \implies$$

$$\phi(t) = \lim_{n\to\infty} \int_0^t f(s, \phi_n(s))\,ds$$

$\Rightarrow$ {since the sequence $\{\phi_n\}$ converges uniformly, we can interchange the operations of integrating and taking the limit

$$= \int_0^t \lim_{n\to\infty} f(s, \phi_n(s))\,ds$$

$\Rightarrow$ {by continuity of f in its 2^{nd} variable $\implies$

$$= \int_0^t f(s, \lim_{n\to\infty} \phi_n(s))\,ds$$

$$= \int_0^t f(s, \phi(s))\,ds.$$

Hence, ϕ satifies the integral equation (2).