2.9 1^{st} Order Difference Equations

* Differential equations are continuous models.
* Difference equations are discrete models.

Def: A 1^{st} order difference equation has the form

$$\boxed{y_{n+1} = f(n, y_n), \quad n = 0, 1, 2, \dots} \quad (1)$$

→ 1^{st} order because y_{n+1} depends on the value of y_n, not on y_{n-1} or y_{n-2}

→ (1) is called linear if f is a linear function of y_n (like differential Equations)

→ (1) is called nonlinear if f is not linear function of y_n.

→ A solution of the difference equation (1) is a sequence of numbers $y_0, y_1, y_2, \dots$ that satisfy the equation $\forall n$.

→ Assume that f depends only on y_n. Thus (1) becomes

$$\boxed{y_{n+1} = f(y_n), \quad n = 0, 1, 2, \dots} \quad (2)$$

an initial condition may be given $y_0 = \alpha$.

→ To construct the solution:

$$y_1 = f(y_0) \qquad 1^{st} \text{ iterate of } (2)$$

$$y_2 = f(y_1) = f(f(y_0)) = f^2(y_0) \qquad 2^{nd} \text{ iterat of } (2)$$

$$y_3 = f(y_2) = f(f^2(y_0)) = f^3(y_0) \qquad 3^{rd} \text{ iterate of } (2)$$

$$\vdots$$

$$y_n = f(y_{n-1}) = f^n(y_0) \qquad n^{th} \text{ iterate of } (2)$$

→ We look to the behavior of y_n as $n \to \infty$, to see if it has a limit. If so, we find it.

* Eq. Solution of Difference equation: occures when y_n has the same value for all n. To find such equilibria we set $y_{n+1} = y_n$. That is $y_n = f(y_n)$

Example (linear Equation): Solve the Difference equation

$$\boxed{y_{n+1} = \rho_n y_n \quad , \; n = 0, 1, 2, \dots} \quad (4)$$

where y_n is the population number in year n, and ρ_n is the reproduction rate in year n.

Note that (4) is linear. Thus it is easy to solve:

$$y_1 = \rho_0 y_0$$

$$y_2 = \rho_1 y_1 = \rho_1 \rho_0 y_0$$

$$y_3 = \rho_2 y_2 = \rho_2 \rho_1 \rho_0 y_0$$

$$\vdots$$

$$\boxed{y_n = \rho_{n-1} \rho_{n-2} \cdots \rho_1 \rho_0 y_0 \qquad n = 1, 2, \dots} \quad (5)$$

Note that when $\rho_n = \rho \;\; \forall n$ then (4) becomes

$$y_{n+1} = \rho y_n$$

with solution obtained from (5) by $\boxed{y_n = \rho^n y_0}$ (6)

→ The eq. solution of (6) is $y_n = 0 \;\; \forall n$.

→ The limiting behavior of y_n in 6 is

$$\lim_{n \to \infty} y_n = \begin{cases} 0 & \text{if } |\rho| < 1 \\ y_0 & \text{if } |\rho| = 1 \\ \text{DNE} & \text{Otherwise.} \end{cases}$$

Hence the eq. solution $y_n = 0$ is asymptotically stable for $|\rho| < 1$ and unstable if $|\rho| > 1$.

Example Solve the Difference equation

$$\boxed{y_{n+1} = \rho y_n + \underset{\text{immigration}}{b_n}, \quad n = 0,1,2,\dots} \quad (7)$$

$$y_1 = \rho y_0 + b_0$$

$$y_2 = \rho y_1 + b_1 = \rho(\rho y_0 + b_0) + b_1 = \rho^2 y_0 + \rho b_0 + b_1$$

$$y_3 = \rho y_2 + b_2 = \rho(\rho^2 y_0 + \rho b_0 + b_1) + b_2 = \rho^3 y_0 + \rho^2 b_0 + \rho b_1 + b_2$$

$$\vdots$$

$$\boxed{y_n = \rho^n y_0 + \rho^{n-1} b_0 + \dots + \rho b_{n-2} + b_{n-1} = \underbrace{\rho^n y_0}_{\text{due to reproduction}} + \underbrace{\sum_{j=0}^{n-1} \rho^{n-1-j} b_j}_{\text{due to immigration}}} \quad (8)$$

* If $b_n = b \ \forall n$, then (7) becomes $\boxed{y_{n+1} = \rho y_n + b}$ — (*)

and it's solution y_n from (8) becomes

$$\boxed{y_n = \rho^n y_0 + (1 + \rho + \rho^2 + \dots + \rho^{n-1}) b} \quad (9)$$

(A) → If $\rho \neq 1$, then the solution (9) can be written as

$$y_n = \underbrace{\rho^n y_0}_{\substack{\text{original}\\ \text{population}}} + \underbrace{\frac{1-\rho^n}{1-\rho} b}_{\text{immigration}}$$

$$= \rho^n \left(y_0 - \frac{b}{1-\rho} \right) + \frac{b}{1-\rho}$$

Thus,

$$\lim_{n\to\infty} y_n = \begin{cases} \dfrac{b}{1-\rho} & \text{if } |\rho| < 1 \\ \text{DNE} & \text{if } |\rho| > 1 \text{ or } \rho = -1 \end{cases}$$

Note that $\frac{b}{1-\rho}$ is an eq. solution of (*). Hence its asymptotically stable for $|\rho| < 1$ and unstable otherwise.

(B) → If $\rho = 1$, then the solution (9) can be written as

$$y_n = y_0 + nb \Rightarrow \lim_{n\to\infty} y_n = \infty$$

Example: \$ 10,000 loan is taken to buy a car. If the interest rate is 12%, what monthly payment is required to payoff the loan in 4 years?

- The intersel rate per month $\rho = 1 + r = 1 + \frac{12}{12}\% = 1.01$
- Let y_n be the loan balance outstanding in the n^{th} month
- Let b be the monthly payment.
- The initial condition is $y_0 = 10,000$

→ Our difference equation is given by

$$\boxed{y_{n+1} = \rho\, y_n + b} \quad (*)$$

→ since $\rho \neq 1$, we have seen the solution of (*) is

$$y_n = \rho^n \left(y_0 - \frac{b}{1-\rho}\right) + \frac{b}{1-\rho}$$

$$= (1.01)^n \left(10,000 + 100b\right) - 100b$$

→ We need to find b s.t $y_{48} = 0$ 4 years $= 4 \times 12 = 48$ months

$$0 = (1.01)^{48}(10,000) + b\left[(1.01)^{48}(100) - 100\right]$$

$$b = -\frac{(1.01)^{48}(100)}{(1.01)^{48} - 1} = -263.34 \text{ per month}$$

The loan is for 48 month. Hence, the total payment is

$$48 \times -263.34 = \$\, 12,640.32$$

Thus the interest $= 12,640.32 - 10,000 = \$\, 2640.32$

Example (Nonlinear Equations) : Logistic Difference Equation

$$\boxed{U_{n+1} = \rho\, u_n (1 - u_n) \qquad \rho > 0} \quad (10)$$

→ Eq. solution $(y_{n+1} = y_n) \Rightarrow U_n = \rho u_n - \rho u_n^2$

$\Rightarrow u_n = 0$ $\qquad$ $u_n = \dfrac{\rho - 1}{\rho}$

→ stability: Nonlinear difference equations are much more complicated so we study the behavior of solution near the equilibria by assuming that we can make (10) linear near the equilibria.

→ For $U_n = 0 \Rightarrow u_n^2$ is small so we neglect it

$\Rightarrow$ (10) becomes linear: $U_{n+1} = \rho U_n$. Thus,

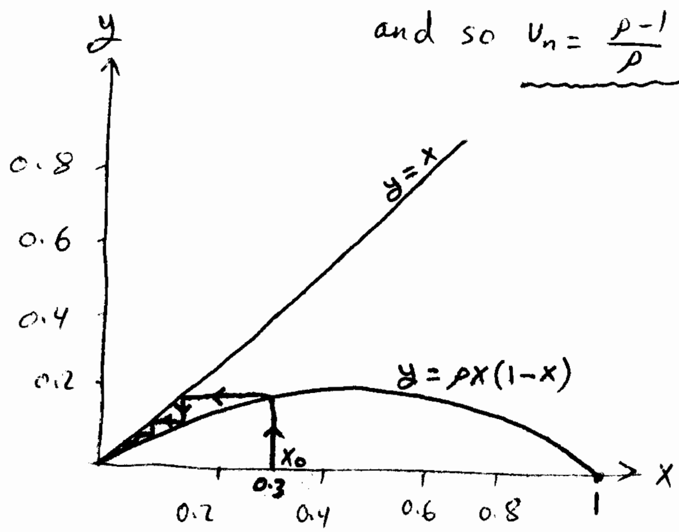
$u_n \to 0$ as $n \to \infty$ if $0 < \rho < 1$ $\Rightarrow u_n = 0$ is asymptotically stable for the linear approximation.

→ For $U_n = \dfrac{\rho - 1}{\rho}$ $\Rightarrow$ let z_n be small, so we study solutions in the neighborhood: $\boxed{U_n = \dfrac{\rho - 1}{\rho} + Z_n}$ (*) substitute (*) in (10)

$\Rightarrow U_{n+1} = (2 - \rho) Z_n - \rho Z_n^2$ since Z_n is small $\Rightarrow Z_n^2$ is smaller and we neglect it

$= (2 - \rho) Z_n$ (linear)

$U_n \to 0$ as $n \to \infty$ if $|2 - \rho| < 1$ i.e, $1 < \rho < 3$ and so $U_n = \dfrac{\rho - 1}{\rho}$ is asymptotically stable.



The sequence converges to origin when $\rho = 0.8$

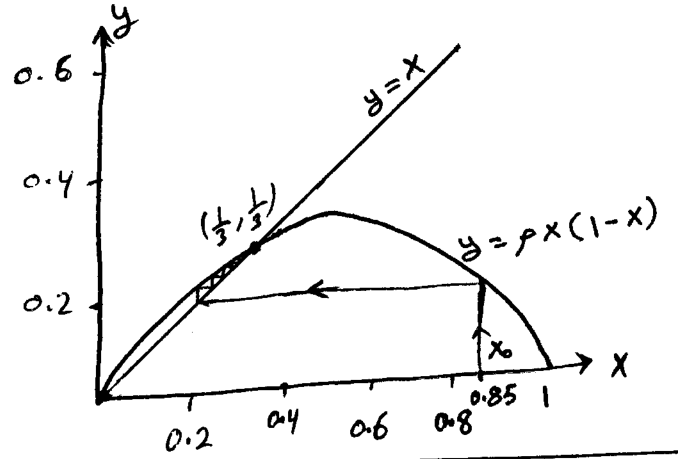
→ The solution sequence is represented by a stairstep

→ The sequence starts at point x_0

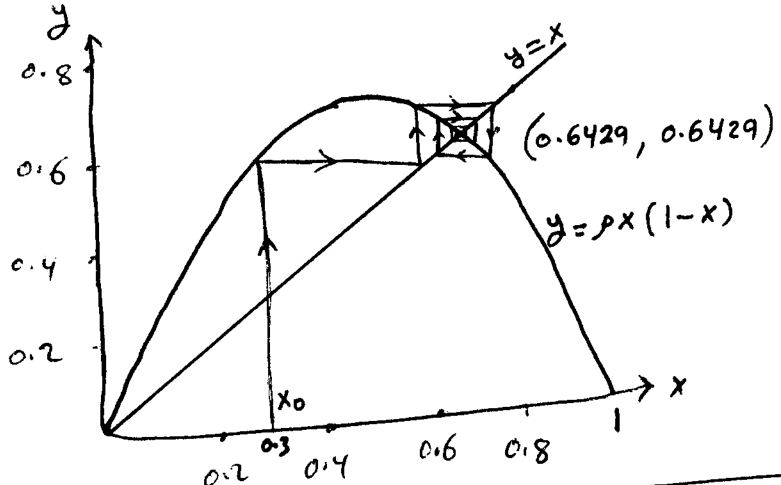
→ Vertical line from x_0 to the parabola $y_1 = \rho x_0 (1 - x_0)$

→ Horizontal line from parabola to the line $y = x$

→ Repeat the process again over again



The sequence converges to nonzero equilibrium when $\rho = 1.5$



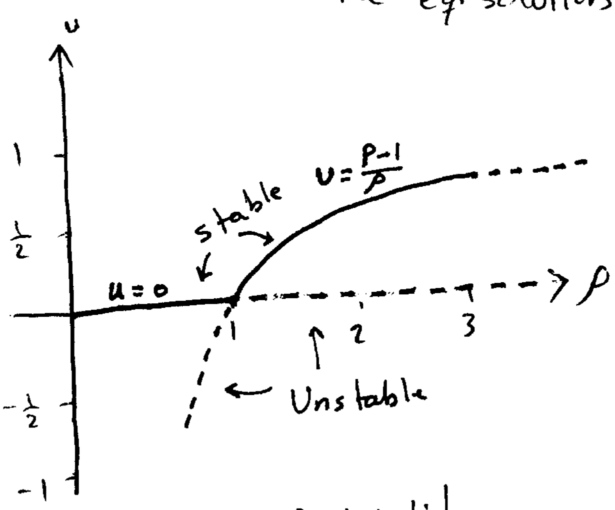
The sequence converges to nonzero equilibrium when $\rho = 2.8$

* Summary: The nonlinear 1st order logistic difference equation (10) has two eq. solutions:

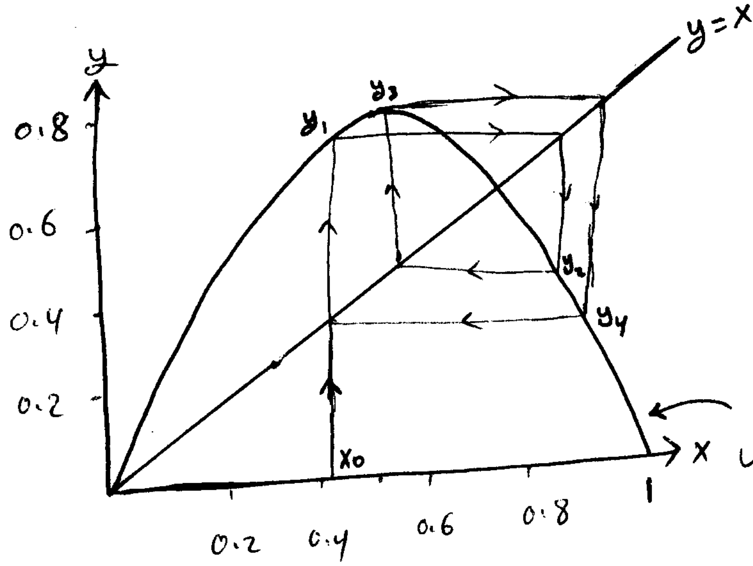
→ $U_n = 0$ which is stable for $0 \le \rho \le 1$

→ $U_n = \frac{\rho - 1}{\rho}$ which is stable for $1 < \rho < 3$

The the eq. solutions intersect at $\rho = 1$ "exchange of stability" where the solution changes from one equilibrium to the other.



Exchange of stability for $U_{n+1} = \rho U_n (1 - U_n)$



$y_{n+1} = \rho\, y_n (1 - y_n)$

→ Here $\rho = 3.5$

→ $\nexists$ stable eq. solution for $\rho > 3$.

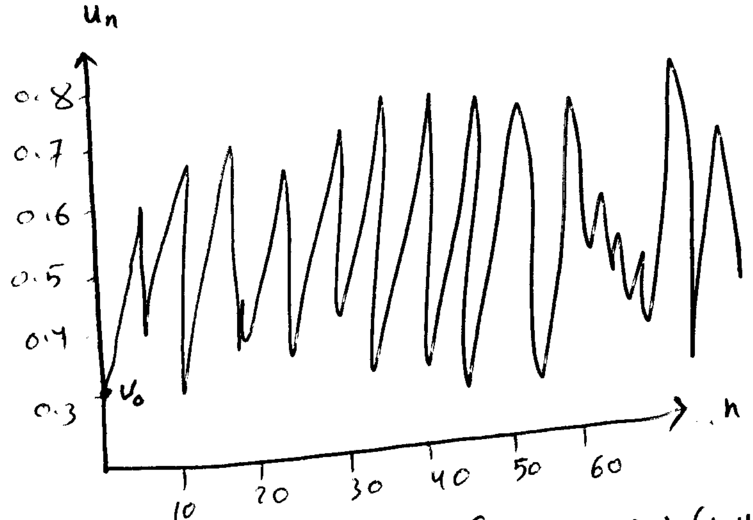
→ steady oscillation of period 4

→ the solution is periodic with period 4

→ If ρ increases more and more than 3.5, then periodic solutions of period 8, 16, ..., appear.

→ The appearence of new solution at a certain parameter value is called a bifurcation.

→ when the solution is oscillation with higher period, the term chaotic is used describe this situation,



A chaotic solution for $u_{n+1} = \rho u_n (1 - u_n)$ with $\rho = 3.65$