3.1 2^{nd} Order linear Homogeneous Equations with Constant Coefficients

A 2^{nd} order ODE has the form

$$y'' = \frac{d^2y}{dt^2} = f(t, y, y'), \quad \text{---} \; (1)$$

where f is some given function, and t is the independent variable.

→ A 2^{nd} order ODE (1) usually appears as

$$P(t)\,y'' + Q(t)\,y' + R(t)\,y = G(t) \quad \text{....} \; (2)$$

→ If $G(t) = 0$ for all t, then the equation (1) is called homogeneous.

→ If $G(t) \neq 0$, then the equation (1) is called nonhomogeneous.

→ If $P(t) \neq 0$, then (2) can be written as

$$y'' + p(t)\,y' + q(t)\,y = g(t), \quad \text{where} \; \text{...} \; (3)$$

$$p(t) = \frac{Q(t)}{P(t)}, \quad q(t) = \frac{R(t)}{P(t)} \quad \text{and} \quad g(t) = \frac{G(t)}{P(t)}$$

→ (3) can be written as

$$(4) \text{....} \quad y'' = g(t) - p(t)\,y' - q(t)\,y = f(t, y, y')$$

→ Equation (1) is called linear if the function f is linear in y and y', i.e. f has the form (4). Otherwise, equation (1) is called nonlinear.

- An IVP consists of DE such as ① together with a pair of initial conditions $y(t_0)=y_0$ and $y'(t_0)=y'_0$.
  - → Thus, the solution passes through (t_0, y_0).
  - → The slope of the solution at (t_0, y_0) is y'_0.
  - → Two initial condition because two integreation are required to find the solution.
- We will focus now on homogeneous equations, because once the homogenous equation is solved, then it is easy to solve the corresponding nonhomogenous equation (section 3.6, 3.7). or at least we write the solution in terms of an integral.
- In this chapter, we focus our attention on equations with constant coefficients, i.e. $P(t)$, $Q(t)$ and $R(t)$ are all constants. We study the variable coefficient case in ch5.
  - → For the homogenous case, equation ② becomes

$$ay'' + by' + cy = 0 \qquad a, b, c \text{ are constants}$$

Example: Consider the DE $\boxed{y''-y=0}$ *. Can you guess the solution?

$a=1$, $b=0$, $c=-1$ $\quad f=f(t,y)=-y$

→ $y_1(t)=e^t$ is a solution

→ $y_2(t)=e^{-t}$ is another solution

→ $y_3(t)=c_1e^t+c_2e^{-t}$ is also a solution "linear combination"

⋮

* has infinitely many solutions. To pick out a particular solution, we introduce 2 initial conditions:

$$y'' - y = 0 \quad , \quad y(0) = 3 \quad , \quad y'(0) = 1$$

$$y(t) = c_1 e^{t} + c_2 e^{-t}$$

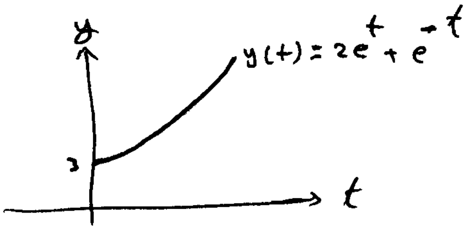
$$y(0) = \boxed{c_1 + c_2 = 3} \rightarrow (1)$$

$$y'(t) = c_1 e^{t} - c_2 e^{-t} \Rightarrow y'(0) = c_1 - c_2 = 1 \Rightarrow$$

$$\boxed{c_1 = 1 + c_2} \rightarrow (2)$$

$$\Rightarrow \boxed{c_1 = 2} \text{ and } \boxed{c_2 = 1}$$

$$y(t) = 2e^{t} + e^{-t}$$



✓ <u>Characteristic Equation</u>

To solve the 2^{nd} order equation with constant coefficients

$$ay'' + by' + cy = 0, \qquad \ldots (5)$$

We start by assuming a solution of the form

$y = e^{rt}$ where r is a parameter need to be determined.

$$\Rightarrow y' = r e^{rt} \text{ and } y'' = r^2 e^{rt} \qquad \ldots (6)$$

$\Rightarrow$ substitute (6) in (5) $\Rightarrow$

$$(ar^2 + br + c) e^{rt} = 0$$

since $e^{rt} \neq 0$, it follows that

$$ar^2 + br + c = 0 \qquad \ldots (7)$$

$\Rightarrow$ Equation (7) is called the characteristic Equation for the DE (5)

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initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$ that satisfy the DE (5). Thus we have the following IVP:

* Equation 7 is a quadratic formula with two roots

$$r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

➡ There are three possible results:

1) $r_1 \neq r_2 \in \mathbb{R}$ (The roots are real and different)
2) $r_1 = r_2 \in \mathbb{R}$ (The roots are real and equal)
3) $r_1, r_2 \in \mathbb{C}$ (The roots are complex conjugates)

- In this section we study the 1st case.
- In section 3.3 we study the complex roots and in section 3.4 we study the repeated roots.

Assuming the roots of the characteristic equation 7 are real and different. Then, the general solution of 5 is

$$y(t) = c_1 \underbrace{e^{r_1 t}}_{\text{this is a solution}} + c_2 \underbrace{e^{r_2 t}}_{\text{this is a solution}}$$

→ To find c_1 and c_2: Recall that we need two initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$ that satisfy the DE 5. Thus we have the following IVP:

$$ay'' + by' + cy = 0 \quad , \quad y(t_0) = y_0 \quad , \quad y'(t_0) = y_0'$$

$$y(t_0) = \boxed{c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0} \quad (1)$$

$$y'(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$$

$$y'(t_0) = \boxed{c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = y_0'} \quad (2)$$

$$\boxed{c_1 = \frac{y_0' - y_0 r_2}{r_1 - r_2} e^{-r_1 t_0}}$$

$$\boxed{c_2 = \frac{y_0 r_1 - y_0'}{r_1 - r_2} e^{-r_2 t_0}}$$

Note that we are assuming that $r_1 \neq r_2$. Thus solution exists for the IVP $\forall$ IC.

Example (a) Find the general solution to the following IVP: $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$

(b) Find the maximum value attained by the solution

Part one: If $y = e^{rt}$, then The characteristic equation is

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0$$

$$r_1 = -2 \text{ and } r_2 = -3$$

(a) Part 2 The general solution of the IVP above is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$y'(t) = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}$$

$$y(0) = \boxed{c_1 + c_2 = 2} \; (1)$$

$$y'(0) = c_1 r_1 + c_2 r_2 = 3$$

$$\boxed{-2c_1 - 3c_2 = 3} \; (2)$$

From (1) and (2) we obtain $\boxed{c_1 = 9}$ $\boxed{c_2 = -7}$

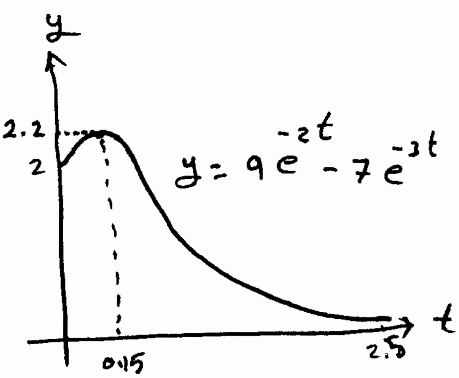
Thus $y(t) = 9e^{-2t} - 7e^{-3t}$

(b) $y'(t) = -18e^{-2t} + 21e^{-3t} = 0$

$$\Leftrightarrow \quad 6e^{-2t} = 7e^{-3t} \quad \Leftrightarrow \quad e^{t} = \frac{7}{6}$$

$$\Leftrightarrow \quad t = \ln\frac{7}{6} \approx 0.15$$

Thus $y(0.15) = 9e^{-2(0.15)} - 7e^{-3(0.15)} \approx 2.2$



Example: Solve the IVP $2y'' + 3y' = 0$, $y(0) = 1$, $y'(0) = 3$

- If $y = e^{rt}$, then the characteristic equation is

$$2r^2 + 3r = 0 \quad \Leftrightarrow \quad r(2r+3) = 0 \quad \Leftrightarrow \quad r_1 = 0 \text{ and } r_2 = -\frac{3}{2}$$

- The general solution $y(t) = c_1 e^{0t} + c_2 e^{-\frac{3}{2}t} = c_1 + c_2 e^{-\frac{3}{2}t}$

$y(0) = 1 = c_1 + c_2 e^{-\frac{3}{2}(0)} \Rightarrow \boxed{c_1 = 1 + 2 = 3}$

$y'(0) = 3 = -\frac{3}{2}c_2 = 3 \Rightarrow \boxed{c_2 = -2}$

Thus $y(t) = 3 - 2e^{-\frac{3}{2}t}$