# 3.2 Solutions of Linear Homogeneous Equations: the Wronskian

- Recall that a $2^{nd}$ order linear homogeneous equation with constant coefficients has the form $ay'' + by' + cy = 0$.
- To develop the theory of linear DE, it is helpful to introduce a differential operator notation.
  - → Let $p$ and $q$ be continous functions on an open interval $I = (\alpha, \beta)$, the interval $I$ could be infinite.
  - → For any function $y$ that is twice differentiable on $I$, we define the differential operator $L$ by:

$$L[y] = y'' + p\,y' + q\,y$$

  - → Note that $L[y]$ is a function defined on $I$, and $L[y]$ at a point $t$ is

$$L[y](t) = y''(t) + p(t)\,y'(t) + q(t)\,y(t)$$

  - → For example: if $p(t) = t^2$, $q(t) = e^{2t}$ and $y(t) = \sin t$ and $I = (0, 2\pi)$, then the differential operator $L[y](t)$ is given by

$$L[y](t) = -\sin t + t^2 \cos t + e^{2t} \sin t.$$

  - → In this section we study the $2^{nd}$ order linear homogeneous equation $L[y](t) = 0$, i.e

$$\left.\begin{array}{l} L[y] = y'' + p(t)\,y' + q(t)\,y = 0 \\ \text{with initial Conditions } \; y(t_0) = y_0, \quad y'(t_0) = y'_0 \quad \text{where } t_0 \in I \end{array}\right\} \quad (1)$$

- Does the IVP ① has a solution? If so, is it unique?
- How does the solution look like (general structure)?
- The proof of the following Theorem may be found in more advanced book.

**Theorem 3.2.1** Consider the IVP (Existence and Uniqueness Theorem)

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \ldots ②$$

where $p, q, g$ are continuous on an open interval $I$. Then, there exists a unique solution $y = \phi(t)$ to the IVP ② over the interval $I$.

Th 3.2.1 says the following

1- The IVP ② has a solution (the solution exists)

2- The IVP ② has only one solution (the solution is unique)

3- The solution $y = \phi(t)$ is defined on the interval $I$ where the coefficients are continuous.

*Example*: Consider the $2^{nd}$ order linear IVP $y'' - y = 0$, $y(0) = 3$, $y'(0) = 1$.
From section 3.1, the solution $y(t) = 2e^{t} + e^{-t}$ satisfies the IVP above. Thus, the solution exists and twice differentiable on $I = (-\infty, \infty)$. Note that the coefficients $p(t) = 0$ and $q(t) = -1$ are continuous on $I$.

Note that in the example above it was easy to find solution. However, problems of the form ②, it is not possible to write down a useful expression for the solution. This is a major difference between $1^{st}$ and $2^{nd}$ order linear equations.

*Example*: Consider the IVP $y'' + p(t)y' + q(t)y = 0$, $y(t_0) = y'(t_0) = 0$, where $p$ and $q$ are continuous on an open interval $I$ containing $t_0$. Find the unique solution of the IVP above.

The function $y = \phi(t) = 0$ $\forall t \in I$ is a solution to the homogeneous IVP above.
By Th 3.2.1 $y = 0$ is the only solution.

**Example:** Find the longest interval in which the IVP

$$(t+1)y'' - (\cos t)\,y' = 1 - 3y \;, \quad y(0) = 1 \;, \; y'(0) = 0$$

has a solution.

$1^{st}$ we write the IVP in the standard form (similar to ②)

$$y'' - \frac{\cos t}{t+1}\, y' + \frac{3}{t+1}\, y = \frac{1}{t+1} \;, y(0) = 1 \;, \; y'(0) = 0$$

- The coefficients are $p(t) = -\dfrac{\cos t}{t+1}$ , $q(t) = \dfrac{3}{t+1}$ and $g(t) = \dfrac{1}{t+1}$
- The only point of discontinuity of the coefficients is $t = -1$
- Thus, the longest interval containing the point $t_0 = 0$ on which the coefficients are continuous is $I = (-1, \infty)$.

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**Theorem 3.2.2** (Principle of Superposition)

If $y_1$ and $y_2$ are solutions the to the differential equation

$$L[y] = y'' + p(t)\, y' + q(t)\, y = 0 \quad ;\cdots ③$$

then the linear combination $c_1 y_1 + c_2 y_2$ is also a solution for any value of the constants $c_1$ and $c_2$.

Proof: Since $y_1$ and $y_2$ are solutions, it follows that $L[y_1] = L[y_2] = 0$. To prove that $c_1 y_1 + c_2 y_2$ is solution we need to show that $L[c_1 y_1 + c_2 y_2] = 0$. That is

$$\begin{aligned}
L[c_1 y_1 + c_2 y_2] &= [c_1 y_1 + c_2 y_2]'' + p\,[c_1 y_1 + c_2 y_2]' + q\,[c_1 y_1 + c_2 y_2] \\
&= c_1 y_1'' + c_2 y_2'' + c_1 p y_1' + c_2 p y_2' + c_1 q y_1 + c_2 q y_2 \\
&= c_1 [y_1'' + p y_1' + q y_1] + c_2 [y_2'' + p y_2' + q y_2] \\
&= c_1 \, L[y_1] + c_2 L[y_2] \\
&= c_1 (0) + c_2 (0) = 0
\end{aligned}$$

. Thus $c_1 y_1 + c_2 y_2$ is a solution for ③

- It follows from Th 3.2.2 that for any two solutions $y_1$ and $y_2$, we can construct an infinite family of solutions, each one is of the form $\boxed{y(t) = c_1 y_1(t) + c_2 y_2(t)}$ — (4)
- The question is weather all solutions of the DE (3) are included in (4), or may be there are other solutions of different form?

  → To answer this question, we use the Wronskian determinant.

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## ✓ The Wronskian determinant

- Suppose that $y_1$ and $y_2$ are solutions to the IVP (1):

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0, \quad y(t_0) = y_0,\ y'(t_0) = y'_0$$

- From Th 3.2.2, we have $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution also that satisfies the initial conditions above. Thus,

$$\left.\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= y_0 \\ c_1 y'_1(t_0) + c_2 y'_2(t_0) &= y'_0 \end{aligned}\right\} \text{(E)}$$

We solve for $c_1$ and $c_2$:

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y'_0 & y'_2(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} = \frac{y_0\, y'_2(t_0) - y'_0\, y_2(t_0)}{y_1(t_0)\, y'_2(t_0) - y'_1(t_0)\, y_2(t_0)}$$

$$c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y'_1(t_0) & y'_0 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{vmatrix}} = \frac{y'_0\, y_1(t_0) - y_0\, y'_1(t_0)}{y_1(t_0)\, y'_2(t_0) - y'_1(t_0)\, y_2(t_0)}$$

(5)

- The Wronskian determinant (or simply Wronskian) of the solutions $y_1$ and $y_2$ is given by

$$W = W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0).$$

- Thus, for $c_1$ and $c_2$ in (5) to make sense, we must have $W \neq 0$.

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**Theorem 3.2.3** suppose that $y_1$ and $y_2$ are solutions of the DE (1):

$$L[y] = y'' + p(t)y' + q(t)y = 0 \quad , \quad y(t_0) = y_0 \,,\; y'(t_0) = y_0'.$$

The Wronskian $W = y_1 y_2' - y_1' y_2$ is not zero at $t_0$. Then, there exist $c_1, c_2$ such that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ satisfies the IVP (1). iff

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**Example:** Recall that $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ are solutions to the DE $y'' + 5y' + 6y = 0$. $*$

→ The Wronskian of $y_1$ and $y_2$ is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = -3\, e^{-2t} e^{-3t} + 2\, e^{-2t} e^{-3t} = -e^{-5t}$$

→ Since $W \neq 0$ for all $t$, it follows that the linear combinations of $y_1$ and $y_2$ can be used to construct solutions of the DE $*$ together with initial conditions given at any value of $t$.

**Theorem 3.2.4** (Fundamental Solutions)

Suppose that $y_1$ and $y_2$ are solutions to the DE ③:

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

There is a point $t_0$ such that $W(y_1, y_2)(t_0) \neq 0$ Then the family of solutions $y = c_1 y_1(t) + c_2 y_2(t)$, with arbitrary coefficients $c_1$ and $c_2$, includes every solution to the DE ③ iff

Proof ⇐ Let $\phi$ be any solution of ③. We need to show that $\phi$ is included in the linear combination $c_1 y_1 + c_2 y_2$ for some choice of the constants $c_1$ and $c_2$.

$\Rightarrow$ by Contradiction: Suppose $W(y_1, y_2)(t_0)$ is zero no matter what $t_0$ is. Then by Th 3.2.3, $\exists\, y_0, y_0'$ s.t Ⓔ has no solution. Select $(y_0, y_0')$ and choose $\phi(t)$ the solution of (3). Such solution $\phi$ exists by Th 3.2.1. However, this solution is not included in $y = c_1 y_1 + c_2 y_2$. Thus, this solution does not include all solutions of (3) if $W(y_1, y_2) = 0$.

→ Let $t_0$ be s.t $W(y_1, y_2)(t_0) \neq 0$.

→ Let $y_0 = \phi(t_0)$ and $y_0' = \phi'(t_0)$

Thus $\phi$ is a solution to the IVP:

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_0' \quad \cdots Ⓐ$$

- By Th 3.2.3, there is $c_1$ and $c_2$ s.t $y = c_1 y_1(t) + c_2 y_2(t)$ is a solution to the IVP Ⓐ, where $c_1$ and $c_2$ are as given in ⑤.
- From the uniqueness part of Th 3.2.1, the two solutions ($\phi$ and $y$) of the same IVP are the same. Thus,

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t).$$

- Thus, $\phi$ is included in the family of function of $c_1 y_1 + c_2 y_2$.
- Since $\phi$ is arbitrary solution of ③, it follows that every solution of ③ is included in this family.

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* The expression $y = c_1 y_1 + c_2 y_2$ is called the <u>general solution</u> of the DE ③.

→ and $y_1$ and $y_2$ are the <u>fundamental set of solutions</u> of the DE ③.

**Example:** Consider the DE $y'' - y = 0$ with two solutions $y_1 = e^t$ and $y_2 = e^{-t}$. Show that $y_1$ and $y_2$ form a fundamental set of solutions to this DE.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -e^t e^{-t} - e^t e^{-t} = -1 - 1 = -2$$

Thus $W \neq 0$ for all $t$. Therefore $y_1$ and $y_2$ form a fundamental set of solution the DE given above, and can be used to construct all of its solution.

- The general solution is $y = c_1 e^t + c_2 e^{-t}$. Not needed.

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**Example:** Let $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ be two solutions of the DE $y'' + p(t) y' + q(t) y = 0$. Show that they form a fundamental set of solutions if $r_1 \neq r_2$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)\, e^{(r_1 + r_2)t} \neq 0 \quad \forall t.$$

Thus $y_1$ and $y_2$ form a fundamental set of solutions to the DE given above.

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**Example:** Show that $y_1 = t^{\frac{1}{2}}$ and $y_2 = t^{-1}$ form a fundamental set of solutions to the DE $2t^2 y'' + 3t y' - y = 0$, $t > 0$.

→ substitute $y_1$ into the equation; $\left(y_1' = \frac{1}{2} t^{-\frac{1}{2}} \text{ and } y_1'' = -\frac{1}{4} t^{-\frac{3}{2}}\right)$

$$2t^2\left(\frac{-1}{4} t^{-\frac{3}{2}}\right) + 3t\left(\frac{1}{2} t^{-\frac{1}{2}}\right) - \left(t^{\frac{1}{2}}\right) = -\frac{1}{2} t^{\frac{1}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0$$

Thus $y_1$ is a solution to the DE above.

→ Substitute $y_2$ into the equation: $\left(y_2' = -t^{-2} \text{ and } y_2'' = 2t^{-3}\right)$

$$2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - \left(t^{-1}\right) = 4t^{-1} - 3t^{-1} - t^{-1} = 0$$

Thus $y_2$ is a solution to the DE above

Now to show that $y_1$ and $y_2$ form a fundamental set of solutions, we find $W(y_1, y_2)$:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2} t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2} t^{-\frac{3}{2}} = -\frac{3}{2} t^{-\frac{3}{2}} = -\frac{3}{2\sqrt{t^3}}$$

Since $t>0$, it follows that $W \neq 0$. Thus, $y_1$ and $y_2$ form a fundamental set of solutions for the DE given above.

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* In several cases: For a given DE, we are able to find the fundamental set of solutions, and therefore the general solution. The question is weather or not a DE of the form (3) has always a fundamental set of solutions?

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**Theorem 3.2.5** (Existence of Fundamental set of Solutions)

Consider the DE (3): $L[y] = y'' + p(t) y' + q(t) y = 0$ whose coefficients $p$ and $q$ are continuous on some interval $I$. Take $t_0 \in I$. Let $y_1$ be the solution of (3) satisfies the initial condition $y_1(t_0) = 1$, $y_1'(t_0) = 0$.

Let $y_2$ be the solution of (3) satisfies the initial condition $y_2(t_0) = 0$, $y_2'(t_0) = 1$.

Then, $y_1$ and $y_2$ form fundamental set of solutions for the DE (3).

Proof: Note that the existance of the solutions $y_1$ and $y_2$ is assured by Th 3.2.1. Now to show that $y_1$ and $y_2$ form a fundamental set of solutions for (3), we find $W(y_1, y_2)(t_0)$:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Thus $W \neq 0$ at $t_0$. Completing the proof.

<u>Example</u>: Find the fundamental set of solutions specified by Th 3.2.5 for the DE

✓end

$\boxed{y'' - y = 0}^{*}$ at the initial point $t_0 = 0$

- We have seen that $y_1(t) = e^{t}$ and $y_2(t) = e^{-t}$ are solutions of $*$. The Wronskian $W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{t} & e^{-t} \\ e^{t} & -e^{-t} \end{vmatrix} = -2 \neq 0$

Thus $y_1$ and $y_2$ are fundamental set of solution for $*$.

- But these two solutions don't satisfy the initial conditions in Th 3.2.5. Therefore, they are not fundamental set of solutions indicated by Th 3.2.5.
- To find the fundamental set of solutions indicated by Th 3.2.5, we need to find solutions $y_3(t)$ and $y_4(t)$ that satisfy the initial conditions: $y_3(0) = 1 \quad y_3'(0) = 0$

$$y_4(0) = 0 \quad y_4'(0) = 1$$

- Since $y_1$ and $y_2$ form a fundamental set of solution for $*$, it follows that: $y_3 = c_1 e^{t} + c_2 e^{-t}$, $y_3(0) = 1$ and $y_3'(0) = 0$

$$y_4 = d_1 e^{t} + d_2 e^{-t}, \quad y_4(0) = 0 \text{ and } y_4'(0) = 1$$

$$\Rightarrow y_3(t) = \frac{1}{2} e^{t} + \frac{1}{2} e^{-t} = \cosh t$$

$$\Rightarrow y_4(t) = \frac{1}{2} e^{t} - \frac{1}{2} e^{-t} = \sinh t$$

- Now the Wronskian of $y_3$ and $y_4$ is

$$W = \begin{vmatrix} y_3 & y_4 \\ y_3' & y_4' \end{vmatrix} = \begin{vmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{vmatrix} = \cosh^2 t - \sinh^2 t = 1 \neq 0 \text{ for all } t.$$

Thus, $y_3$ and $y_4$ form the fundamental set of solutions indicated by Th 3.2.5 with general solution $y(t) = k_1 \cosh t + k_2 \sinh t$.

- We conclude that

$$S_1 = \{e^{t}, e^{-t}\} \text{ and } S_2 = \{\cosh t, \sinh t\}$$

both form fundamental sets of solutions to the DE

$y'' - y = 0$ with initial point $t_0 = 0$

- In general, a DE has infinitely many different fundamental solution sets. We should choose the most convenient set.

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## Summary

To find a general solution of the DE

$$y'' + p(t) + q(t) = 0, \quad \alpha < t < \beta \quad \text{.... (10)}$$

- We must 1) find two solutions $y_1$ and $y_2$ satisfy (10).
  2) make sure that $\exists$ a point $t_0 \in (\alpha, \beta)$ s.t

$$W(y_1, y_2)(t_0) \neq 0$$

- It follows that $y_1$ and $y_2$ form a fundamental set of solutions to (10), with general solution

$$y = c_1 y_1(t) + c_2 y_2(t).$$

- If the initial conditions prescribed (معطاة) at point $t_0 \in (\alpha, \beta)$ where $W \neq 0$, then $c_1$ and $c_2$ can be chosen to satisfy those conditions.

# Linear Independence and the Wronskian



* The general solution, $y(t) = c_1 y_1(t) + c_2 y_2(t)$, of a $2^{nd}$ order linear homogeneous DE: $L[y] = y'' + p(t)y' + q(t)y = 0$ is a linear combination of two solutions, $y_1$ and $y_2$, whose Wronskian is not zero. This representation is related to the concept of linear independence of two functions.

* Two functions $f$ and $g$ are linearly dependent if $\exists$ constants $c_1$ and $c_2$, not both zero, s.t $\boxed{c_1 f(t) + c_2 g(t) = 0}^{*}$ $\forall t \in I$.

  → This tells us weather $f$ and $g$ are multiple of each other.

* The functions $f$ and $g$ are linearly independent on $I$ if they are not linearly dependent. That is if the only solution to * is $c_1 = c_2 = 0$.

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**Example**: Determine weather the function $f(t) = \sin 2t$ and $g(t) = \sin t \cos t$ are linearly independent or lineary dependent on an arbitrary interval $I$.

The two given functions are linearly dependent on any interval $I$ because $c_1 \sin 2t + c_2 \sin t \cos t = 0$ for all $t \in I$

if we choose $c_1 = 1$ and $c_2 = -2$.

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* The solution of the 2x2 system of equations
$$c_1 x_1 + c_2 x_2 = a$$
$$c_1 y_1 + c_2 y_2 = b$$
is:

$$c_1 = \frac{a y_2 - b x_2}{D} \quad \text{and} \quad c_2 = \frac{-a y_1 + b x_1}{D} \quad \text{where} \quad D = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

* Assume $D \neq 0$. Now if $a = b = 0$, then the only solution to this system is $c_1 = c_2 = 0$.

**Example**: Show that the functions $e^t$ and $e^{-t}$ are linearly independent on any interval $I$.

Suppose that $\boxed{c_1 e^t + c_2 e^{-t} = 0}$ Ⓐ $\forall\, t \in I$.

We need to show that $c_1 = c_2 = 0$.

Choose two points $t_0$ and $t_1$ in $I$ s.t $t_0 \neq t_1$.

Thus,

$$\left.\begin{aligned} c_1 e^{t_0} + c_2 e^{-t_0} &= 0 \\ c_1 e^{t_1} + c_2 e^{-t_1} &= 0 \end{aligned}\right\} \text{Ⓑ}$$

This because Ⓐ holds $\forall\, t \in I$

Now $D = \begin{vmatrix} e^{t_0} & e^{-t_0} \\ e^{t_1} & e^{-t_1} \end{vmatrix} = e^{t_0 - t_1} - e^{t_1 - t_0}$. Since $t_0 \neq t_1$, it follows that $D \neq 0$.

Thus, the only solution of Ⓑ is $c_1 = c_2 = 0$.

Hence, $e^t$ and $e^{-t}$ are linearly independent.

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**Theorem 3.3.1**: If $f$ and $g$ are differentiable functions on an open interval $I$ and if $W(f,g)(t_0) \neq 0$ for some point $t_0 \in I$, then $f$ and $g$ are linearly independent on $I$.

Moreover, if $f$ and $g$ are linearly dependent on $I$, then $W(f,g)(t) = 0$ $\forall\, t \in I$.

Proof ($1^{st}$ part): Suppose that $c_1 f(t) + c_2 g(t) = 0 \quad \forall\, t \in I$

Choose the point $t_0 \in I$. Thus, we have

$$\left.\begin{aligned} c_1 f(t_0) + c_2 g(t_0) &= 0 \\ c_1 \underline{f'(t_0)} + c_2 \underline{g'(t_0)} &= 0 \end{aligned}\right\} \text{Ⓒ}$$

Since $W(f,g)(t_0) \neq 0$ ($D \neq 0$), it follows that the only solution of Ⓒ is $c_1 = c_2 = 0$. Thus, $f$ and $g$ are linearly independent.

($2^{nd}$ part): Suppose that $f$ and $g$ are linearly dependent and $W(f,g)(t_0) \neq 0$ for some point $t_0 \in I$. By the $1^{st}$ part, $f$ and $g$ are linearly independent. ※ Contradiction. Completing the proof.

## Theorem 3.3.2 (Abel's Theorem)

Suppose that $y_1$ and $y_2$ are solutions of DE

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0, \quad \ldots\text{(D)}$$

where $p$ and $q$ are continuous on an open interval $I$.

Then, the Wronskian $W(y_1,y_2)(t)$ is given by

$$W(y_1,y_2)(t) = c\, e^{-\int p(t)\,dt},$$

where $c$ is a constant that depends on $y_1$ and $y_2$, but not on $t$.

Further, $W(y_1,y_2)(t)$ is either zero $\forall t \in I$ (if $c=0$) or $W(y_1,y_2)(t)$ is never zero in $I$ (if $c \neq 0$).

Proof: Since $y_1$ and $y_2$ are solution to the DE (D), we have

$$y_1'' + p(t)\,y_1' + q(t)\,y_1 = 0$$
$$y_2'' + p(t)\,y_2' + q(t)\,y_2 = 0$$

Multiply the 1st equation by $-y_2$ and the 2nd equation by $y_1$ and add the results:

$$\left(y_1 y_2'' - y_1'' y_2\right) + p(t)\left(y_1 y_2' - y_1' y_2\right) = 0$$

$$\Leftrightarrow \quad W' + p(t)\,W = 0 \qquad \text{(separable)}$$

$$\Leftrightarrow \quad \frac{W'}{W} = -p(t) \quad \Rightarrow W(t) = c\,e^{-\int p(t)\,dt}, \quad c \text{ is constant.}$$

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Example: Consider the following DE $\boxed{2t^2y'' + 3ty' - y = 0}$ $*$, $t>0$ with two solutions $y_1(t) = t^{\frac{1}{2}}$ and $y_2(t) = t^{-1}$.

Find the Wronskian and compare it with Abel's Th[redacted]rem.

- $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}} = -\frac{3}{2}t^{-\frac{3}{2}}$$
- Now we write $*$ in the standard form $y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0$

Thus $W = c\,e^{-\int p(t)\,dt} = c\,e^{-\int \frac{3}{2}\frac{1}{t}\,dt} = c\,e^{-\frac{3}{2}\ln t} = c\,t^{-\frac{3}{2}}. \;\ldots\ldots 8$

8 gives the Wronskian for any pair of solutions of $*$. For this example we choose $c = -\frac{3}{2}$.

**Theorem 3.3.3** Suppose that $y_1$ and $y_2$ are solutions to the DE

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad \cdots (15)$$

where $p$ and $q$ are continuous on an open interval $I$.

Then, $\boxed{1}$ $y_1$ and $y_2$ are linearly dependent on $I$ iff $W(y_1, y_2)(t) = 0 \ \forall t \in I$.

Also, $\boxed{2}$ $y_1$ and $y_2$ are linearly independent on $I$ iff $W(y_1, y_2)(t) \neq 0 \ \forall t \in I$.

Proof: Note that Th 3.3.2 provides that $W(y_1, y_2)(t) = 0 \ \forall t \in I$ or $W(y_1, y_2)(t) \neq 0 \ \forall t \in I$.

Now to prove $\boxed{1}$ $(\Rightarrow)$: since $y_1$ and $y_2$ are linearly dependent, it follows by Th 3.3.1 that $W(y_1, y_2)(t) = 0 \ \forall t \in I$.

$(\Leftarrow)$: Assume that $W(y_1, y_2)(t) = 0 \ \forall t \in I$, we need to show that $y_1$ and $y_2$ are linearly dependent.

Let $t_0$ be any point in $I$. Thus, $W(y_1, y_2)(t_0) = 0$, and so the system

$$\left.\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= 0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= 0 \end{aligned}\right\} (16)$$

has nontrivial solution for $c_1$ and $c_2$.

Let $\phi(t) = c_1 y_1(t) + c_2 y_2(t)$. Then $\phi$ is a solution to (15) and by (16), $\phi$ satisfies the initial condition $\phi(t_0) = 0$ and $\phi'(t_0) = 0$.

By the uniqueness part of Th 3.2.1, $\phi(t) = 0 \ \forall t \in I$

Thus $\phi(t) = c_1 y_1(t) + c_2 y_2(t) = 0$ for $c_1$ and $c_2$ not both zero. This means that $y_1$ and $y_2$ are linearly dependent.

The proof of $\boxed{2}$ follows immediately, i.e. by contradiction.

# Summary

Let $y_1$ and $y_2$ be solutions to the DE

$$y'' + p(t)\,y' + q(t) = 0,$$

where $p$ and $q$ are continuous on an open interval $I$.

Then, the following 4 statements are equivalent (each one implies the other three):

$\boxed{1}$ The functions $y_1$ and $y_2$ are fundamental set of solutions on $I$.

$\boxed{2}$ The functions $y_1$ and $y_2$ are linearly independent on $I$.

$\boxed{3}$ $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$.

$\boxed{4}$ $W(y_1, y_2)(t) = 0 \quad \forall\, t \in I$.

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## linear Algebra Aspects

- Let $V$ be the set: $V = \{ y : y'' + p(t)\,y' + q(t)\,y = 0,\ t \in I = (\alpha, \beta) \}$

  Then $V$ is a vector space of dimension two with basis given by any fundamental set of solutions $y_1$ and $y_2$.

- For example the solution space $V$ to the DE $y'' - y = 0$ has basis $S_1 = \{ e^{t}, e^{-t} \}$, $S_2 = \{ \cosh t, \sinh t \}$

  with $$V = \operatorname{span} S_1 = \operatorname{Span} S_2$$

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* $\operatorname{Span}(v_1, v_2, \ldots, v_n)$ : is the set of all vectors that can be represented as the linear combination of $v_1, v_2, \ldots, v_n$.