3.3 Complex Roots of the Characteristic Eq.

Recall that a 2^{nd} order linear homogeneous DE with constant coefficients is $\boxed{ay'' + by' + cy = 0}^{*}$. Assuming an exponential solution $y = e^{rt}$ to $*$, then r must be a root of the characteristic equation:

$ar^2 + br + c = 0$. Thus, we have two solutions:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$*$ If the discriminant $b^2 - 4ac$ is positive, then the general solution is $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ $(r_1 \neq r_2)$

$*$ If $b^2 - 4ac < 0$, then the roots are conjugate complex numbers $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, where $\lambda, \mu \in \mathbb{R}$.

Thus, $y_1(t) = e^{(\lambda + i\mu)t}$ and $y_2(t) = e^{(\lambda - i\mu)t}$ .....①

Taylor Series: f can be represented as an infinit sum of the function's derivative at point a:

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Example $(a=0)$: and $-\infty < x < \infty$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \cdots ②$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \cdots ③$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \cdots ④$$

Thus, assume that we can substitute it in ④:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

using that:
$i^2 = -1$
$i^3 = -i$
$i^4 = +1$

$$\boxed{e^{it} = \cos t + i \sin t}$$

using ② and ③

Euler's Formula

$$\Rightarrow e^{-it} = \cos(-t) + i \sin(-t)$$
$$= \cos t - i \sin t$$

$$e^{2+\frac{\pi}{2}i} = e^2 e^{\frac{\pi}{2}i} = e^2 [\cos\tfrac{\pi}{2} + i \sin\tfrac{\pi}{2}] = e^2[0+i] = e^2 i \approx 7.39\, i$$

Example: Use Euler's formula to write $e^{2+\frac{\pi}{2}i}$ in the form $a+bi$.

Therefore, we can write ① as:

- $y_1(t) = e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}[\cos\mu t + i \sin \mu t]$

$$\boxed{y_1(t) = e^{\lambda t}\cos\mu t + i e^{\lambda t} \sin \mu t} \quad ⑤$$

- $y_2(t) = e^{(\lambda-i\mu)t} = e^{\lambda t} e^{-i\mu t} = e^{\lambda t}[\cos\mu t - i \sin \mu(t)]$

$$\boxed{y_2(t) = e^{\lambda t}\cos\mu t - i e^{\lambda t} \sin \mu t} \quad ⑥$$

Real Valued Solutions: The solutions ⑤ and ⑥ are complex valued functions because the roots r_1 and r_2 are complex numbers.

Since our DE * has real coefficients, we prefere to have real solutions. By Th 3.2.2, any linear combinations of ⑤ and ⑥ is also a solution:

$$y_1(t) + y_2(t) = 2e^{\lambda t}\cos\mu t = y_3(t)$$
$$y_1(t) - y_2(t) = 2ie^{\lambda t}\sin\mu t = y_4(t)$$

* Since we are looking for real valued solutions, let us ignore the constants multipliers from y_3 and y_4:

$$y_3(t) = e^{\lambda t}\cos\mu t \quad \text{and} \quad y_4(t) = e^{\lambda t}\sin\mu t$$

* Note that y_3 is the real part of y_1 and y_2 while y_4 is the imaginary part of y_1.

* The Wronskian $W = \begin{vmatrix} y_3 & y_4 \\ y_3' & y_4' \end{vmatrix} = \begin{vmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu(t) \\ e^{\lambda t}(\lambda\cos\mu t - \mu\sin\mu t) & e^{\lambda t}(\lambda\sin\mu t + \mu\cos\mu t) \end{vmatrix}$

$$W = \mu e^{2\lambda t} \neq 0$$

Because $\mu \neq 0$. If $\mu = 0$, then the roots are real.

Thus, y_3 and y_4 form a fundamental solution set for the DE *. Therefore, the general solution of the DE * is

$$\boxed{y(t) = c_1 e^{\lambda t}\cos\mu t + c_2 e^{\lambda t}\sin\mu t} \qquad c_1, c_2 \in \mathbb{R}$$

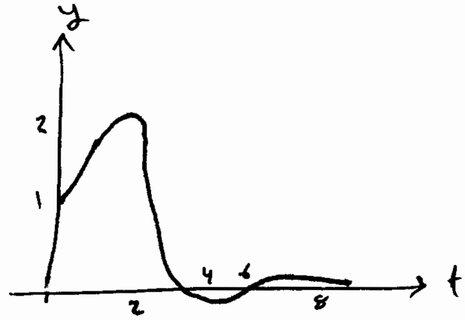
<u>Example</u>: Find the general solution of $y'' + y' + y = 0$.

If $y(t) = e^{rt}$, then the characteristic equation is $r^2 + r + 1 = 0$ with roots

$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i. \quad \text{Thus, } \lambda = \frac{-1}{2},\ \mu = \frac{\sqrt{3}}{2}$$

and the general solution is

$$y(t) = c_1 e^{-\frac{t}{2}}\cos\frac{\sqrt{3}t}{2} + c_2 e^{-\frac{t}{2}}\sin\frac{\sqrt{3}t}{2}.$$



Oscillation (decay exp.)
$\lambda = -\frac{1}{2} < 0$

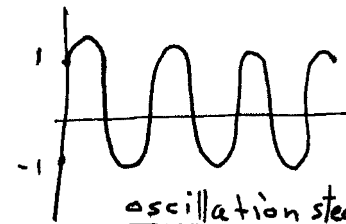
Example: Find the general solution of $y'' + 4y = 0$

If $y = e^{rt}$, then the characteristic equation is $r^2 + 4 = 0$

$\Leftrightarrow r = \pm 2i$. Thus, $\lambda = 0$ and $\mu = 2$

The general solution is:

$y(t) = c_1 \cos 2t + c_2 \sin 2t$



oscillation steadily ($\lambda = 0$)

Example: Find the solution of the IVP

$$16y'' - 8y' + 145y = 0, \quad y(0) = -2, \quad y'(0) = 1$$

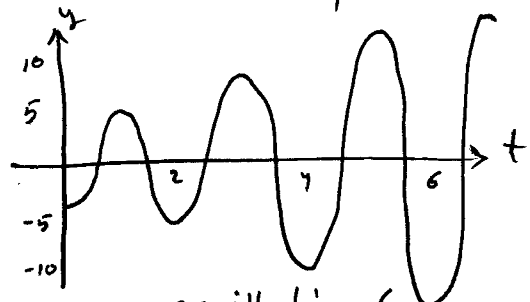
If $y(t) = e^{rt}$, then the characteristic equation is

$16r^2 - 8r + 145 = 0$ with roots $r = \frac{1}{4} \pm 3i$

Thus, $\lambda = \frac{1}{4}$ and $\mu = 3$.

The general solution is

$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t$.



oscillation (grows exp.) $\lambda = \frac{1}{4} > 0$

$y(0) = \boxed{-2 = c_1}$ and $y'(0) = 1 = \frac{1}{4} c_1 + 3c_2$

$\Rightarrow \boxed{c_2 = \frac{1}{2}}$

Hence, the general solution is $y(t) = -2 e^{\frac{t}{4}} \cos 3t + \frac{1}{2} e^{\frac{t}{4}} \sin 3t$
