# 3.4 Repeated Roots; Reduction of Order



Recall that a $2^{nd}$ order linear homogeneous DE with constant coefficient is $\boxed{ay''+by'+cy=0}^{*}$. If $y=e^{rt}$, then the characteristic equation is $ar^2+br+c=0$. The roots are $r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

$\Rightarrow$ If $r_1\neq r_2\in\mathbb{R}$, then the general solution is $y=c_1e^{r_1t}+c_2e^{r_2t}$.

$\Rightarrow$ If $r_1$ and $r_2$ are complex, then the general solution is

$$y(t)=c_1e^{\lambda t}\cos\mu t+c_2e^{\lambda t}\sin\mu t$$

$\Rightarrow$ Now If $b^2-4ac=0$, then $r_1=r_2=-\frac{b}{2a}$ (Repeated Roots):

→ This gives one solution $y_1(t)=e^{-\frac{b}{2a}t}$ for the DE $*$.

→ To find the general solution, we need to find the second solution

→ Note that $y_2(t)=c\,e^{-\frac{b}{2a}t}$ is a solution for $*$ for any $c\in\mathbb{R}$.

The idea is generalize this by replacing $c$ by a function $V(t)$ so that $\boxed{y_2(t)=V(t)\,e^{-\frac{b}{2a}t}}$ (20) is a solution for $*$ (D'Alembert Method)

$$y_2'(t)=v'(t)\,e^{-\frac{b}{2a}t}-\frac{b}{2a}v(t)\,e^{-\frac{b}{2a}t}$$

$$y_2''(t)=v''(t)\,e^{-\frac{b}{2a}t}-\frac{b}{2a}v'(t)\,e^{-\frac{b}{2a}t}-\frac{b}{2a}v'(t)e^{-\frac{b}{2a}t}+\left(\frac{b}{2a}\right)^2v(t)\,e^{-\frac{b}{2a}t}$$

Substitute $y_2, y_2', y_2''$ in $*$ and simplify things to arrive:

$$av''(t)-\left(\frac{b^2-4ac}{4a}\right)v(t)=0$$

but $b^2-4ac=0$. Thus $v''(t)=0$ and $\boxed{v(t)=k_3t+k_4}$ (21)

$\Rightarrow$ Using (21), the solution (20) becomes $y_2(t)=(k_3t+k_4)\,e^{-\frac{b}{2a}t}$

Thus, the general solution becomes:

$$y(t)=k_1y_1(t)+k_2y_2(t)=k_1e^{-\frac{b}{2a}t}+k_2(k_3t+k_4)e^{-\frac{b}{2a}t}$$

$$=(k_1+k_4)\,e^{-\frac{b}{2a}t}+k_2k_3\,t\,e^{-\frac{b}{2a}t}=c_1e^{-\frac{b}{2a}t}+c_2t\,e^{-\frac{b}{2a}t}.$$

So, the general solution for the repeated roots is $\boxed{y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}}$

- To check that $y_1$ and $y_2$ are linearly independent, we calculate the Wronskian:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{b}{2a}t} & t e^{-\frac{b}{2a}t} \\ -\frac{b}{2a} e^{-\frac{b}{2a}t} & \left(1 - \frac{bt}{2a}\right) e^{-\frac{b}{2a}t} \end{vmatrix}$$

$$= \left(1 - \frac{bt}{2a}\right) e^{-\frac{bt}{a}} + \frac{bt}{2a} e^{-\frac{bt}{a}} = e^{-\frac{bt}{2a}} \neq 0 \quad \forall t.$$

Thus, $y_1$ and $y_2$ form a fundamental set of solutions for $*$.

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Example: Solve the ODE $y'' + 2y' + y = 0$, $y(0) = 1$, $y'(0) = 1$

If $y = e^{rt}$, then the characteristic equation is

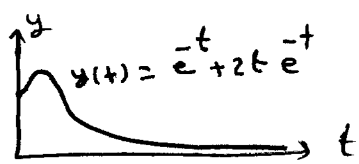
$$r^2 + 2r + 1 = 0 \iff (r+1)^2 = 0 \iff r = -1$$

Thus, the general solution is $y(t) = c_1 e^{-t} + c_2 t e^{-t}$

$y(0) = 1 = c_1 + 0 \Rightarrow \boxed{c_1 = 1}$

$y'(t) = -e^{-t} - c_2 t e^{-t} + c_2 e^{-t} \Rightarrow y'(0) = 1 = -1 + c_2 \Rightarrow \boxed{c_2 = 2}$

Thus, $\boxed{y(t) = e^{-t} + 2t e^{-t}}$

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- Note that in the previous example $\lim_{t \to \infty} y(t) = 0$

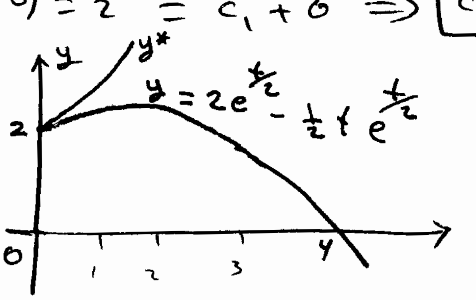
**Example**: Find the solution of the following IVP:

$$y'' - y' + \frac{1}{4} y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}$$

Compare the result with the case when $y'(0) = \frac{3}{2}$

- If $y = e^{rt}$, then the characteristic equation is $r^2 - r + \frac{1}{4} = 0$
with the repeated root $(r - \frac{1}{2})^2 = 0 \Leftrightarrow r = \frac{1}{2}$.
- Thus, the general solution is $\boxed{y(t) = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}}$
- $y(0) = 2 = c_1 + 0 \Rightarrow \boxed{c_1 = 2}$, $y'(t) = \frac{1}{2} c_1 e^{\frac{1}{2}t} + c_2 e^{\frac{1}{2}t} + c_2 \frac{t}{2} e^{\frac{1}{2}t}$

$y'(0) = \frac{1}{2} = \frac{1}{2}(2) + c_2 \Rightarrow \boxed{c_2 = -\frac{1}{2}}$.




Thus, the general solution is

$$\boxed{y(t) = 2e^{\frac{t}{2}} - \frac{1}{2} t e^{\frac{t}{2}}}.$$

Now if $y'(0) = \frac{3}{2} = \frac{1}{2}(2) e^{\frac{1}{2}(0)} + c_2 \Rightarrow \boxed{c_2 = \frac{1}{2}}$

Thus, the general solution becomes $\boxed{y^*(t) = 2e^{t/2} + \frac{1}{2} t e^{t/2}}$

- Note that the behavior of solution depends on the initial slope.

## Reduction of Order

We can apply D'Alembert method to equations with nonconstant coefficients:
$$y'' + p(t) y' + q(t) y = 0 \quad \text{---- (F)}$$

$\Rightarrow$ Given the $1^{st}$ solution $y_1(t)$. To find the second solution, Let

$$y_2 = v(t) y_1(t) \quad \Rightarrow y_2'(t) = v'(t) y_1(t) + v(t) y_1'(t)$$

$$\Rightarrow y_2''(t) = v''(t) y_1(t) + 2v'(t) y_1'(t) + v(t) y_1''(t)$$

$\Rightarrow$ Substitute $y_2, y_2', y_2''$ in (F) and collect terms, we obtain

$$y_1 v'' + (2y_1' + p y_1) v' + \underbrace{(y_1'' + p y_1' + q y_1)}_{\text{This is zero since } y_1 \text{ is a solution}} = 0$$

$$\Rightarrow \quad \boxed{y_1 v'' + (2y_1' + py_1)v' = 0} \text{ (H)}$$

Note that the $2^{nd}$ order linear homogeneous DE (F) reduces to $1^{st}$ order linear homogeneous DE (H), i.e (F) is $2^{nd}$ order in $y$ and (H) is $1^{st}$ order in $v'$. That is why this method is called the method of reduction of order.

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**Example**: Given that $y_1(t) = \frac{1}{t}$ is a solution of

$$t^2 y'' + 3t y' + y = 0 \quad , \quad t > 0 \quad \cdots *^1$$

Find a second solution (Use the method of reduction of order to find the $2^{nd}$ solution).

$$y_2(t) = v(t)\, y_1 = v(t)\, t^{-1} \quad \cdots\cdot \text{ (A)}$$

$$y_2'(t) = v'(t)\, t^{-1} - v(t)\, t^{-2}$$

$$y_2''(t) = v''(t)\, t^{-1} - 2v'(t)\, t^{-2} + 2v(t)\, t^{-3}$$

- Substituting $y_2, y_2', y_2''$ in $*^1$

$$t^2(v'' t^{-1} - 2v' t^{-2} + 2v t^{-3}) + 3t(v' t^{-1} - v t^{-2}) + v t^{-1} = 0$$

$$\Leftrightarrow v'' t - 2v' + \cancel{2v t^{-1}} + 3v' - \cancel{3v t^{-1}} \cancel{+ v t^{-1}} = 0 \quad *^2$$

$$\Leftrightarrow t v'' + v' = 0 \qquad \boxed{\text{let } u(t) = v'(t)}$$

$$\Leftrightarrow t u' + u = 0 \quad (\text{separation of variables})$$

$$\int \frac{u'}{u} = \int \frac{-1}{t} \quad \Leftrightarrow \quad \ln|u| = -\ln|t| + c$$

$$\Leftrightarrow |u| = |t|^{-1} e^{c} \Leftrightarrow \boxed{u = k_4 t^{-1}} \quad \text{since } t>0.$$

Thus, by $*2$, we have $v' = \frac{k_4}{t} \Leftrightarrow \boxed{v(t) = k_4 \ln t + k_3}$

$$\Rightarrow y_2(t) = k_4 t^{-1} \ln t + k_3 t^{-1}$$

Hence, the general Solution for the DE $*^1$ is

$$y(t) = k_1 y_1(t) + k_2 y_2(t)$$

$$= k_1 t^{-1} + k_2 k_3 t^{-1} + k_2 k_4 t^{-1} \ln t$$

$$= (k_1 + k_2 k_3) t^{-1} + k_2 k_4 t^{-1} \ln t$$

$$= c_1 t^{-1} + c_2 t^{-1} \ln t$$

Hence

$$y(t) = \frac{c_1}{t} + \frac{c_2 \ln t}{t}$$

$$y(t) = \frac{c_1 + c_2 \ln t}{t}$$