# 3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

- Recall that a $2^{nd}$ order linear nonhomogeneous equation:

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad \dots \boxed{1}$$

where $p, q, g$ are continuous functions on an open interval $I$.

- The corresponding homogeneous equation is

$$L[y] = y'' + p(t)y' + q(t)y = 0 \quad \dots \boxed{2}$$

**Theorem 3.5.1**: If $Y_1$ and $Y_2$ are solutions of the nonhomogeneous equation $\boxed{1}$, then the difference $Y_1 - Y_2$ is a solution to the corresponding homogeneous equation $\boxed{2}$.

Furthermore, if $y_1$ and $y_2$ form a fundamental set of solutions of the homogeneous equation $\boxed{2}$, then $\exists\, c_1, c_2 \in \mathbb{R}$ s.t. $\boxed{Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t)}$. $\boxed{3}$

Proof: • Since $Y_1$ and $Y_2$ are solutions to $\boxed{1}$, we have

$$L[Y_1](t) = g(t) \quad \text{and} \quad L[Y_2](t) = g(t).$$

Thus,

$$L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0 \Rightarrow$$

$$L[Y_1 - Y_2](t) = 0.$$ Thus, $Y_1 - Y_2$ is solution to $\boxed{2}$.

- Since $y_1$ and $y_2$ form a fundamental set of solutions of $\boxed{2}$, any solution of $\boxed{2}$ can be then written as a linear combinations of $y_1$ and $y_2$ (Th. 3.2.4)

Since $Y_1 - Y_2$ is a solution of $\boxed{2} \Rightarrow Y_1 - Y_2 = c_1 y_1 + c_2 y_2$ and $\boxed{3}$ holds.

**Theorem 3.5.2 (General Solution)**: The general solution of the nonhomogeneous equation $\boxed{1}$ has the form: $y(t) = y_h(t) + y_p(t)$, where $y_h(t) = c_1 y_1(t) + c_2 y_2(t)$, $y_1$ and $y_2$ form a fundamental set of solutions for the corresponding homogeneous equation $\boxed{2}$; and $y_p(t)$ is a particular solution of the nonhomog equation $\boxed{1}$.

Proof. The proof follows by Th 3.6.1: Note that $\boxed{3}$ holds if we replace $Y_1(t)$ by an arbitrary solution $y(t)$ and $Y_2(t)$ by a particular solution $y_p(t)$.

Thus $y(t) - y_p(t) = c_1 y_1(t) + c_2 y_2(t)$. Hence, $y(t) = y_h + y_p$.

- Since $y(t)$ is an arbitrary solution of $\boxed{1}$, it follows that $y_h + y_p$ includes all solution of $\boxed{1}$. Thus $y(t)$ is a general solution.

---

Th. 3.6.2 provides away to find the general solution $y(t) = y_h(t) + y_p(t)$ of the nonhomoganuous equation $\boxed{1}$, where

- $y_h = c_1 y_1(t) + c_2 y_2(t)$ is the homogenuous solution that results from solving the homogenuous equation 2. It's also called the complementary solution $y_c(t) = y_h(t) = c_1 y_1(t) + c_2 y_2(t)$.
- $y_p$ is a particular solution for the nonhomogenuous equation $\boxed{1}$.

There are two methods of finding the particular solution $y_p(t)$ for a nonhomogenuous equations:

1) **Method of Undetermined Coefficients**
2) **Method of Variation Parameters** (next section)

---

## Method of Undetermined Coefficients

- Recall the nonhomogenuous equation $\boxed{1}$ $y'' + p(t)y' + q(t)y = g(t)$ with general solution $y(t) = y_h(t) + y_p(t)$, where $y_h = c_1 y_1 + c_2 y_2$.
- The method of undetermined coefficients is used to find a particular solution $y_p(t)$ for a nonhomogenuous equation $\boxed{1}$ by making an initial assumption about the form of the particular solution $y_p(t)$.
- The method of undetermined coefficients is limitted for which $p$ and $q$ are constants and $g(t)$ is a polynomial, exponential, sine or cosine functions.

**Example 1** ($g(t)$ is exponential) Find the general solution of the IVP

$$\boxed{5} \cdots \quad y'' - 3y' - 4y = 3e^{2t} \qquad y(0) = \frac{9}{2}, \quad y'(0) = -1$$

The general solution is $y(t) = y_h(t) + y_p(t)$.

→ To find $y_h(t)$; we solve $y'' - 3y' - 4y = 0$.

If $y = e^{rt}$, then the characteristic equation is

$$r^2 - 3r - 4 = 0 \Leftrightarrow (r-4)(r+1) = 0 \Leftrightarrow r_1 = 4, \; r_2 = -1$$

Thus, $\boxed{y_h(t) = c_1 e^{4t} + c_2 e^{-t}}$

→ To find $y_p(t)$, we start with $y_p(t) = A e^{2t}$ $\Rightarrow y_p' = 2Ae^{2t}$ $\Rightarrow y_p'' = 4Ae^{2t}$

Substituting $y_p, y_p', y_p''$ in 5, we obtain

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t} \Leftrightarrow \boxed{A = -\frac{1}{2}}.$$ Thus, the particular solution is $\boxed{y_p(t) = -\frac{1}{2}e^{2t}}$. Hence, the general solution becomes $y(t) = y_h(t) + y_p(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t}$

→ To find $c_1$ and $c_2$: $y(0) = c_1 + c_2 - \frac{1}{2} = \frac{9}{2} \Rightarrow \boxed{c_1 + c_2 = 5}$

$y'(0) = 4c_1 - c_2 - 1 = -1 \Rightarrow \boxed{c_2 = 4c_1}$

Hence, $\boxed{c_1 = 1}$ and $\boxed{c_2 = 4}$. Thus, $y(t) = e^{4t} + 4e^{-t} - \frac{1}{2}e^{2t}$.

---

**Example 2**: Find a particular solution of $y'' - 3y' - 4y = 2\sin t \cdots \boxed{6}$

- Note that $g(t) = 2\sin t$. So we start with a particular solution of the form $y_p(t) = A\sin t$ $\Rightarrow y_p' = A\cos t$ and $y_p'' = -A\sin t$
- Substitute $y_p, y_p', y_p''$ in $\boxed{6}$, we obtain:

$$-A\sin t - 3A\cos t - 4A\sin t = 2\sin t \Leftrightarrow (2+5A)\sin t + 3A\cos t = 0$$

Since $\sin t$ and $\cos t$ are linearly independent $\Rightarrow 3A = 0$ and $2 + 5A = 0$ $\Rightarrow A = 0$ and $2 = 0$. ✗.

- So we make a another form of $y_p$ to start with:

$\Rightarrow$ Consider $y_p(t) = A \sin t + B \cos t$

$\Rightarrow y_p' = A \cos t - B \sin t$ and $y_p'' = -A \sin t - B \cos t$

- Substitute $y_p, y_p', y_p''$ in $\boxed{6}$, we obtain:

$$-A \sin t - B \cos t - 3A \cos t + 3B \sin t - 4A \sin t - 4B \cos t = 2 \sin t$$

$$(-5A + 3B) \sin t + (-3A - 5B) \cos t = 2 \sin t$$

$\boxed{-5A + 3B = 2}$ and $\boxed{-3A - 5B = 0}$ $\Rightarrow A = -\frac{5}{17}$ and $B = \frac{3}{17}$

Thus a particular solution to the nonhomogenuous ODE 6 is

$$y_p(t) = \frac{-5}{17} \sin t + \frac{3}{17} \cos t$$

---

Example 3: Find a particular solution of $\boxed{y'' - 3y' - 4y = 4t^2 - 1}$. $-\boxed{7}$

- Note that $g(t) = 4t^2 - 1$ (polynomial of degree 2).
- We start with a particular solution of the form $y_p(t) = At^2 + Bt + C$
- Substitute $y_p, y_p', y_p''$ in $\boxed{7}$, we get:

$$y_p' = 2At + B$$
$$y_p'' = 2A$$

$$2A - 3(2At + B) - 4(At^2 + Bt + C) = 4t^2 - 1$$

$$-4At^2 - (6A + 4B)t + (2A - 3B - 4C) = 4t^2 - 1$$

$\boxed{A = -1}$, $\boxed{B = \frac{3}{2}}$, $\boxed{C = -\frac{11}{8}}$. Thus a particular solution

$$y_p(t) = -t^2 + \frac{3}{2} t - \frac{11}{8}$$

---

Example 4: Find a particular solution of $\boxed{y'' - 3y' - 4y = -8e^t \cos 2t}$ $-\boxed{8}$

- Note that $g(t) = -8e^t \cos 2t$. We start with a particular solution of the form $y_p(t) = A e^t \cos 2t + B e^t \sin 2t$

$$y_p'(t) = (A + 2B) e^t \cos 2t + (B - 2A) e^t \sin 2t$$

$$y_p''(t) = (4B - 3A) e^t \cos 2t + (-4A - 3B) e^t \sin 2t$$

- Substitute $y_p, y_p', y_p''$ into the ODE $\boxed{8}$ and solve for $A$ and $B$, we get

$\boxed{A = \frac{10}{13}}$ and $\boxed{B = \frac{2}{13}}$. Thus the particular solution is

$$y_p(t) = \frac{10}{13} e^t \cos 2t + \frac{2}{13} e^t \sin 2t.$$

**Lemma.** Let $g(t) = g_1(t) + g_2(t)$. Suppos that $Y_1$ and $Y_2$ are solutions of the equations

$$y'' + a y' + b y = g_1(t) \quad \cdots \boxed{11}$$
$$y'' + a y' + b y = g_2(t) \quad \cdots \boxed{12}$$

Then $Y_1 + Y_2$ is a solution of the equation

$$y'' + ay' + by = g(t) \quad \cdots \boxed{13}$$

Proof: $(Y_1 + Y_2)'' + a(Y_1 + Y_2)' + b(Y_1 + Y_2) =$

$Y_1'' + Y_2'' + aY_1' + aY_2' + bY_1 + bY_2 =$

$\underbrace{Y_1'' + aY_1' + bY_1}_{Y_1 \text{ is a solution to } \boxed{11}} + \underbrace{Y_2'' + aY_2' + bY_2}_{Y_2 \text{ is a solution to } \boxed{12}} = g_1(t) + g_2(t) = g(t)$

Thus, $Y_1 + Y_2$ is a solution to $\boxed{13}$.

---

<u>Example</u> Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t. \quad \cdots \boxed{14}$$

We split $\boxed{14}$ into three equations

$y'' - 3y' - 4y = 3e^{2t}$ with a particular solution $y_{p_1}(t) = -\frac{1}{2}e^{2t}$ (Example $\boxed{1}$)

$y'' - 3y' - 4y = 2\sin t$ " " " $y_{p_2}(t) = \frac{-5}{17}\sin t + \frac{3}{17}\cos t$ (Example $\boxed{2}$)

$y'' - 3y' - 4y = -8e^{t}\cos 2t$ " " " $y_{p_3}(t) = \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$ (Example $\boxed{4}$)

Thus, a particular solution of $\boxed{14}$ is then given by

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t) + y_{p_3}(t)$$
$$= -\frac{1}{2}e^{2t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$$

**Example**: Find a particular solution of

$$y'' + 4y = 3\cos 2t \quad \ldots \boxed{15}$$

- Note that $g(t) = 3\cos 2t$. Thus, we start with a particular solution of the form $\boxed{y_p(t) = A\sin 2t + B\cos 2t} - \boxed{16}$

$\Rightarrow y_p'(t) = 2A\cos 2t - 2B\sin 2t$ and $y_p''(t) = -4A\sin 2t - 4B\cos 2t$

- Substitute $y_p, y_p', y_p''$ in $\boxed{15}$, we get

$$(-4A\cancel{\sin 2t} - 4B\cancel{\cos 2t}) + 4(A\cancel{\sin 2t} + B\cancel{\cos 2t}) = 3\cos 2t = 0 \quad ?!$$

Thus, no particular solution exists of the form $\boxed{16}$. Why?!

- Because the homogeneous solution of $\boxed{y'' + 4y = 0}^{\boxed{17}}$ is $\boxed{y_h = c_1\cos 2t + c_2\sin 2t}$

Thus our particular solution $y_p(t)$ solves $\boxed{17}$ and not $\boxed{16}$.

- We choose another particular solution $y_p^*(t) = At\sin 2t + Bt\cos 2t$

$\Rightarrow y_p^{*\prime}(t) = A\sin 2t + 2At\cos 2t + B\cos 2t - 2Bt\sin 2t$

$y_p^{*\prime\prime}(t) = 4A\cos 2t - 4B\sin 2t - 4At\sin 2t - 4Bt\cos 2t$

- Substitute $y_p^*, y_p^{*\prime}, y_p^{*\prime\prime}$ in $\boxed{15}$ and arrange terms to arrive:

$$4A\cos 2t - 4B\sin 2t = 3\cos 2t \iff \boxed{B = 0}, \boxed{A = \frac{3}{4}}$$

Thus, a particular solution of $\boxed{15}$ is $y_p^*(t) = \frac{3}{4}t\sin 2t$

---

- Recall that the method of undetermined coefficients works if the function $g(t)$ is an exponential functions, sines, cosines, polynomials, or sum or product of such functions.
- If this is not the case, then we use the method of variations of parameters (see section 3.7).