3.6 Variation of Parameters

Recall that there are two methods to find a particular solution $y_p(t)$ to the nonhomogeneous ODE:

$$y'' + p(t)\,y' + q(t)\,y = g(t),$$

where p, q, g are continuous functions on an open interval I:

1) the method of undetermined coefficients: Usually used when $g(t)$ is exponential or sine or cosin or polynomial or sum or product of such functions with an initial assumption about the solution form

2) the method of variation of Parameters: This is more general method. No initial assumption is required about the form of the solution. However, a certain integrals need to be evaluated.

Theorem 3.6.1 Consider the DE's

$$y'' + p(t)\,y' + q(t)\,y = g(t) \quad \cdots \boxed{1}$$
$$y'' + p(t)\,y' + q(t)\,y = 0, \quad \cdots \boxed{2}$$

where p, q, g are continuous on an open interval I.

If y_1 and y_2 form a fundamental set of solutions to equation $\boxed{2}$, then a particular solution of equation $\boxed{1}$ is

$$y_p(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)\,g(s)}{W(y_1,y_2)(s)}\,ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s)\,g(s)}{W(y_1,y_2)(s)}\,ds$$

and the general solution is then given by

$$y(t) = y_h + y_p = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

Proof: The general solution of equation $\boxed{2}$ is

$$y_c = y_h = c_1 y_1 + c_2 y_2 .$$

- The idea in the method of variation of parameters is to find a particular solution $y_p(t)$ by replacing c_1 and c_2 by functions $v_1(t)$ and $v_2(t)$ so that

$$\boxed{y_p(t) = v_1(t)\, y_1(t) + v_2(t)\, y_2(t)} \quad (A)$$

is a solution to Eq. $\boxed{1}$

- To find v_1 and v_2 we need to substitute y_p in equation $\boxed{1}$.

$$y_p'(t) = \underline{v_1'(t)\, y_1(t)} + v_1(t)\, y_1'(t) + \underline{v_2'(t)\, y_2(t)} + v_2(t)\, y_2'(t)$$

To be able to solve for v_1 and v_2, we assume that:

$$\boxed{v_1'(t)\, y_1(t) + v_2'(t)\, y_2(t) = 0} \quad \boxed{3}$$

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Thus

$$y_p'(t) = v_1(t)\, y_1'(t) + v_2(t)\, y_2'(t)$$

$$y_p''(t) = v_1'(t)\, y_1'(t) + v_1(t)\, y_1''(t) + v_2'(t)\, y_2'(t) + v_2(t)\, y_2''(t)$$

- Substitute y_p, y_p', y_p'' in equation $\boxed{1}$:

$$v_1' y_1' + \underline{v_1 y_1''} + v_2' y_2' + \underline{v_2 y_2''} + p(t)\left[\underline{v_1 y_1'} + \underline{v_2 y_2'}\right] + q(t)\left[\underline{v_1 y_1} + \underline{v_2 y_2}\right] = g(t)$$

$$v_1' y_1' + v_2' y_2' + v_1 \underbrace{\left[y_1'' + p(t)\, y_1' + q(t)\, y_1\right]}_{\text{zero since } y_1 \text{ is solution to } \boxed{2}} + v_2 \underbrace{\left[y_2'' + p(t)\, y_2' + q(t)\, y_2\right]}_{\text{zero since } y_2 \text{ is solution } \boxed{2}} = g(t)$$

$$\boxed{v_1' y_1' + v_2' y_2' = g(t)} \quad \boxed{4}$$

- From equations $\boxed{3}$ and $\boxed{4}$, we have

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2(t)\, g(t)}{W(y_1, y_2)(t)} \iff v_1(t) = -\int_{t_0}^{t} \frac{y_2(s)\, g(s)}{W(y_1, y_2)(s)}\, ds$$

$$V_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1(t)\, g(t)}{W(y_1,y_2)(t)} \Leftrightarrow V_2(t) = \int_{t_0}^{t} \frac{y_1(s)\, g(s)}{W(y_1,y_2)(s)}\, ds$$

• Thus, a particular solution Ⓐ becomes:

$$y_p(t) = V_1(t)\, y_1(t) + V_2(t)\, y_2(t)$$

$$= -y_1(t) \int \frac{y_2(s)\, g(s)}{W(y_1,y_2)(s)}\, ds + y_2(t) \int \frac{y_1(s)\, g(s)}{W(y_1,y_2)(s)}\, ds$$

<u>Example</u> (a) Find a particular solution of $y'' + 4y = 3\csc t$. — $\boxed{6}$ $\left(\csc t = \frac{1}{\sin t}\right)$

(b) Find the general solution also.

(a) • Note that we can't use the method of undetermined coefficients here because $g(t) = 3\csc t$ (not e^t, $\sin t$, $\cos t$, polynomial ...) يحتوي على قسمة

• To find y_1 and y_2 we solve the characteristic equation for the homogeneous equation $y'' + 4y = 0$, i.e $r^2 + 4 = 0 \Leftrightarrow$ $r = \pm 2i \Leftrightarrow \lambda = 0, \mu = 2$

$\Leftrightarrow$ $y_1(t) = \cos 2t$

$y_2(t) = \sin 2t$

• $W(y_1,y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

$$= \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2$$

• $V_1(t) = -\int \frac{y_2(s)\, g(s)}{W(y_1,y_2)(s)}\, ds = -\frac{1}{2}\int \sin 2t \, \frac{3}{\sin t}\, dt = -\frac{3}{2}\int \frac{2\cancel{\sin t}\cos t}{\cancel{\sin t}}\, dt$

$$= -3\int \cos t\, dt = -3\sin t + k_1$$

• $V_2(t) = \int \frac{y_1(s)\, g(s)}{W(y_1,y_2)(s)}\, ds = \frac{1}{2}\int \cos 2t \, \frac{3}{\sin t}\, dt = \frac{3}{2}\int \frac{1 - 2\sin^2 t}{\sin t}\, dt$

$$= \frac{3}{2}\overset{*}{\int} \csc t\, dt - 3\int \sin t\, dt$$

$$= \frac{3}{2}\ln\left|\csc t - \cot t\right| + 3\cos t + k_2$$

See the last page

* see on the back how to integrate this

• Thus, a particular solution to the nonhomogeneous equation $\boxed{6}$ is $y_p(t) = v_1(t)\, y_1(t) + v_2(t)\, y_2(t) \Rightarrow$

$$y_p(t) = (-3\sin t + k_1)\cos 2t + \left(\tfrac{3}{2}\ln|\csc t - \cot t| + 3\cos t + k_2\right)\sin 2t$$

(b) The general solution of the nonhomogeneous equation $\boxed{6}$ is then given by: $y(t) = y_h(t) + y_p(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$

$$y(t) = c_3\cos 2t + c_4\sin 2t - 3\sin t\cos 2t + 3\sin 2t\cos t + \tfrac{3}{2}\ln|\csc t - \cot t|\sin 2t$$

$$y(t) = c_3\cos 2t + c_4\sin 2t + 3\sin t + \tfrac{3}{2}\ln|\csc t - \cot t|\sin 2t$$

$c_3 = c_1 + k_1$

$c_4 = c_2 + k_2$

Note that $-3\sin t\cos 2t + 3\sin 2t\cos t =$

$-3\sin t\,[2\cos^2 t - 1] + 3[2\sin t\cos t]\cos t =$

$-6\cancel{\sin t}\cos^2 t + 3\sin t + 6\cancel{\sin t}\cos^2 t = 3\sin t$

$$\int \csc x\, dx = \int \frac{dx}{\sin x} = \int \frac{\sin x}{\sin^2 x}\, dx = \int \frac{\sin x}{1-\cos^2 x}\, dx$$

$$= \int \frac{\sin x \quad dx}{(1-\cos x)(1+\cos x)}$$

$u = \cos x$

$du = -\sin x\, dx$

$$= -\int \frac{du}{(1-u)(1+u)}$$

use the partial fractions:

$$= -\left[\int \frac{\frac{1}{2}}{1+u}du + \int \frac{\frac{1}{2}}{1-u}du\right]$$

$$\frac{1}{(1-u)(1+u)} = \frac{A}{(1+u)} + \frac{B}{(1-u)}$$

$$1 = A(1-u) + B(1+u)$$

$\boxed{B-A=0}$ $\boxed{A+B=1}$

$A = B = \frac{1}{2}$

$$= \frac{1}{2}\left[\ln|1+u| + \ln|1-u| + c\right]$$

$$= -\frac{1}{2}\left[\ln|1+\cos x| + \ln|1-\cos x| + c\right]$$

$$= -\frac{1}{2}\left[\ln|1-\cos^2 x| + c\right]$$

$$= -\frac{1}{2}\left[\ln|\cos^2 x| + c\right]$$

assume $c = 0$

$$= \frac{1}{2}\ln|\csc^2 x|$$

$$= \ln|\csc^2 x|^{\frac{1}{2}}$$

$$= \ln|\csc x|$$

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or $\int \csc x\, dx = \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x}\, dx = -\int \frac{du}{u} = -\ln|u| + c$

$= -\ln|\csc x + \cot x|$

let $u = \csc x + \cot x$

$du = (-\csc x \cot x - \csc^2 x)\, dx$