Chapter 4

4.1 General Theory of n^{th} Order Linear Equations

- An n^{th} order ODE has the general form

$$P_0(t)\frac{d^ny}{dt^n} + P_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + P_{n-1}(t)\frac{dy}{dt} + P_n(t)y = G(t) \quad \cdots \boxed{1}$$

- We assume that $P_0, P_1, \ldots, P_n$ are continuous real-valued functions on some interval $I = (\alpha, \beta)$, and that P_0 is nowhere zero on I.
- Divid by P_0, equation $\boxed{1}$ becomes:

$$L[y] = y^{(n)} + p_1(t)\,y^{(n-1)} + \cdots + p_{n-1}(t)\,y' + p_n(t)\,y = g(t) \quad \cdots \boxed{2}$$

- For an n^{th} order ODE, there are n initial conditions:

$$y(t_0) = y_0,\quad y'(t_0) = y_0',\quad \ldots,\quad y^{(n-1)}(t_0) = y_0^{(n-1)},\quad t_0 \in I \quad - \boxed{3}$$

- Since the linear differential operator L of order n defined in $\boxed{2}$ is similar to the 2^{nd} order linear differential operator introduced in chapter 3, it follows that the mathematical theory associated with eqation $\boxed{2}$ is completely analogous to that for 2^{nd} order equation.
- We will state the results here without proof.
- The following Theorem is similar to Th 3.2.1 (Existence and Uniqueness).

Theorem 4.1.1 If the functions $p_1, p_2, \ldots, p_n$ and g are continuous on an open interval I, then $\exists$ a unique solution $y = \phi(t)$ of the differential equation $\boxed{2}$ that satisfies the initial conditions $\boxed{3}$. This solution exists throughout the interval I.

Example: Determind an interval on which the solution is sure to exist for the DE

$$t^2 y^{(4)} + t y^{(3)} + 5y = \sin t \quad \Rightarrow \quad y'''' + \frac{1}{t}y''' + \frac{5}{t^2}y = \frac{\sin t}{t}$$

The discontinuity of p_1, p_4 is at $t = 0$. Thus, the interval $I = (-\infty, 0) \cup (0, \infty)$ is where the solution exists.

$\boxed{A}$ Homogenuous Equations: If the functions $y_1, y_2, \cdots, y_n$ are solutions of the DE

$$L[y] = y^{(n)} + p_1(t)\, y^{(n-1)} + \cdots + p_{n-1}(t)\, y' + p_n(t)\, y = 0, \quad \cdots \boxed{4}$$

then the linear combination

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t) \quad \cdots \boxed{5}$$

is also a solution of $\boxed{4}$, where $c_1, c_2, \cdots, c_n$ are constants.

- To determine the constants $c_1, c_2, \cdots, c_n$ using the initial conditions $\boxed{3}$, the following equations must be satisfied:

$$\begin{aligned}
c_1 y_1(t_0) + \cdots + c_n y_n(t_0) &= y_0 \\
c_1 y_1'(t_0) + \cdots + c_n y_n'(t_0) &= y_0' \\
&\vdots \\
c_1 y_1^{(n-1)}(t_0) + \cdots + c_n y_n^{(n-1)}(t_0) &= y_0^{(n-1)}
\end{aligned} \quad \cdots \boxed{6}$$

- The system of equations $\boxed{6}$ has a unique solution iff its determinant or Wronskian is not zero at t_0:

$$W(y_1, y_2, \cdots, y_n)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{vmatrix} \neq 0$$

- Since t_0 can be any point in I, the Wronskian determinant needs to be nonzero at every point in I.
- As the 2^{nd} order linear equation, if $y_1, y_2, \cdots, y_n$ are solutions of the DE $\boxed{4}$ then the Wronskian $W(y_1, y_2, \cdots, y_n)(t) = 0 \;\; \forall\, t \in I$ or
$W(y_1, y_2, \cdots, y_n)(t) \neq 0 \;\; \forall\, t \in I$.

Theorem 4.1.2 If a) the functions $p_1, p_2, \ldots, p_n$ are continuous on an open interval I,

b) the functions $y_1, y_2, \ldots, y_n$ are solutions of Eq. $\boxed{4}$,

c) $W(y_1, y_2, \ldots, y_n)(t) \neq 0$ for at least one point in I,

then every solution of Eq. $\boxed{4}$ can be expressed as a linear combination of $y_1, y_2, \ldots, y_n$ $\boxed{5}$: $y(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$

- Recall that a set of solutions $y_1, \ldots, y_n$ of Eq. $\boxed{4}$ whose Wronskian is non zero forms a **fundamental set of solutions**.
- Since all solutions of Eq. $\boxed{4}$ has the form $\boxed{5}$, we call the form $\boxed{5}$ a general solution.
- The functions $f_1, f_2, \ldots, f_n$ are **linearly dependent** on I if $\exists$ $k_1, k_2, \ldots, k_n$ not all zero s.t $k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0$.
- The functions $f_1, f_2, \ldots, f_n$ are **linearly independent** on I if they are not linearly dependent, i.e: $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ iff $c_1 = c_2 = \cdots = c_n = 0$.
- If $y_1, y_2, \ldots, y_n$ are solutions of Eq $\boxed{4}$, then $y_1, y_2, \ldots, y_n$ are linearly independent if $W(y_1, y_2, \ldots, y_n)(t_0) \neq 0$ for some $t_0 \in I$. Hence, $y_1, y_2, \ldots, y_n$ form a fundamental set of solutions of Eq $\boxed{4}$.

Example: Verify that the functions $1, t, t^3$ are solutions of the DE

$$-t^2 y^{(4)} + t y''' = 0 \quad .$$

Find their Wronskian.

$$y_1 = 1 \Rightarrow y_1' = y_1'' = y_1''' = 0 \Rightarrow -t^2(0) + t(0) = 0$$

$$y_2 = t \Rightarrow y_2' = 1,\ y_2'' = y_2''' = 0 \Rightarrow -t^2(0) + t(0) = 0$$

$$y_3 = t^3 \Rightarrow y_3' = 3t^2,\ y_3'' = 6t,\ y_3''' = 6 \Rightarrow -t^2(6) + t(6t) = 0$$

$$W(y_1, y_2, y_3)(t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{vmatrix} = 6t \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} = 6t. \text{ Thus } W \neq 0 \text{ iff } t \neq 0$$

Example: Determine weather the functions $f(t) = \sin 2t$ and $g(t) = \sin t \cos t$ are linearly independent or linearly dependent on an arbitrary interval I.

The two given functions are linearly dependent on I because $c_1 \sin 2t + c_2 \sin t \cos t = 0$ for all $t \in I$. If we choose $c_1 = 1$ and $c_2 = -2$

Example: Determine weather the functions $f_1(t) = 1$, $f_2(t) = t$, $f_3(t) = t^2$ are linearly indep. or dep. on the interval $I: -\infty < t < \infty$.

$$K_1 f_1 + K_2 f_2 + K_3 f_3 = K_1 + K_2 t + K_3 t^2 = 0$$

$$\left.\begin{array}{lll} \text{when } t=0 & \Rightarrow & K_1 = 0 \\ t=1 & \Rightarrow & K_1 + K_2 + K_3 = 0 \\ t=-1 & \Rightarrow & K_1 - K_2 + K_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} K_1 = 0 \\ K_2 = K_3 = 0 \end{array}$$

Thus, f_1, f_2, f_3 are linearly indep.

Th 4.13 If $y_1, y_2, \dots, y_n$ is a fundemental set of solutions of Eq. $\boxed{4}$: $L[y] = y^{(n)} + p_1(t) y^{(n-1)} + \dots + p_{n-1}(t) y' + p_n(t) y = 0$ on an interval I, then $y_1, y_2, \dots, y_n$ are linearly indep. on I.

Conversely, if $y_1, y_2, \dots, y_n$ are linearly indep. solutions of Eq $\boxed{4}$ on I, then they form a fundemental set of solutions.

Proof $\Rightarrow$ If $y_1, y_2, \dots, y_n$ is a fundemental set of solutions of $\boxed{4}$ on I, then $W(y_1, y_2, \dots, y_n)(t) \neq 0$ $\forall\, t \in I$. Thus, $y_1, y_2, \dots, y_n$ are linearly indep. since the solution for $k_1 y_1 + k_2 y_2 + \dots + k_n y_n = 0$ is $k_1 = k_2 = \dots = k_n = 0$

$\Leftarrow$ (see the book).

$\boxed{13}$ Nonhomogenuous Equations: Recall the nonhomogenuous Equation $\boxed{2}$: $L[y] = y^{(n)} + p_1(t)\, y^{(n-1)} + \cdots + p_{n-1}(t)\, y' + p_n(t)\, y = g(t)$.

- If Y_1 and Y_2 are solutions to the nonhomogenuous equation $\boxed{2}$, then $Y_1 - Y_2$ is a solution to the homogenuous equation $\boxed{4}$.

  Because $L[Y_1 - Y_2] = L[Y_1] - L[Y_2] = g(t) - g(t) = 0$

- Since $Y_1 - Y_2$ is a solution to the homogenuous equation $\boxed{4}$, it can be written as a linear combination of a fundamental set of solutions $y_1, y_2, \ldots, y_n$, i.e $\exists\ c_1, c_2, \ldots, c_n$ s.t

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t).$$

- Thus, the general solution of the nonhomogenuous ODE $\boxed{2}$ is

$$y(t) = y_h + y_p$$

where y_h is the homogenuous or complementary solution: $y_h = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t)$ and y_p is the particular solution for the nonhomogenuous Eq. $\boxed{2}$.