$\boxed{4.2}$ Homogenuous Equations with Constant Coefficients

- Consider the n^{th} order linear homogenuous DE

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0 , \quad \cdots \boxed{I}$$

where $a_0, a_1, \ldots, a_n$ are real constants.

- Assuming an exponential solution e^{rt} to equation $\boxed{I}$: Then

$$L[e^{rt}] = e^{rt}\left[a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n\right] = 0 \quad \Leftrightarrow$$

$$\overbrace{a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n}^{Z(r)} = 0$$

- The polynomial $Z(r)$ is called the **characteristic polynomial.**
- The equation $Z(r)=0$ is called the **characteristic equation** of $\boxed{I}$
- By the Fundamental Theorem of Algebra, a polynomial of degree n has n roots: $r_1, r_2, \ldots, r_n$. Henc,

$$Z(r) = a_0(r-r_1)(r-r_2)\cdots(r-r_n)$$

$\boxed{A}$ Real and Unequal Roots

- If the roots of the characteristic equation are real and unequal, then there are n distinct solutions $e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}$ of Eq. $\boxed{I}$
- If these functions are linearly independent, then the general solution of Eq. $\boxed{I}$ is $\quad y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t}$
- The Wronskian can be used to determine linear independence of solutions.

Example: Find the general solution of the IVP:

$$y^{(4)} + 2y''' - 13y'' - 14y' + 24y = 0, \quad y(0) = 1, \; y'(0) = -1, \; y''(0) = 0,$$
$$y'''(0) = -1.$$

- Assuming an exponential solution $y(t) = e^{rt}$, then the characteristic equation is $r^4 + 2r^3 - 13r^2 - 14r + 24 = 0 \iff$

$$(r-1)(r+2)(r-3)(r+4) = 0 \iff$$

- The general solution $y(t) = c_1 e^{t} + c_2 e^{-2t} + c_3 e^{3t} + c_4 e^{-4t}$
- Using the initial conditions:

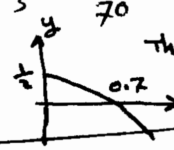
$$y(0) = 1 \implies c_1 + c_2 + c_3 + c_4 = 1$$

$$y'(0) = -1 \implies c_1 - 2c_2 + 3c_3 - 4c_4 = -1$$

$$y''(0) = 0 \implies c_1 + 4c_2 + 9c_3 + 16c_4 = 0$$

$$y'''(0) = -1 \implies c_1 - 8c_2 + 27c_3 - 64c_4 = -1$$

- Solve the system above to get $c_1 = \frac{1}{2}$, $c_2 = \frac{4}{5}$, $c_3 = -\frac{11}{70}$, $c_4 = \frac{-1}{7}$
- Thus, $y(t) = \frac{1}{2} e^{t} + \frac{4}{5} e^{-2t} - \frac{11}{70} e^{3t} - \frac{1}{7} e^{-4t}$



The effect of the largest root.

B Complex Roots

- If the characteristic equation has complex roots, then must occur in conjugate pairs $\lambda \pm i\mu$.
- Note that not all the roots need to be complex.
- Solutions corresponding to complex roots have the form

$$e^{(\lambda + i\mu)t} = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$

$$e^{(\lambda - i\mu)t} = e^{\lambda t} \cos \mu t - i e^{\lambda t} \sin \mu t$$

- As in Section 3.4, we use the real-valued solutions

$$e^{\lambda t} \cos \mu t, \quad e^{\lambda t} \sin \mu t$$

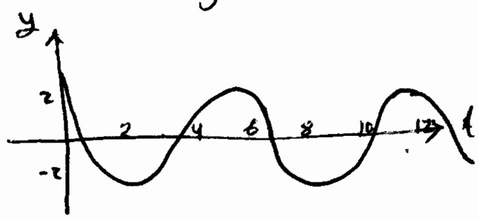
Example: Find the general solution for the IVP:

$$y^{(4)} - y = 0 \ , \quad y(0) = \frac{7}{2} \ , \ y'(0) = -4 \ , \ y''(0) = \frac{5}{2} \ , \ y'''(0) = -2$$

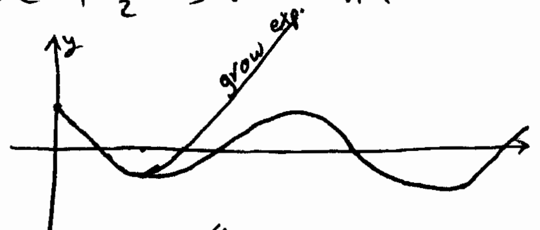
- Assuming an exponential solution e^{rt}, then the characteristic equation is $r^4 - 1 = 0 \Leftrightarrow (r^2-1)(r^2+1) = 0 \Leftrightarrow$ The roots are

$$r_1 = 1 \ , \ r_2 = -1 \ , \ r_3 = i \ , \ r_4 = -i \qquad \boxed{\lambda = 0} \ \boxed{\mu = 1}$$

- The general solution of is $y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$
- Using the initial conditions, we obtain $c_1 = 0 \ , \ c_2 = 3 \ , \ c_3 = \frac{1}{2} \ , \ c_4 = -1$
- Thus, the general solution is $y(t) = 3e^{-t} + \frac{1}{2}\cos t - \sin t$



$c_1 = 0$



If we let $y'''(0) = -\frac{15}{8}$, then $c_1 = \frac{1}{32}$

C Repeated Roots

- If a root r_K of the characteristic equation is a repeated root with multiplicity s, then the linearly independent solutions corresponding to this repeated root have the form:

$$e^{r_K t} \ , \ t\, e^{r_K t} \ , \ t^2 e^{r_K t} \ , \ \dots \ , \ t^{s-1} e^{r_K t}$$

- If a complex root $\lambda + i\mu$ is repeated s times, then so its conjugate $\lambda - i\mu$. There are $2s$ corresponding linearly independent solutions derived from real and imaginary parts of:

$$e^{(\lambda+i\mu)t} \ , \ t\, e^{(\lambda+i\mu)t} \ , \ t^2 e^{(\lambda+i\mu)t} \ \dots \ t^{s-1} e^{(\lambda+i\mu)t}$$

$$e^{\lambda t}\cos\mu t, \ e^{\lambda t}\sin\mu t, \ t e^{\lambda t}\cos\mu t, \ t e^{\lambda t}\sin\mu t, \ \dots, \ t^{s-1}\cos\mu t, \ t^{s-1}\sin\mu t$$

Example: Find the general solution of the ODE

$$y^{(iv)} + 2y'' + y = 0$$

- Assuming $y = e^{rt}$, then the characteristic equation is

$$r^4 + 2r^2 + 1 = 0 \iff (r^2+1)^2 = 0 \iff (r^2+1)(r^2+1) = 0$$

$\iff$ The roots are $r_1 = i$, $r_2 = i$, $r_3 = -i$, $r_4 = -i$. $\boxed{\begin{matrix}\lambda = 0\\ \mu = 1\end{matrix}}$

- Thus, the general solution is $y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$

Example: Find the general solution of the DE $y^{(4)} + y^{(2)} = 0$

- If $y = e^{rt}$, then the characteristic equation is $r^4 + r^2 = 0 \iff$

$r^2(r^2+1) = 0 \iff$ The roots are $r_1 = r_2 = 0$, $r_3 = i$, $r_4 = -i$

- Thus, the general solution is $y(t) = \underset{\searrow c_1 e^{0t}}{c_1} + \underset{\searrow c_2 e^{0t}(t)}{c_2 t} + c_3 \cos t + c_4 \sin t$

Example: Find the general solution of the DE: $y^{(iv)} - y^{(iii)} - y'' + y' = 0$

- If $y = e^{rt}$, then the characteristic equation is $r^4 - r^3 - r^2 + r = 0$

$$\iff r\left[r^3 - r^2 - r + 1\right] = 0 \iff r\left[r^2(r-1) - (r-1)\right] = 0 \iff$$

$$r(r-1)(r^2-1) = 0 \iff \text{The roots are } r_1 = 0,\ r_2 = r_3 = 1,\ r_4 = -1$$

- Thus, the general solution is $y(t) = c_1 + c_2 e^{t} + c_3 t e^{t} + c_4 e^{-t}$

Example: Write the complex number $-1 + \sqrt{3}\, i$ in the form

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$ cos is negative and sin is positive, الربع ٢, the polar angle is $120 = \frac{2\pi}{3}$

$$2\left(\cos\left(\frac{2\pi}{3} + 2\pi m\right) + i \sin\left(\frac{2\pi}{3} + 2\pi m\right)\right) = 2 e^{i\left(\frac{2\pi}{3} + 2\pi m\right)}$$

Example*: Find the general solution of $y^{(4)} + y = 0$

- If $y = e^{rt}$, then the characteristic equation is

$$r^4 + 1 = 0 .$$

- To solve this equation we must compute the four roots of -1.

- Think of -1 as a complex number $-1 + 0i$

sin is zero and cos is -1, Thus the polar angle is π

Euler's formula $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

$$-1 = \cos\pi + i\sin\pi = e^{i\pi}$$

- The angle is determined only up to multiple of $\underline{\underline{2\pi}}$ since we have sin and cos with period $= 2\pi$.

$$-1 = \cos(\pi + 2\pi m) + i\sin(\pi + 2\pi m) = e^{i(\pi + 2\pi m)}$$

where m is any integer.

- Thus, $(-1)^{\frac{1}{4}} = \cos\left(\frac{\pi}{4} + \frac{\pi m}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi m}{2}\right) = e^{i\left(\frac{\pi}{4} + \frac{\pi m}{2}\right)}$

- The roots are when $m = 0, 1, 2, 3$

$m = 0 \Rightarrow \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \overset{\lambda}{\frac{1}{\sqrt{2}}} + i\overset{\mu}{\frac{1}{\sqrt{2}}} = \frac{1+i}{\sqrt{2}}$

$m = 1 \Rightarrow \frac{-1+i}{\sqrt{2}} = \lambda\boxed{\frac{-1}{\sqrt{2}}} + \overset{\mu}{\frac{1}{\sqrt{2}}}i$

$m = 3 \Rightarrow \frac{-1-i}{\sqrt{2}} = \boxed{\frac{-1}{\sqrt{2}}} - \overset{\mu}{\frac{1}{\sqrt{2}}}i$

$m = 4 \Rightarrow \frac{1-i}{\sqrt{2}} = \overset{\lambda}{\frac{1}{\sqrt{2}}} - \overset{\mu}{\frac{1}{\sqrt{2}}}i$

The general solution is then given by

$$y(t) = e^{t/\sqrt{2}}\left(c_1\cos\frac{t}{\sqrt{2}} + c_2\sin\frac{t}{\sqrt{2}}\right) + e^{\frac{-t}{\sqrt{2}}}\left(c_3\cos\frac{t}{\sqrt{2}} + c_4\sin\frac{t}{\sqrt{2}}\right)$$

Example: Express the given complex number in the form

$$R(\cos\theta + i\sin\theta) = Re^{i\theta}$$

① $-1+\sqrt{3}\,i \qquad 2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\right)i$

Polar angle is $\theta = \frac{2\pi}{3}$ 120

since $\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

$$2\left[\cos\left(\frac{2\pi}{3}+2\pi m\right) + i\sin\left(\frac{2\pi}{3}+2\pi m\right)\right] = 2e^{i\left(\frac{2\pi}{3}+2\pi m\right)}$$

③ $-3 = -3+0i = 3(-1+0i)$

Polar angle is π since $\cos(\pi) = -1$ and $\sin(\pi) = 0$

$$3\left[\cos(\pi+2\pi m) + i\sin(\pi+2\pi m)\right] = 3e^{i(\pi+2\pi m)}$$

Example: Follow the procedure of Example* to determine the two roots of $(1-i)^{\frac{1}{2}}$

We must compute the two roots of $1-i$

The complex number has the polar angle $-\frac{\pi}{4} = -\frac{\pi}{4}+2\pi = \frac{7\pi}{4}$

$$1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

$$1-i = \sqrt{2}\left[\cos\left(\frac{7\pi}{4}+2\pi m\right) + i\sin\left(\frac{7\pi}{4}+2\pi m\right)\right] = \sqrt{2}\,e^{i\left(\frac{7\pi}{4}+2\pi m\right)}$$

$$(1-i)^{\frac{1}{2}} = (\sqrt{2})^{\frac{1}{2}}\left[\cos\left(\frac{7\pi}{8}+\pi m\right) + i\sin\left(\frac{7\pi}{8}+\pi m\right)\right] = 2^{\frac{1}{4}}e^{i\left(\frac{7\pi}{8}+\pi m\right)}$$

The two roots are when $m = 0, -1$

$$m=0 \Rightarrow r_1 = 2^{\frac{1}{4}}e^{i\frac{7\pi}{8}} = 2^{\frac{1}{4}}\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = 2^{\frac{1}{4}}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

$$m=-1 \Rightarrow r_2 = 2^{\frac{1}{4}}e^{\frac{-\pi}{8}i} = \ldots$$