4.3

The Method of Undetermined Coefficients

The method of undetermined coefficients can be used to find a particular solution, $y_p(t)$, of an n^{th} order linear nonhomogenuous ODE with constant coefficients:

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t)$$

If the function $g(t)$ is an exponential function or sine or cosine or polynomial or sum or product of these functions.

Example 1 Find the general solution of $y''' - 3y'' + 3y' - y = 4e^t$ $\ast$

The general solution of $\ast$ is $y(t) = y_h(t) + y_p(t)$.

- To find $y_h(t)$: If an exponential solution e^{rt} is assumed, then the characteristic equation of the homogenuous equation is $r^3 - 3r^2 + 3r - 1 = 0 \Leftrightarrow (r-1)^3 = 0$
$\Leftrightarrow r_1 = r_2 = r_3 = 1$. Thus, the homogenuous solution is
$y_h(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$ $\boxed{s=3}$ multiplicity of $r=1$

- To find $y_p(t)$: we start with $y_p(t) = At^3e^t$. To find $A \Rightarrow$
$\Rightarrow y_p' = At^3e^t + 3At^2e^t$
$\Rightarrow y_p'' = At^3e^t + 6At^2e^t + 6Ate^t$
$\Rightarrow y_p''' = At^3e^t + 9At^2e^t + 18Ate^t + 6Ae^t$

- Substitute y_p, y_p', y_p'', y_p''' in $\ast$ and arrange terms, we obtain $6Ae^t = 4e^t$. Hence, $A = \frac{2}{3}$. Thus, $y_p(t) = \frac{2}{3} t^3 e^t$

- The general solution is then given by $y(t) = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t$

Example $\boxed{2}$ Find the general solution of the ODE :

$$y^{(4)} + 8y'' + 16y = 2\sin t - 3\cos t \quad *'$$

The general solution is $y(t) = y_h(t) + y_p(t)$

- To find $y_h(t)$: Assuming an exponential solution e^{rt}, then the characteristic solution is $r^4 + 8r^2 + 16 = 0 \iff$

$$(r^2+4)(r^2+4) = 0 \iff r_1 = 2i,\ r_2 = -2i,\ r_3 = 2i,\ r_4 = -2i$$

$\boxed{\lambda = 0, \mu = 2}$

Thus, the homogenuous solution $y_h(t) = c_1\cos 2t + c_2 \sin 2t + (c_3\cos 2t + c_4 \sin 2t)t$

- To find the particular solution $y_p(t)$:

We start with $y_p(t) = A\sin t + B\cos t \Rightarrow$ To find A and B:

$\Rightarrow y_p' = A\cos t - B\sin t \quad \Rightarrow \quad y_p'' = -A\sin t - B\cos t$

$\Rightarrow y_p''' = -A\cos t + B\sin t \quad \Rightarrow \quad y_p'''' = A\sin t + B\cos t$

- Substitue y_p, y_p'', y_p'''' in $*' \Rightarrow 9A\sin t + 9B\cos t = 2\sin t - 3\cos t$

$\iff A = \frac{2}{9}$ and $B = \frac{-1}{3}$. Thus, $y_p(t) = \frac{2}{9}\sin t - \frac{1}{3}\cos t$

- Thus, the general solution of $*'$ is then given by :

$$y(t) = c_1\cos 2t + c_2\sin 2t + c_3 t\cos 2t + c_4 t\sin 2t + \frac{2}{9}\sin t - \frac{1}{3}\cos t.$$

Example $\boxed{3}$: Find the general solution of the ODE

$$y^{(4)} + 8y'' + 16y = 2\sin 2t - 3\cos 2t \quad *^2$$

- From Example $\boxed{2}$ the homogenous solution is

$$y_h(t) = c_1\cos 2t + c_2\sin 2t + c_3 t\cos 2t + c_4 t\sin 2t$$

- To find $y_p(t)$, we start with $y_p(t) = At^2\sin 2t + Bt^2\cos 2t$
- Substitue y_p, y_p'', y_p'''' in $*^2$: we get : $A = \frac{-1}{16}$ and $B = \frac{3}{32}$
- Thus, $y_p(t) = \frac{-1}{16}t^2\sin 2t + \frac{3}{32}t^2\cos 2t$. Thus, the general solution is $y(t) = y_h(t) + y_p(t)$

Example: Find a particular solution of $y''' - 4y' = t + 3\cos t + e^{-2t}$ $*^3$ (110)

- 1^{st} we solve the homogeneous equation $y''' - 4y' = 0$.
- Assuming exponential solution e^{rt}, then the characteristic equation is

$$r^3 - 4r = 0 \Leftrightarrow r(r^2-4) = 0 \Leftrightarrow r_1 = 0,\ r_2 = 2,\ r_3 = -2.$$

- Thus, the homogeneous solution is $y_h(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}$.
- To find a particular solution $y_p(t)$: we start by separation $*^3$

$y''' - 4y' = t$, (A)

$y_{p_1}(t) = (A_0 t + A_1)t$,

$y'_{p_1} = 2A_0 t + A_1$

$y''_{p_1} = 2A_0$, $y'''_{p_1} = 0$

- Substitute y'_{p_1}, y'''_{p_1} in (A)

$0 - 4(2A_0 t + A_1) = t$

$-8A_0 t - 4A_1 = t$

$\Leftrightarrow A_0 = \frac{-1}{8}$ and $A_1 = 0$

Thus, $\boxed{y_{p_1}(t) = -\frac{1}{8}t^2}$

$y''' - 4y' = 3\cos t$, (B)

$y_{p_2}(t) = B_1\cos t + B_2 \sin t$,

$y'_{p_2} = -B_1 \sin t + B_2 \cos t$

$y''_{p_2} = -B_1 \cos t - B_2 \sin t$

$y'''_{p_2} = B_1 \sin t - B_2 \cos t$

- Substitute y'_{p_2}, y'''_{p_2} in (B)

$5B_1 \sin t - 5B_2 \cos t = 3\cos t$

$\boxed{B_1 = 0}$ and $\boxed{B_2 = \frac{-3}{5}}$

Thus, $\boxed{y_{p_2}(t) = \frac{-3}{5}\sin t}$

$y''' - 4y' = e^{-2t}$ (D)

$y_{p_3}(t) = D t e^{-2t}$

$y'_{p_3} = -2Dte^{-2t} + De^{-2t}$

$y''_{p_3} = 4Dte^{-2t} - 4De^{-2t}$

$y'''_{p_3} = -8Dte^{-2t} + 12De^{-2t}$

- Substitute y'_{p_3}, y'''_{p_3} in (D)

$8De^{-2t} = e^{-2t}$

$\Leftrightarrow D = \frac{1}{8}$

Thus, $\boxed{y_{p_3}(t) = \frac{1}{8}te^{-2t}}$

Hence, a particular solution of equation $*^3$ is then given by

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t) + y_{p_3}(t)$$

$$y_p(t) = \frac{-1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}te^{-2t}$$