4.4 The method of Variation of Parameters For

- The method of variation of parameters can be used to find a particular solution for a nonhomogenuous n^{th} order linear DE:

$$L[y] = y^{(n)} + P_1(t)\, y^{(n-1)} + \cdots + P_{n-1}(t)\, y' + P_n(t)\, y = g(t) \quad \cdots \boxed{1}$$

P_i and g are continuous

- If we know that $y_1, y_2, \cdots, y_n$ form a fundamental set of solutions of the homogenuous equation crrosponding to $\boxed{1}$, then the general solution of the homogenuous equation is

$$y_h(t) = y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t).$$

- We assume that a particular solution, $y_p(t)$, of Eq. $\boxed{1}$ has the form

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t) + \cdots + u_n(t)\, y_n(t),$$

where $u_1, u_2, \cdots, u_n$ are functions need to be found.

- Thus, we need n equations to find n functions.
- As in section 3.7:

$$y_p'(t) = \left(u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n\right) + \left(u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'\right)$$

- The first condition we require is

$$u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n = 0$$

$$y_p''(t) = \left(u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n'\right) + \left(u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n''\right)$$

- The second condition we require is

$$u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$y_p^{(n-1)}(t) = \left(u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + \cdots + u_n' y_n^{(n-2)}\right) + \left(u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + \cdots + u_n y_n^{(n-1)}\right)$$

- The $(n-1)^{th}$ condition we require is

$$u_1' y_1^{(n-2)} + u_2' y_2^{(n-2)} + \cdots + u_n' y_n^{(n-2)} = 0$$

$$y_p^{(n)}(t) = \left(u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}\right) + \left(u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)}\right)$$

- The n^{th} condition we get by substituting $y_p, y_p', \cdots, y_p^{(n)}$ in Eq $\boxed{1}$ and noting that $L[y_i]=0$, $i=1,\cdots,n$ is

$$u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} = g(t)$$

- Thus, the n equations needed in order to find the n functions $u_1, u_2, \cdots, u_n$ are

$$\left.\begin{aligned} u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n &= 0 \\ u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n' &= 0 \\ &\vdots \\ u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)} &= g(t) \end{aligned}\right\} \quad \boxed{2}$$

- The system $\boxed{2}$ is a linear algebraic system for the unknown quantities $u_1', u_2', \cdots, u_n'$.
- Since $y_1, y_2, \cdots, y_n$ are linearly independent solutions of the homogeneous equation corresponding to $\boxed{1}$, it follows that $W(y_1, y_2, \cdots, y_n)$ is nowhere zero. This is a sufficient condition for the existence of a solution for the system $\boxed{2}$.
- Using Cramer's Rule and for each $m = 1, 2, \cdots, n$

$$u_m'(t) = \frac{g(t) W_m(t)}{W(t)}$$ where $W(t) = W(y_1, y_2, \cdots, y_n)(t)$ and $W_m(t)$ is the determinant obtained by replacing the m^{th} column of $W(t)$ by $(0, 0, \cdots, 1)$.

- Integrat to obtain $u_1, u_2, \cdots, u_n$: $$u_m(t) = \int_{t_0}^{t} \frac{g(s) W_m(s)}{W(s)} ds, \quad m=1,2,\dots,n$$

where t_0 is arbitrary

- Thus, a particular solution of the ODE $\boxed{1}$ is then given by

$$y_p(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{g(s)\, W_m(s)}{W(s)}\, ds \quad \text{where } t_0 \text{ is arbitrary}$$

__Example__: Given that $y_1(t) = e^t$, $y_2(t) = te^t$, $y_3(t) = e^{-t}$ are solutions of the homogenuous equation corresponding to $y''' - y'' - y' + y = e^{2t}$ *. Determine a particular solution of * in terms of an integral.

The particular solution $y_p(t)$ is given by

$$y_p(t) = \sum_{m=1}^{3} y_m(t) \int_{t_0}^{t} \frac{e^{2s}\, W_m(s)}{W(s)}\, ds \quad , \text{ where}$$

- $W(s) = W(y_1, y_2, y_3)(s) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^s & se^s & e^{-s} \\ e^s & (1+s)e^s & -e^{-s} \\ e^s & (2+s)e^s & e^{-s} \end{vmatrix} = 4e^s$

- $W_1(s) = \begin{vmatrix} 0 & se^s & e^{-s} \\ 0 & (1+s)e^s & -e^{-s} \\ 1 & (2+s)e^s & e^{-s} \end{vmatrix} = -2s - 1$

- $W_2(s) = \begin{vmatrix} e^s & 0 & e^{-s} \\ e^s & 0 & -e^{-s} \\ e^s & 1 & e^{-s} \end{vmatrix} = -2$ and $W_3(s) = \begin{vmatrix} e^s & se^s & 0 \\ e^s & (1+s)e^s & 0 \\ e^s & (2+s)e^s & 1 \end{vmatrix} = e^{2s}$

- Thus, $y_p(t) = e^t \int_{t_0}^{t} \frac{e^{2s}(-2s-1)}{4e^s}\, ds + te^t \int_{t_0}^{t} \frac{-2e^{2s}}{4e^s}\, ds + e^{-t} \int_{t_0}^{t} \frac{e^{2s} e^{2s}}{4e^s}\, ds$

$$y_p(t) = \frac{-e^t}{4} \int_{t_0}^{t} e^s(2s+1)\, ds - \frac{te^t}{2} \int_{t_0}^{t} e^s\, ds + \frac{e^{-t}}{4} \int_{t_0}^{t} e^{3s}\, ds$$