Chapter 5

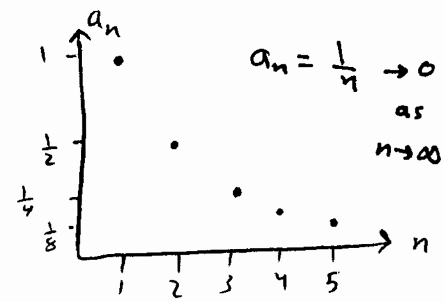
Series Solutions of 2^{nd} Order linear Equations

$\boxed{5.1}$ Review of Power Series

- Recall that to find the general solution of a linear DE, we need to find the fundamental set of solutions of the corresponding homogenuous equation.
- In this chapter, we study the use of power series to construct fundamental sets of solutions of 2^{nd} order linear DE whose coefficients are functions of the independent variable.

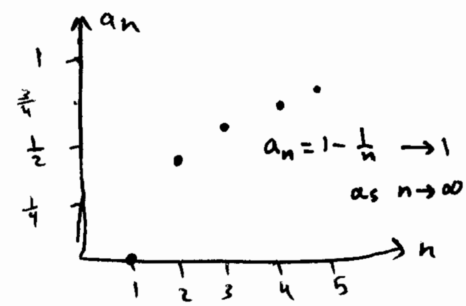
- Sequence : function from $\mathbb{N} \to \mathbb{R}$
  - $a_n = \dfrac{1}{n}$, $n = 1, 2, 3, \dots$

    $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_4 = \frac{1}{4}$, ...



  - $a_n = 1 - \frac{1}{n}$, $n = 1, 2, 3, \dots$

    $a_1 = 0$, $a_2 = \frac{1}{2}$, $a_3 = \frac{2}{3}$, $a_4 = \frac{3}{4}$, ...



  - The sequence $a_n \to L$ iff

    $\forall \epsilon > 0, \exists N > 0$ s.t $|a_n - L| < \epsilon \quad \forall n \geq N$.

    " $\lim_{n \to \infty} a_n = L$ "

- Series: is the sum of a sequence.
  - If the sequence a_n is infinite, then the corresponding series is $\sum_{n=0}^{\infty} a_n$
  - If the sequence a_n is finite (with k terms), then the corresponding series is $\sum_{n=0}^{k} a_n$

- **Power series:** • a power series about the point x_0 has the form $\boxed{\sum_{n=0}^{\infty} a_n (x-x_0)^n}$ *

  • a power series * **converges** at point x if $\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x-x_0)^n$ exists for that x.

  • Note that the series converges for $x = x_0$.

  • a power series * **converges absolutely** at point x if the series $\sum_{n=0}^{\infty} |a_n (x-x_0)^n| = \sum_{n=0}^{\infty} |a_n| \, |x-x_0|^n$ converges.

  • Note that if the series converges absolutely, then the series also converges. The convers is not true (see Example $\boxed{2}$)

- Ratio Test: $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, $a_n \neq 0$

  • If $L < 1$, then the series converges absolutely
  • If $L > 1$, then the series diverges.
  • If $L = 1$, then the test is inconclusive.

- If $a_n \neq 0$, and if for a fixed value of x

$$\lim_{n\to\infty} \left| \frac{a_{n+1} (x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = |x-x_0| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L\,|x-x_0|,$$

then the power series *

  • converges absolutely at that value of x if $|x-x_0| < \frac{1}{L}$
  • diverges if $|x-x_0| > \frac{1}{L}$.

If $|x-x_0| = \frac{1}{L}$, then the test is inconclusive.

Example 1 For which values of x does the power series $\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n$ converges?

To test the convergence, we use the ratio test:

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+2} (n+1) (x-2)^{n+1}}{(-1)^{n+1} n (x-2)^n} \right| = |x-2| \underbrace{\left(\lim_{n\to\infty} \frac{n+1}{n} \right)}_{L=1} = |x-2| < \frac{1}{L}$$

$\Rightarrow |x-2| < 1 \iff 1 < x < 3$. Thus, the series converges absolutely for $1<x<3$, and diverges for $|x-2|>1$.

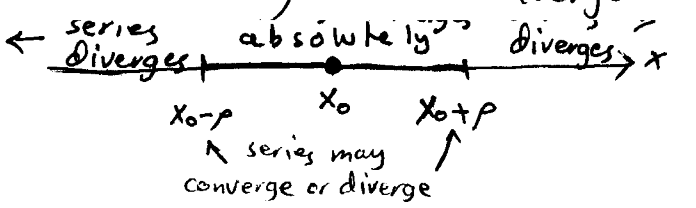
- A $x=1$ and at $x=3$, the series diverges since

at $x=1 \Rightarrow a_n = (-1)^{2n+1} n$ and the series becomes $\sum_{n=1}^{\infty} (-1)^{2n+1} n$

at $x=3 \Rightarrow a_n = (-1)^{n+1} n$ and the series becomes $\sum_{n=1}^{\infty} (-1)^{n+1} n$

diverge since the nth term does not go to 0 as $n\to\infty$

- Radius of Convergence:
  - There is a nonnegative number ρ, called the radius of convergence, s.t the power series $* \sum_{n=0}^{\infty} a_n (x-x_0)^n$
  - converges absolutely for $|x-x_0| < \rho$ and
  - diverges for $|x-x_0| > \rho$.
  - If the series converges only at x_0, then $\rho = 0$
  - If the series converges for all x, then $\rho = \infty$
- If $\rho > 0$, then the interval $|x-x_0| < \rho$ is called the interval of convergence. The series may either converge or



Example 2 Determine the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x+1)^n}{n\,2^n}$

Apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n\,2^n}{(x+1)^n} \right| = \frac{|x+1|}{2} \overbrace{\left(\lim_{n\to\infty} \frac{n}{n+1}\right)}^{1=1} = \frac{|x+1|}{2}.$$

Thus, the series converges absolutely for $|x+1|<2 \iff -3<x<1$ and diverges for $|x+1|>2$. The radius of convergence of the power series is $\rho=2$.

- To check the end points of the interval of the convergence:
  - At $x=1 \Rightarrow$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic series) which diverges.
  - At $x=-3 \Rightarrow$ the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ($\frac{1}{n} \to 0$ as $n\to\infty$) which **converges but does not converge absolutely**. It converges conditionally at $x=-3$. Thus, the power series:
    - converges for $-3 \le x < 1$ and diverges otherwise;
    - converges absolutely for $-3<x<1$.
    - has radius of convergence 2.

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* Suppose that $\displaystyle\sum_{n=0}^{\infty} a_n (x-x_0)^n$ convergs to $f(x)$ for $|<\rho,\ \rho>0$

and $\displaystyle\sum_{n=0}^{\infty} b_n (x-x_0)^n$ converges to $g(x)$ for $|x-x_0|<\rho,\ \rho>0$

Then: ① $\displaystyle f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-x_0)^n$ converges at least for $|x-x_0|<\rho$.

② $\displaystyle f(x)\,g(x) = \left[\sum_{n=0}^{\infty} a_n (x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n (x-x_0)^n\right] = \sum_{n=0}^{\infty} c_n (x-x_0)^n,$

converges at least for $|x-x_0|<\rho$, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

③ If $g(x_0) \neq 0$, then $\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n (x-x_0)^n$ and the radius of convergence $\underset{\text{may be}}{<} \rho$. To find $d_n \Rightarrow$

$$f(x) = g(x) \sum_{n=0}^{\infty} d_n (x-x_0)^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} d_k b_{n-k} \right) (x-x_0)^n$$

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n =$$

④ f is continuous and has derivatives of all orders for $|x-x_0|<\rho$

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$$

$\vdots$

converges absolutely for $|x-x_0|<\rho$.

⑤ The value of a_n is given by $a_n = \frac{f^{(n)}(x_0)}{n!}$ "Talyor series" about $x=x_0$

⑥ • If $\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{\infty} b_n (x-x_0)^n$ $\forall x$ in some interval with center x_0, then $a_n = b_n$ $\forall n = 0,1,2,\dots$

• If $\sum_{n=0}^{\infty} a_n (x-x_0)^n = 0$ $\forall x$, then $a_0 = a_1 = \cdots = a_n = 0$

* A function f that has a Taylor series expansion about $x=x_0$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n,$$ with radius of convergence

$\rho > 0$ is called analytic function at $x = x_0$.

Example: • $\sin x$, e^x are analytic everywhere

• $\frac{1}{x}$ is analytic except at $x=0$

• $\tan x$ " " " odd multiple of $\frac{\pi}{2}$

* If f and g are analytic at x_0, then so are $f \pm g$, fg, $\frac{f}{g}$.

Shifting Index of Summation

It is not important which letter is used for the index of summation $\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{k=0}^{\infty} a_k (x-x_0)^k$

Example: $\sum_{n=1}^{\infty} a_n (x+1)^{n-1} = \sum_{m=0}^{\infty} a_{m+1} (x+1)^m$ by letting $m = n-1$

$n=1 \Rightarrow m=0$

$= \sum_{n=0}^{\infty} a_{n+1} (x+1)^n$

Example: write the series $\sum_{n=0}^{\infty} (n+1) a_n x^{n+3}$ as sum involves x^n

let $m = n+3$. when $n=0 \Rightarrow m=3$

when $n=n+1 \Rightarrow m = n+3 \Rightarrow n = m-3$

$\Rightarrow n+1 = m-2$

$$\sum_{n=0}^{\infty} (n+1)\, a_n\, x^{n+3} = \sum_{m=3}^{\infty} (m-2)\, a_{m-3}\, x^m$$

$$= \sum_{n=3}^{\infty} (n-2)\, a_{n-3}\, x^n$$

harmonic series is a divergent series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \dots + \frac{1}{16}\right)}_{> \frac{8}{16} = \frac{1}{2}} + \dots$$

$\Rightarrow$ there is no upper bound

المتتالية الحسابية ح_ن = أ + (ن − ١) د

sequence : $a_n = a_0 + (n-1) r$

Example: 3, 7, 11, ... Find a_7

$a_0 = 3 \quad r = 4$

$a_7 = 3 + (7-1)4 = 27$

مجموع متسلسلة حسابية   Sum of Gometric Series

• ج_ن = ن/٢ (أ + ل)

عدد الحدود ، الحد الأول ، الحد الأخير

• ج_ن = ن/٢ [٢أ + (ن − ١) د]

المتتالية الهندسية $a_n = a_0 r^{n-1}$

Geometric sequence   ح_ن = أ ر^(ن−١)

Example: 5, 15, 45, 135, ... Find a_6

$r = \frac{15}{5} = 3$

$a_6 = 5(3)^{6-1} = 1215$

مجموع متسلسلة هندسية

ج_ن = أ(١ − رⁿ) / (١ − ر)   ر ≠ ١

ج_ن = ن أ   ر = ١

مثال: أوجد مجموع متسلسلة هندسية حدها الأول ٤٨ وأساسها ½ وعدد حدودها ٥

أ = ٤٨   ر = ½   ∴ ج_٥ = ٤٨(١ − (½)^٥) / (١ − ½) = ٩٣

مجموع أول ن حد من متسلسلة هندسية حدها الأول أ وأساسها ر وحدها الأخير (العام) ل هو

ج_ن = (أ − ل ر) / (١ − ر)   ر ≠ ١

مثال: جد مجموع المتتالية ٣، −٦، ١٢، ...، −٩٦

ر = −٦/٣ = −٢   ل = −٩٦   أ = ٣

ج = (٣ − (−٩٦ × −٢)) / (١ − −٢) = −٦٣

مجموع متسلسلة هندسية لا نهائية حدها الأول أ وأساسها ر هو

ج = أ / (١ − ر) |ر| < ١

مثال: أوجد مجموع ٢٥ + ٢٠ + ١٦ + ١٢٫٨ + ...

ر = ٠٫٨   أ = ٢٥   ∴ ج = ٢٥ / (١ − ٠٫٨) = ١٢٥