5.2 Series Solutions Near an Ordinary Point I

* In ch3, we examined methods of solving 2^{nd} order linear differential equations with constant coefficients.

* Now we consider the case where the coefficients are functions of the independent variable x

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \quad \text{---} \ (1)$$

Note that the case is similar for the nonhomogeneous case.

* We will consider the case where P, Q, R are polynomials (thus continuous) and have no common factors

* The point x_0 is called an ordinary point if $P(x_0) \neq 0$.

* Since $P(x)$ is continuous, $P(x) \neq 0$ for all x in some interval about x_0, for x in this interval we can write (1) as

$$y'' + p(x)y' + q(x)y = 0 \qquad p(x) = \frac{Q(x)}{P(x)}$$

$$q(x) = \frac{R(x)}{P(x)}$$

* Since p and q are continuous, by Th 3.2.1, there is a unique solution satisfies the initial conditions $y(x_0) = y_0$, $y'(x_0) = y'_0$.

* In this section and the following section, we discuss the solution of eq. (1) in the neighborhood of an ordinary point

* Note that if $P(x_0) = 0$, then x_0 is called singular point of eq. (1). Since P, Q, R have no common factors $\Rightarrow Q(x_0) \neq 0$ or $R(x_0) \neq 0$ or both non zero $\Rightarrow p$ and q become unbounded as $x \to x_0$ $\Rightarrow$ Th 3.2.1 does not apply. Section 5.4–5.7 deal with finding solutions

of eq. 1 in the neighborhood of singular point.

* To solve eq. 1, near an ordinary point we assume a series representation of the unknown solution y:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + \cdots a_n (x-x_0)^n + \cdots$$

as long as we are within the interval of convergence $|x - x_0| < \rho$ for some $\rho > 0$.

- We substitute y, y', y'' in eq 1.

Example: Find a series solution of the equation

$$y'' + y = 0, \quad -\infty < x < \infty. \quad \cdots (2)$$

- Note that the fundamental set of solutions (2) are $\sin x$ and $\cos x$.
- $P(x) = 1 = R(x)$, $Q(x) = 0$. Thus $P(x) \neq 0 \ \forall x \Rightarrow$ any point is an ordinary point.
- Take $x_0 = 0$ (the simplest choice).
- Assume a series solution as a power solution about $x_0 = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

- Substitutes y, y', y'' in (2) $\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

- Shifting Indices: $\displaystyle\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$

$$\Rightarrow \sum_{n=0}^{\infty}\left((n+2)(n+1)a_{n+2} + a_n\right)x^n = 0$$

For this equation to be satisfied $\forall x$, the coefficients of each power of x must be zero:

$$(n+2)(n+1)a_{n+2} + a_n = 0 \quad , \quad n = 0, 1, 2, \cdots \qquad \cdots (3)$$

• Relation (3) is called recurrence relation (علاقة)

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)} \quad , \quad n = 0, 1, 2, \cdots$$

<u>Even Coefficients:</u> To find $a_2, a_4, a_6, \cdots$

$$a_2 = \frac{-a_0}{(2)(1)}$$

$$a_4 = \frac{-a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)(1)}$$

$$a_6 = \frac{-a_4}{(6)(5)} = \frac{-a_0}{(6)(5)(4)(3)(2)(1)}$$

$$\vdots$$

$$a_{2K} = \frac{(-1)^K a_0}{(2K)!} \quad , \quad K = 1, 2, 3, \cdots$$

<u>Odd Coefficients:</u> To find $a_3, a_5, a_7, \cdots$

$$a_3 = \frac{-a_1}{(3)(2)} \quad , \quad a_5 = \frac{-a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)(1)}$$

$$a_7 = \frac{-a_5}{(7)(6)} = \frac{-a_1}{(7)(6)(5)(4)(3)(2)(1)}$$

$$\vdots$$

$$a_{2K+1} = \frac{(-1)^K a_1}{(2K+1)!} \quad , \quad K = 1, 2, 3, \cdots$$

Now our solution is

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \text{with} \quad a_{2k} = \frac{(-1)^k}{(2k)!} a_0 \,, \quad a_{2k+1} = \frac{(-1)^k}{(2k+1)!} a_1$$

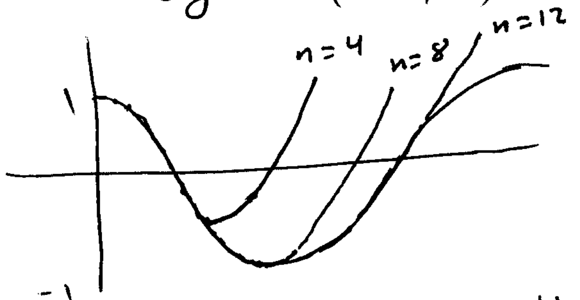
Thus,

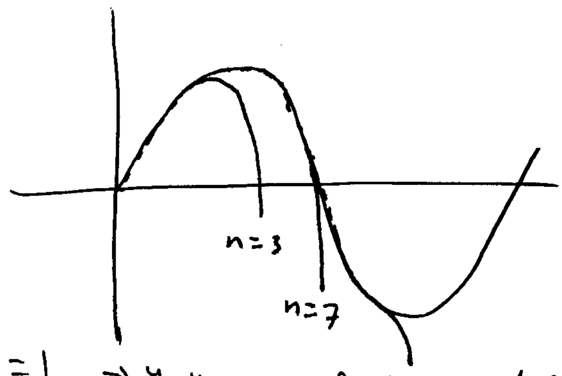
$$y(x) = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}_{y_1(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}_{y_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}$$

Note that a_0 and a_1 are determined by the initial conditions. $y(0)$, $y'(0)$

$$y(x) = a_0 \cos x + a_1 \sin x$$

- We can use Ratio test to see the two series converge absolutely on $(-\infty, \infty)$.





we can see $W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \Rightarrow y_1, y_2$ are fundamental set of solution

Example 2. Find a series solution in powers of x of Airy's equation

$$y'' - xy = 0, \qquad -\infty < x < \infty. \quad \cdots (3)$$

$P(x) = 1$, $Q(x) = 0$, $R(x) = -1$. Hence every point is an ordinary point. We take $x_0 = 0$ to get powers of x.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \,, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \,, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute y, y', y'' in (3) $\Rightarrow$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \quad \Rightarrow$$

shifting Indices: $\displaystyle\sum_{n=0}^{\infty}(n+2)(n+1)\,a_{n+2}\,x^{n} - \sum_{n=1}^{\infty} a_{n-1}\,x^{n} = 0$

$$(2)(1)a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)\,a_{n+2}\,x^{n} - \sum_{n=1}^{\infty} a_{n-1}\,x^{n} = 0$$

$$(2)(1)\,a_2 + \sum_{n=1}^{\infty}\Big((n+2)(n+1)\,a_{n+2} - a_{n-1}\Big)\,x^{n} = 0$$

$\boxed{a_2 = 0}$ and the recurrence relation is $a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}$, $n = 1, 2, 3, \ldots$

- Since $a_2 = 0$, we have $a_5 = a_8 = a_{11} = \cdots = 0$
- We need to find $a_0, a_3, a_6, \ldots$ by finding formula a_{3n}, $n = 1, 2, 3, \ldots$

  $a_1, a_4, a_7, \ldots$ by finding formula a_{3n+1}, $n = 1, 2, 3, \ldots$

To find the formula a_{3n}, $n = 1, 2, 3, \ldots$

$$a_3 = \frac{a_0}{(2)(3)}, \quad a_6 = \frac{a_3}{(5)(6)} = \frac{a_0}{(2)(3)(5)(6)}$$

$$a_9 = \frac{a_6}{(8)(9)} = \frac{a_0}{(2)(3)(5)(6)(8)(9)}$$

$$\vdots$$

$$a_{3n} = \frac{a_0}{(2)(3)(5)(6)\cdots(3n-4)(3n-3)(3n-1)(3n)}, \quad n = 1, 2, 3, \ldots$$

To find formula for a_{3n+1} , $n=1,2,3,\ldots$

$$a_4 = \frac{a_1}{(3)(4)} \quad , \quad a_7 = \frac{a_4}{(6)(7)} = \frac{a_1}{(3)(4)(6)(7)}$$

$$a_{10} = \frac{a_7}{(9)(10)} = \frac{a_1}{(3)(4)(6)(7)(9)(10)}$$

$$\vdots$$

$$a_{3n+1} = \frac{a_1}{(3)(4)(6)(7)\cdots(3n-3)(3n-2)(3n)(3n+1)} \quad , n=1,2,3,\ldots$$

• $$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \underbrace{a_2 x^2}_{zero} + \sum_{n=3}^{\infty} a_n x^n$$

$$y(x) = a_0\left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{(2)(3)\cdots(3n-1)(3n)}\right] + a_1\left[x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(3)(4)\cdots(3n)(3n+1)}\right]$$

$$y(x) = a_0\, y_1(x) + a_1 y_2(x)$$

• If $\begin{cases} y(0)=1,\ y'(0)=0 \iff a_0=1,\ a_1=0 \\ y(0)=0,\ y'(0)=1 \iff a_0=0,\ a_1=1 \end{cases}$ $\Rightarrow$ y_1 and y_2 are linearly independent since $W(y_1,y_2)=1\neq 0$

~~and y_1 and y_2 form fundamental set of solution for Airy's equation.~~

*We can find a series solution for Airy's equation in powers of $x-1$

The point $x=1$ is an ordinary point

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n \ , \quad y' = \sum_{\substack{n=1\\ n=0}}^{\infty} n\, a_n (x-1)^{n-1} \ , \quad y'' = \sum_{\substack{n=2\\ n=0}}^{\infty} n(n-1)\, a_n (x-1)^{n-2}$$

Substitutes y, y', y'' $\Rightarrow$

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} (x-1)^n = x \sum_{n=0}^{\infty} a_n (x-1)^n$$

We write x as $x = 1 + (x-1)$

$$\sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} (x-1)^n = \left(1 + (x-1)\right) \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^{n+1}$$

$$= \sum_{n=0}^{\infty} a_n (x-1)^n + \sum_{n=1}^{\infty} a_{n-1} (x-1)^n$$

$$2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1)\, a_{n+2} (x-1)^n = a_0 + \sum_{n=1}^{\infty} a_n (x-1)^n + \sum_{n=1}^{\infty} a_{n-1} (x-1)^n$$

Thus, the recurrence relation is

$$a_{n+2} = \frac{a_n + a_{n-1}}{(n+2)(n+1)}, \quad n = 1, 2, 3, \dots, \quad \boxed{a_2 = \frac{a_0}{2}}$$

$$n=1 \Rightarrow a_3 = \frac{a_1 + a_0}{(3)(2)} = \frac{a_1}{6} + \frac{a_0}{6}$$

$$n=2 \Rightarrow a_4 = \frac{a_2 + a_1}{(4)(3)} = \frac{a_2}{12} + \frac{a_1}{12} = \frac{a_0}{24} + \frac{a_1}{12}$$

$\vdots$

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$$

it's difficult to find a general formula for a_n. Thus, we can not use ratio test.

$$= a_0 \left[1 + \frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \dots\right] +$$

$$a_1 \left[(x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \dots\right] = a_0 y_1(x) + a_1 y_2(x)$$

$y_1(1) = 1$, $y_2(1) = 0$, $y_1'(1) = 0$, $y_2'(1) = 1$ $\Rightarrow$ $W(y_1, y_2)(1) = 1 \neq 0 \Rightarrow y_1$ and y_2 form a fundamental set of solutions.