$\boxed{5.3}$ Series Solutions Near an Ordinary Point II.

- We have considered the problem of finding solutions of

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad \cdots \text{①}$$

where P, Q, R are polynomials, in the neighborhood of an ordinary point x_0.

- If eq. ① has a solution $y = \phi(x)$ that has a Talyor series

$$\text{②} \cdots \quad y = \phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$ which converges $|x - x_0| < \rho$, $\rho > 0$

then by differentiating ② m times and setting $x = x_0$, we get

$$m!\, a_m = \phi^{(m)}(x_0)$$

since $$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

$$\phi'' = y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

$$\phi''' = y''' = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n (x - x_0)^{n-3}$$

$$\vdots$$

$$\phi^{(m)} = y^{(m)} = \sum_{n=m}^{\infty} n(n-1)(n-2)\cdots(n-m+1) a_n (x - x_0)^{n-m}$$

$$= \underbrace{m(m-1)(m-2)\cdots(m-m+1)}\, a_m \underbrace{(x - x_0)^{m-m}}_{1} + \sum_{n=m+1}^{\infty} n(n-1)(n-2)\cdots(n-m+1) a_n (x - x_0)^{\boxed{n-m}}$$

$$\phi^{(m)}(x_0) = y^{(m)}(x_0) = m!\, a_m\,(1) + 0$$

$$y^{(m)}(x_0) = m!\, a_m$$

- To determine $\phi^{(n)}(x_0)$ and a_n, $n = 2, 3, \cdots$

(Note that $a_0 = y(x_0) = y_0$ and $a_1 = y'(x_0) = y_0'$)

we have ϕ is a solution to eq ① $\Rightarrow$

$$P(x)\,\phi''(x) + Q(x)\,\phi'(x) + R(x)\,\phi(x) = 0$$

$$\Rightarrow \boxed{\phi''(x) = -p(x)\,\phi'(x) - q(x)\,\phi(x)}, \quad p(x) = \frac{Q(x)}{P(x)}, \quad q(x) = \frac{R(x)}{P(x)}$$

$$\phi''(x_0) = -p(x_0)\,\phi'(x_0) - q(x_0)\,\phi(x_0)$$

Hence, $\boxed{2!\, a_2 = \phi''(x_0) = -p(x_0)\,a_1 - q(x_0)\,a_0}$

$\Rightarrow$ To find a_3 : $\phi''' = -p\phi'' - p'\phi' - q\phi' - q'\phi$

$$3!\, a_3 = \phi'''(x_0) = -\left[p(x_0)\,\underline{\underline{\phi''(x_0)}} + \left(p'(x_0) + q(x_0)\right)\phi'(x_0) + q'(x_0)\,\phi(x_0)\right]$$

$$= -\left[p(x_0)\left(2!\, a_2\right) + \left(p'(x_0) + q(x_0)\right)a_1 + q'(x_0)\,a_0\right]$$

$$= -\left[p(x_0)\left(-p(x_0)\,a_1 - q(x_0)\,a_0\right) + \left(p'(x_0) + q(x_0)\right)a_1 + q'(x_0)\,a_0\right]$$

so we write all in terms of a_0 and a_1

Since P, Q and R are polynomials (continuous) and $P(x_0) \neq 0$, then all derivatives of p and q exists at x_0. Hence, we can continue to find $a_4, a_5, \cdots$ in terms of a_0 and a_1

* A functions p, q ~~is~~ are analytic at x_0 ($P(x_0) \neq 0$) if it has a Taylor series expansion that converges to p in som interval about x_0 :

$$p(x) = p_0 + p_1(x - x_0) + \cdots + p_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} p_n (x - x_0)^n$$

$$q(x) = q_0 + q_1(x - x_0) + \cdots + q_n(x - x_0)^n + \cdots = \sum_{n=0}^{\infty} q_n (x - x_0)^n$$

Example: Determine $\phi''(x_0)$, $\phi'''(x_0)$, $\phi^{(4)}(x_0)$ for $x_0 = 0$ if $y = \phi(x)$ is a solution of

$$y'' + xy' + y = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 0$$

$a_0 = \phi(0) = y(0) = 1$

$a_1 = \phi'(0) = y'(0) = 0$

$y'' = -xy' - y$ since $y = \phi$ is a solution $\Rightarrow$

$$\boxed{\phi''(x) = -x\phi'(x) - \phi(x)} \quad (1)$$

Thus, $\phi''(x_0) = \phi''(0) = -(0)\phi'(0) - \phi(0) = -\phi(0) = -1$ ✓

To find $\phi'''(x_0)$, we differentiate (1)

$$\phi'''(x) = -x\phi''(x) - \phi'(x) - \phi'(x)$$
$$= -x\phi''(x) - 2\phi'(x) \quad --- (2)$$

Thus, $\phi'''(x_0) = \phi'''(0) = -(0)\phi''(0) - \phi'(0) - \phi'(0) = 0$ ✓

To find $\phi''''(0)$, we differentiate (2)

$$\phi^{(4)}(x) = -x\phi'''(x) - \phi''(x) - 2\phi''(x)$$

$$\phi^{4}(0) = -(0)\phi'''(0) - \phi''(0) - 2\phi''(0) = 1 - 2(-1)$$
$$= 3$$ ✓

Th 5.3.1: If x_0 is an ordinary point of the DE

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad \cdots (1)$$

that is $p = \frac{Q}{P}$ and $q = \frac{R}{P}$ are analytic at x_0, then the general solution of eq. (1) is

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0\, y_1(x) + a_1\, y_2(x), \quad \text{where}$$

a_0 and a_1 are arbitrary, y_1 and y_2 are two power series solutions that are analytic at x_0. The solutions y_1 and y_2 form a fundamental set of solutions. Furthermore, the radius of convergence for each of the series solutions y_1 and y_2 is at least as large as the minimum of the radii of the convergence of the series p and q. P,Q,R are بغض النظر بشكل عام سواء Polynomials or not

Example: What is the radius of convergence of the Talyor series for $(1 + x^2)^{-1}$ about $x_0 = 0$?

S1: The Talyor series of $f(x) = (1 + x^2)^{-1}$ about $x_0 = 0$ is

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

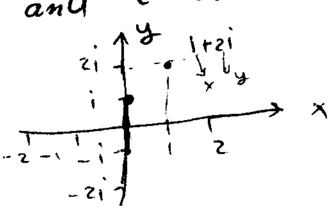
Using ratio test: $\lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+2}}{(-1)^n x^{2n}} \right| = \lim_{n\to\infty} x^2 = x^2 < 1$

$\Leftrightarrow |x| < 1 \Rightarrow$ Thus, $\rho = 1$

S2: the zeros of $1 + x^2$ are $\pm i$.

The distance between $x_0 = 0$ and i or $-i$ is 1

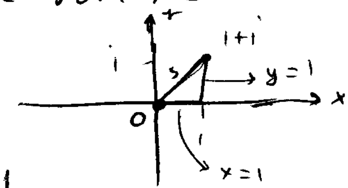
Thus, $\rho = 1$.



- when P, Q, R are polynomials in the theory of functions of a complex variables It's shown when $P(x_0) \neq 0$ that p has a convergent power series about x_0.
- Further, if Q and P have no common factors, the the radius of convergent for the power series p and q about x_0 is the distance between x_0 and the nearest zero of P.

Example: What is the radius of convergence of the Talyor series for $(x^2-2x+2)^{-1}$ about $x=0$? about $x=1$?

Note that $x^2-2x+2=0$ has solutions $x=1\pm i$.
The distance in the complex plane from $x=0$ to either $x=1+i$ or $x=1-i$ is $\sqrt{2}$



$s^2=x^2+y^2$
$=1+1$
$=2$
$s=\sqrt{2}$

Hence the radius of convergence of the Talyor series $\sum_{n=0}^{\infty} a_n x^n$ about $x=0$ is $\sqrt{2}$

The distance in the complex plane from $x=1$ to either $x=1+i$ or $x=1-i$ is 1. Hence the radius of convergence of the Taloyr series expansion $\sum_{n=0}^{\infty} b_n(x-1)^n$ about $x=1$ is 1.

Example: Determine a lower bound for the radius of convergence of the series solutions ① about $x_0=1$ for $y''-xy=0$.

$P(x)=1$, $Q(x)=0$, $R(x)=-x$ (Polynomials) ✓ (Airy's equation)

Thus, every point x is an ordinary point because $p(x)=0$ and $q(x)=-x$ are both analytic everywhere

Thus the radius of convergence for p and q is infinite.

Therefore, by Theorem 5.3.1, the radius of convergence of the series solution about $x_0=1$ is infinite.

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② about $x_0=-1$ for $(x^2+3x)y''+y'+y=0$

- $P(x)=x(x+3)$, $Q(x)=1$, $R(x)=1$ (Polynomials) ✓. Thus, $x_0=-1$ is an ordinary point since $p(x)=\frac{1}{x(x+3)}$ and $q(x)=\frac{1}{x(x+3)}$ are analytic at $x_0=-1$.
- p and q have singular points at $x=0$ and $x=-3$

Thus ρ for p and q about $x_0=-1$ is 1. Thus, by Th. 5.3.1 ρ for the series solution $\sum a_n(x+1)^n$ is at least $\rho=1$

Example --- ③ about $\boxed{x_0 = 0}$ for the legendre equation $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

$P, Q, R \to$ (Polynomial) α constant

- $P(x) = 1 - x^2$, $Q(x) = -2x$, $R(x) = \alpha(\alpha+1)$
- Thus, $x_0 = 0$ is an ordinary point since $p(x) = \dfrac{-2x}{1-x^2}$ and $q(x) = \dfrac{\alpha(\alpha+1)}{1-x^2}$ are analytic at $x_0 = 0$.
- P and q have singular points at $x = \pm 1$.
- Thus, the radius of convergence for the Taylor series expansion of p and q about $x_0 = 0$ is $\rho = 1$
- Therefore, by Th. 5.3.1 the radius of convergence for the series solution about $x_0 = 0$ is at least $\rho = 1$

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④ about ① $x_0 = 0$ for the equation

$$(1+x^2)y'' + 2xy' + 4x^2 y = 0$$

- $P(x) = 1 + x^2$, $Q(x) = 2x$, $R(x) = 4x^2$ (Polynomials)
- Thus, $x_0 = 0$ is an ordinary point since $p(x) = \dfrac{2x}{1+x^2}$ and $q(x) = \dfrac{4x^2}{1+x^2}$ are analytic at $x_0 = 0$.
- P and q have singular points at $x = \pm i$.
- Thus, the radius of convergence for the Taylor series expansion of p and q about $x_0 = 0$ is $\rho = 1$.

radius of convergence for the series solution $\sum_{n=0} a_n x^n$ about $x_0 = 0$ is at least $\rho \geq 1$

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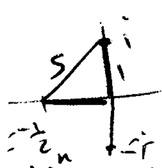
② $x_0 = -\frac{1}{2}$

$$s = \sqrt{1^2 + \tfrac{1}{4}} = \sqrt{\tfrac{5}{4}} = \tfrac{\sqrt{5}}{2}$$



$\sum_{n=0} a_n (x + \frac{1}{2})^n$ about $x_0 = -\frac{1}{2}$ is at least $\rho = \frac{\sqrt{5}}{2}$ $\quad |x + \tfrac{1}{2}| < \tfrac{\sqrt{5}}{2}$

- suppose $y(0) = y_0$ and $y'(0) = y'_0$ are given.

Since $1+x^2 \neq 0$ for all x, by Th 3.2.1 $\exists$ a unique solution of the IVP on $-\infty < x < \infty$.

- However, Th. 5.3.1 only guarantees a solution of the form $\sum_{n=0}^{\infty} a_n x^n$ for $-1 < x < 1$ where $a_0 = y_0$ and $a_1 = y'_0$

- Thus, the unique solution on $-\infty < x < \infty$ may not have a power series about $x_0 = 0$ that converges for all x.

Example: Determine a lower bound for the radius of convergence of series solution about $x_0 = 0$ for the equation:

$$y'' + (\sin x)\, y' + (1+x^2)\, y = 0$$

- $P(x) = 1$, $Q(x) = \sin x$, $R(x) = 1 + x^2$ (not Polynomials)
- $p(x) = \sin x$ is not a polynomial, but it has a Taylor series about $x_0 = 0$ that converges for all x.
- $q(x) = 1 + x^2$ has a Taylor series about $x_0 = 0$ namely, $(1+x^2)$, which converges for all x.
- Therefore, by Th.5.3.1, the radius of convergence for the series solution about $x_0 = 0$ is infinite.
