5.4 Euler Equations; Regular Singular Points

- Recall that the point x_0 is an ordinary point of

① -- $P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$ if

$p(x) = \dfrac{Q(x)}{P(x)}$ and $q(x) = \dfrac{R(x)}{P(x)}$ are analytic at x_0.

Otherwise x_0 is a singular point.

- If P, Q, R are polynomials with no common factors, then the singular points of the DE ① are the points where $P(x) = 0$.

- How to solve eq. ① in the neighborhood of a singular point.

Euler Equation is a simple DE that has a singular point (at 0)

② --- $L[y] = x^2 y'' + \alpha x y' + \beta y = 0$ α, β constant and real.

- $P(x) = x^2$, so $x = 0$ is the only singular point
- all other points are ordinary points.

Assume ② has a solution of the form $y = x^r$.

$\Rightarrow y' = r x^{r-1}$ and $y'' = r(r-1)x^{r-2}$

② becomes $x^2\, r(r-1)x^{r-2} + \alpha x\, r x^{r-1} + \beta x^r = 0$

$x^r\left[r(r-1) + \alpha r + \beta\right] = 0$

Case 1 $x > 0$
Case 2 $x < 0$

③ ----- $r^2 + (\alpha - 1) r + \beta = 0$

- If r is a root for ③, then $y = x^r$ is a solution of ②
- The roots of ③ are $r_1, r_2 = \dfrac{-(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$

So, we have three cases: If

① The roots r_1 and r_2 are real and different, then the general solution of ② is $y(x) = c_1 |x|^{r_1} + c_2 |x|^{r_2}$

② the roots r_1 and r_2 are real and equal ($r_1 = r_2$), then the general solution of ② is $y(x) = c_1 |x|^{r_1} + c_2 \ln|x| \, |x|^{r_2}$

③ The roots r_1 and r_2 are complex conjugates ($r_1, r_2 = \lambda \pm \mu i$) then the general solution of ② is

$$y(x) = c_1 |x|^{\lambda} \cos(\mu \ln|x|) + c_2 |x|^{\lambda} \sin(\mu \ln|x|)$$

~~Example~~ ①

* If r_1 and r_2 are real roots with $r_1 \neq r_2$, then $y_1(x) = x^{r_1}$ and $y_2(x) = x^{r_2}$ are solutions for ② Since

$$W(y_1, y_2)(x) = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1 - 1} & r_2 x^{r_2 - 1} \end{vmatrix} = (r_2 - r_1) \, x^{r_1 + r_2 - 1}$$

which is not zero for $r_1 \neq r_2$ and $x > 0$. Thus y_1 and y_2 are linearly indep. and the general solution of ② in this case is $y = c_1 x^{r_1} + c_2 x^{r_2}$, $x > 0$

② ~~see the pages on the back:~~ ③ ~~see the pages on the back~~

Example: Solve $2x^2 y'' + 3xy' - y = 0$, $x > 0$.

Substitute $y = x^r$, $y' = r x^{r-1}$, $y'' = r(r-1) x^{r-2}$

$x^r [2r(r-1) + 3r - 1] = 0 \iff 2r^2 + r - 1 = 0$

$\iff (2r-1)(r+1) = 0$. Hence $r_1 = \frac{1}{2}$ and $r_2 = -1$,

So the general solution is $y = c_1 x^{\frac{1}{2}} + c_2 x^{-1}$, $x > 0$.

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* $t = \ln x \implies e^t = e^{\ln x} = x \implies e^{rt} = x^r$

* see Q34 and Q35-41 page 165

② * If $r_1 = r_2$ are real, then we have one solution $y_1(x) = x^{r_1}$

To get the second solution, we use the reduction of order, or instead, we consider an alternative method:

Euler equation: $L[y] = x^2 y'' + \alpha x y' + \beta y = 0$

$t = \ln x \Rightarrow e^t = e^{\ln x}$

$\Rightarrow e^t = x$

$\Rightarrow x^r = e^{rt}$

$\Rightarrow \boxed{x^r = e^{r \ln x}}$

$\frac{\partial}{\partial r} x^r = \ln x \, \boxed{e^{r \ln x}}$

$= \ln x \, x^r$

$y = x^r \Rightarrow$

$L[x^r] = x^r [r(r-1) + \alpha r + \beta]$

$= x^r [\underbrace{r^2 + (\alpha - 1) r + \beta}_{F(r)}]$

$= x^r F(r)$

when F has repeated roots "double roots" $\Rightarrow$

$F(r) = (r - r_1)^2$ with $F'(r_1) = 0$

r_1 forces F and F' to be zero

$\Rightarrow L[x^r] = x^r F(r) = x^r (r^2 + (\alpha - 1) r + \beta) = x^r (r - r_1)^2$

$\frac{\partial}{\partial r} L[x^r] = \frac{\partial}{\partial r} x^r (r - r_1)^2$

$L[x^r \ln x] = \boxed{x^r \ln x \underbrace{(r - r_1)^2}_{\text{constant}} + 2 \underbrace{(r - r_1) x^r}_{\text{constant}}}$

constant $\| 0$ when $r = r_1$

$\Rightarrow y_2(x) = x^{r_1} \ln x, \quad x > 0$

$\rightarrow x^r \ln x$ is solution

$\Rightarrow W(y_1, y_2) = \begin{vmatrix} x^{r_1} & x^{r_1} \ln x \\ r_1 x^{r_1 - 1} & x^{r_1 - 1} (r_1 \ln x + 1) \end{vmatrix} = x^{2 r_1 - 1} \neq 0$ for all $x > 0$

for eq ② when $x > 0$

Thus, y_1 and y_2 are linearly indpendent and the general solution is $y(x) = c_1 x^{r_1} + c_2 x^{r_1} \ln x = (c_1 + c_2 \ln x) x^{r_1}, \quad x > 0$

③ If $F(r)$ has a complex roots $r = \lambda \pm i\mu$ with $\mu \neq 0$, then

$$x^r = e^{\ln x^r} = e^{r \ln x} = e^{(\lambda + i\mu)\ln x}$$
$$= e^{\lambda \ln x} e^{i\mu \ln x}$$
$$= e^{\ln x^\lambda} e^{i\mu \ln x}$$
$$= x^\lambda \left[\cos(\mu \ln x) + i \sin(\mu \ln x)\right], \quad \underline{x>0}$$

- Thus, x^r is defined for complex r and it can be shown that $y(x) = c_1 x^{\lambda + i\mu} + c_2 x^{\lambda - i\mu}$, $x > 0$ is the general solution.

- However, the solution is complex, and we seek to real valued solution, so it can be shown that $\underline{y_1(x) = x^\lambda \cos(\mu \ln x)}$ and $\underline{y_2(x) = x^\lambda \sin(\mu \ln x)}$ are solutions as well.

- Using the Wronskian, it can be shown that $W(y_1, y_2)(x) \neq 0$. Thus, y_1, y_2 are linearly independent, and the general solution is

$$y(x) = c_1 x^\lambda \cos(\mu \ln x) + c_2 x^\lambda \sin(\mu \ln x), \quad x > 0$$

Example Solve $x^2 y'' + 5xy' + 4y = 0$, $x>0$.

substituting $y = x^r$, $y' = r x^{r-1}$, $y'' = r(r-1)x^{r-2}$

$$x^r\{r(r-1) + 5r + 4\} = 0 \iff x^r[r^2 + 4r + 4] = 0$$

$$\iff (r+2)^2 = 0 \iff r_1 = r_2 = -2$$

Thus, the general solution is $y(x) = c_1 x^{-2} + c_2 x^{-2} \ln x$, $x>0$

Example: Solve $x^2 y'' + xy' + y = 0$

substitute $y = x^r$, $y' = r x^{r-1}$, $y'' = r(r-1) x^{r-2}$

$$x^r[r(r-1) + r + 1] = 0 \iff x^r(r^2+1) = 0$$

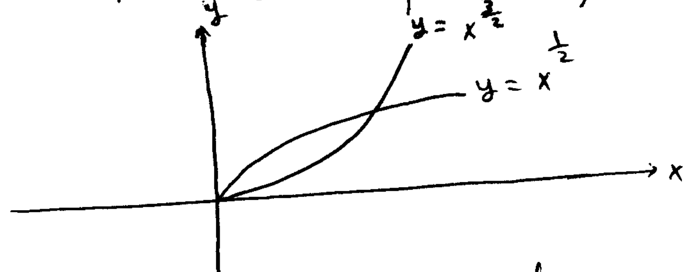
$r_{1,2} = \pm i$ $\quad \lambda = 0$ and $\mu = 1$. Thus, the general

Here $x^\lambda = 1$

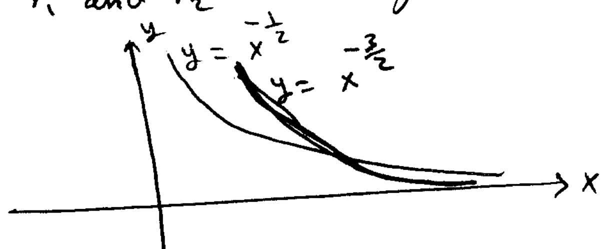
solution is $y(x) = c_1 \cos(\ln x) + c_2 \sin(\ln x)$, $x>0$

- Qualitative behavior of the solutions of Euler Equation (2) near the singular point $x=0$ depends totally on r_1 and r_2:

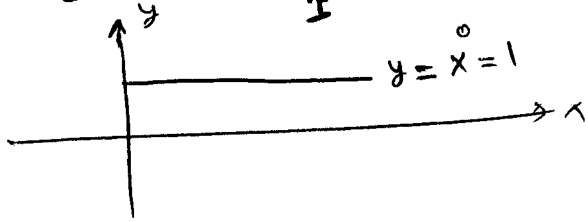
1) If r_1 and r_2 are positive real, then $x^r \to 0$ as $x \to 0$



2) If r_1 and r_2 are negative real, then $x^r \to \infty$ as $x \to 0$



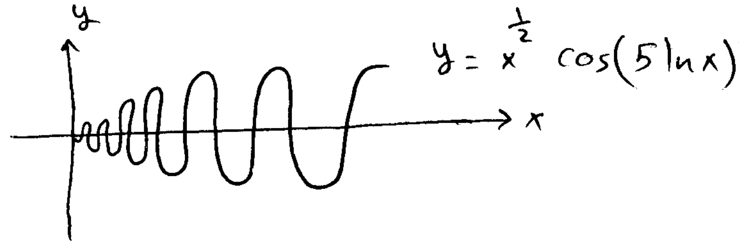
3) If $r_1 = r_2 = 0$, then $y = x^r = 1$, and so $x^r \to 1$ as $x \to 0$



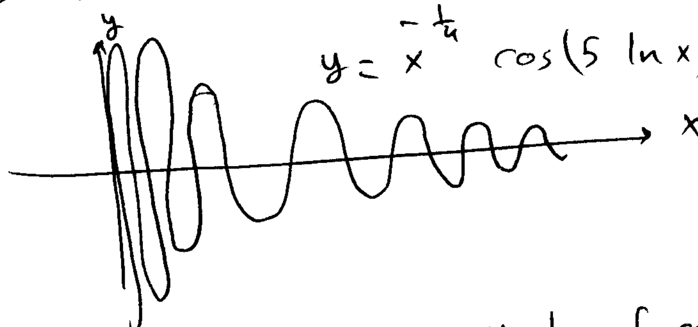
4) If r_1 and r_2 are complex : $r_{1,2} = \lambda \pm \mu i$

Then we have 3 cases:

1) $\lambda > 0 \Rightarrow$ the solution $y \to 0$ as $x \to 0$

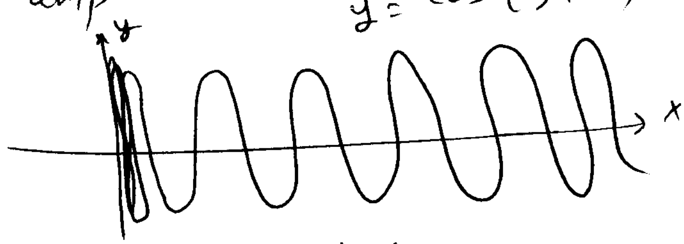


2) $\lambda < 0 \Rightarrow$ the solution $y \to \infty$ as $x \to 0$



3) $\lambda = 0 \Rightarrow$ the solution y oscillate of constant amplitude.

$y = \cos(5 \ln x)$

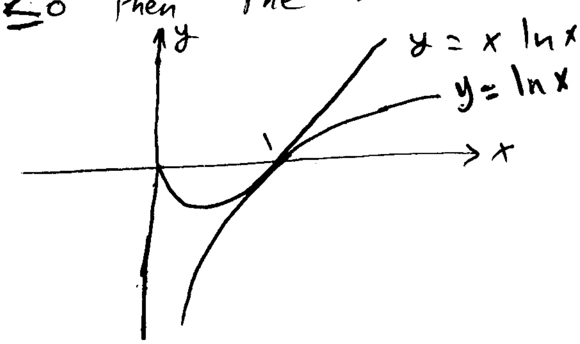


goes with 3)

5) If $r_1 = r_2$ are real repeated then ~~one~~ second solution is $y_2 = x^r \ln x$

a) If $r > 0$ then the solution $y \to 0$ as $x \to 0$

b) If $r \leq 0$ then the solution $y \to -\infty$ as $x \to 0$



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see one more example or quize

Example Solve the IVP: $x > 0$

$$2x^2 y'' + 6xy' + 10y = 0 \qquad y(1)=1, \quad y'(1)=1$$

substitute $y = x^r$, $y' = r x^{r-1}$, $y'' = r(r-1)x^{r-2}$

$$x^r\left[2r(r-1) + 6r + 10\right] = 0 \iff 2r^2 + 4r + 10 = 0 \iff r^2 + 2r + 5 = 0$$

$$\iff r = \frac{-2 \pm \sqrt{4-20}}{2} = \underbrace{-1}_{\lambda} \pm \underbrace{2}_{\mu} i$$

The general solution is

$$y(x) = c_1 |x|^{-1} \cos(2\ln|x|) + c_2 |x|^{-1} \sin(2\ln|x|)$$

$$y(x) = \frac{c_1}{x}\cos(2\ln x) + \frac{c_2}{x}\sin(2\ln x) \qquad \text{since } \underline{x_0 = 1}$$

$y(1) = 1 \Rightarrow \boxed{c_1 = 1}$

$y'(1) = 1 \Rightarrow 2c_2 = c_1 \iff \boxed{c_2 = 1}$

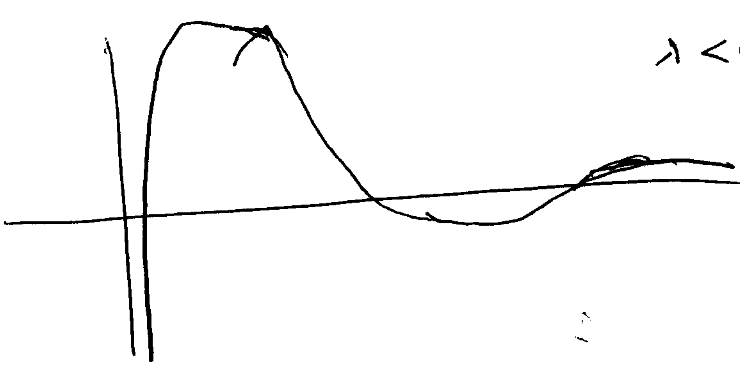
Thus, the general solution is

$$y(x) = x^{-1}\cos(2\ln x) + x^{-1}\sin(2\ln x)$$

$$= \frac{\cos(2\ln x) + \sin(2\ln x)}{x}$$

$x = 0$ is a singular point

$\lambda < 0$



as $x \to 0$, the solution y oscillates and becomes unbounded

⊛

Solution Behavior and Singular Points:

* **Bessel Equation** of order ν: $\quad x^2y'' + xy' + (x^2 - \nu^2)y = 0$
  - The point $x=0$ is a singular point since $P(x) = x^2 = 0 \Leftrightarrow x = 0$.
  - All other points are ordinary points.

* legendre Equation: $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

  The points $x = \pm 1$ are singular points since $P(x) = 1 - x^2$ is zero there.
  All other points are ordinary points.

If we try to find a series solution (like in the preceding two sections) to solve the DE in a neighborhood of a singular point x_0, we will find that it's not possible.

→ Because the solution may not be analytic at x_0, and hence will not have Taylor series expansion about x_0.

→ Thus, without more information about $\frac{Q}{P}$ and $\frac{R}{P}$ in the neighborhood of a singular point x_0, it may be impossible to describe solutions behavior near x_0.

* classifying Singular Points:
  - Our goal is to extend the methods already developed for solving $P(x)y'' + Q(x)y' + R(x)y = 0$ near an ordinary point so that it applies to the neighborhood of a singular point x_0.
  - To do so, we restrict ourselves to cases in which singularities in $\frac{Q}{P}$ and $\frac{R}{P}$ at x_0 are not too severe (حاد).
    that is "weak singularties".

Consider the DE: $P(x)y'' + Q(x)y' + R(x)y = 0$

- If P and Q, R are polynomials, then a singular point x_0 is called regular if

$$\lim_{x \to x_0} (x-x_0)\frac{Q(x)}{P(x)} \text{ is finite and } \lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} \text{ is finite.}$$

- If P and Q, R are more general functions than polynomials, then a singular point x_0 is called regular if

$$(x-x_0)\frac{Q(x)}{P(x)} \text{ and } (x-x_0)^2\frac{R(x)}{P(x)} \text{ are analytic at } x = x_0$$

- A singular point that is not regular is called irregular singular point.

Example: Determine the singular points of the Bessel equation $x^2y'' + xy' + (x^2 - v^2)y = 0$ and determine weather they are regular or irregular?

The point $x=0$ is a regular singular point since the following two limits are finite

$P(x) = x^2$
$Q(x) = x$
$R(x) = x^2 - v^2$
Polynomial

$$\lim_{x \to x_0} (x-x_0)\frac{Q(x)}{P(x)} = \lim_{x \to 0} x \frac{x}{x^2} = 1$$

$$\lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to 0} x^2\left(\frac{x^2-v^2}{x^2}\right) = -v^2$$

Example: Determine the singular points of the Legendre equation $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$ and determine weathe they are regular or irregular?

$P(x) = 1-x^2$, $Q(x) = -2x$, $R(x) = \alpha(\alpha+1)$ Polynomials

$x = \pm 1$ singular points. $\lim_{x \to x_0} (x-x_0)\frac{Q(x)}{P(x)} = \lim_{x \to 1} (x-1)\frac{-2x}{1-x^2} = \lim_{x \to 1} \frac{2x}{x+1} = 1$

$\lim_{x \to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} = \lim_{x \to 1} (x-1)^2\left(\frac{\alpha(\alpha+1)}{1-x^2}\right) = \lim_{x \to 1} (x-1)\left(\frac{\alpha(\alpha+1)}{x+1}\right) = 0$

$\Rightarrow x=1$ is regular singler point since the two limits are finite. Simillarly $x=-1$ is a regular singular point.

Example: Determine the singular points of the DE

$$2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$$

and classify them as regular or irregular.

Divide by $2x(x-2)^2 \Rightarrow$

$$y'' + \frac{3x}{2x(x-2)^2} y' + \frac{1}{2x(x-2)} y = 0$$

P and Q are polynomials

$$p(x) = \frac{Q(x)}{P(x)} = \frac{3}{2(x-2)^2} \quad \text{and} \quad q(x) = \frac{R(x)}{P(x)} = \frac{1}{2x(x-2)}$$

The singular points are $x=0$ and $x=2$ (where $P(x)=0$)

➡ $x=0$ is a regular singular point because

$$\lim_{x\to x_0} (x-x_0)\frac{Q(x)}{P(x)} = \lim_{x\to 0} x \frac{3}{2(x-2)^2} = \lim_{x\to 0} \frac{3x}{2(x-2)^2} = 0 < \infty$$

$$\lim_{x\to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} = \lim_{x\to 0} x^2 \frac{1}{2x(x-2)} = \lim_{x\to 0} \frac{x}{2(x-2)} = 0 < \infty$$

➡ $x=2$ is irregular singular point since the following limit does not exist

$$\lim_{x\to x_0} (x-x_0)\frac{Q(x)}{P(x)} = \lim_{x\to 2} (x-2)\frac{3}{2(x-2)^2} = \lim_{x\to 2} \frac{3}{2(x-2)}$$

Example:

Determine the singular points of $\left(x-\frac{\pi}{2}\right)^2 y'' + \cos x\, y' + \sin x\, y = 0$ and classify them as regular or irregular.

- $x = \frac{\pi}{2}$ is the only singular point. $P = \left(x-\frac{\pi}{2}\right)^2$, $Q = \cos x$ not polynomials

- We consider the function

$$(x-x_0)\frac{Q(x)}{P(x)} = \left(x-\frac{\pi}{2}\right)\frac{\cos x}{\left(x-\frac{\pi}{2}\right)^2} = \frac{\cos x}{x-\frac{\pi}{2}} \quad \text{and}$$

$$(x-x_0)^2 \frac{R(x)}{P(x)} = \left(x-\frac{\pi}{2}\right)^2 \frac{\sin x}{\left(x-\frac{\pi}{2}\right)^2} = \sin x$$

we need to show that these functions are analytic about $x_0 = \frac{\pi}{2}$.

* Using methods of calculus, we can show that the Talyor series of $\cos x$ about $\frac{\pi}{2}$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x-\frac{\pi}{2}\right)^{2n+1}$$

Thus,
$$\frac{\cos x}{x-\frac{\pi}{2}} = -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!}\left(x-\frac{\pi}{2}\right)^{2n}$$

which converges for all x.
Hence, it's analytic at $\frac{\pi}{2}$

* Similarly, $\sin x$ is analytic at $\frac{\pi}{2}$ with Taylor series $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{2}\right)^{2n}$

* Thus $\frac{\pi}{2}$ is a regular singular point of the DE.