5.5 Series Solutions Near a Regular Singular Point Part I

- We will solve the 2^{nd} order linear equation

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad \text{----} (1)$$

in the neighborhood of a regular singular point $x = x_0$.

- To simplify things we consider $x_0 = 0$. Otherwise, we can do transformation for the equation into a one with a regular singular point at the origin by letting $x - x_0 = t$.

- Since $x=0$ is a regular singular point of (1) $\Rightarrow$

$$(x - x_0)\frac{Q(x)}{P(x)} = x\,p(x) \quad \text{and} \quad (x - x_0)^2 \frac{R(x)}{P(x)} = x^2 q(x)$$

are analytic at $x=0$ and have finite limits:

$$x\,p(x) = \sum_{n=0}^{\infty} p_n x^n \quad \text{and} \quad x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n \quad \text{----} (2)$$

That is $x\,p(x)$ and $x^2 q(x)$ have convergent power series expansion on some interval $|x| < \rho$ about the origin.

- Divid (1) by $P(x)$ and multiply (1) by x^2

$$x^2 y'' + x^2 \frac{Q(x)}{P(x)} y' + x^2 \frac{R(x)}{P(x)} y = 0$$

$$x^2 y'' + x^2 p(x)\,y' + x^2 q(x)\,y = 0$$

$$x^2 y'' + x\,(x\,p(x))\,y' + x^2 q(x)\,y = 0$$

- Using (2) $\Rightarrow$

$$x^2 y'' + x\left(\sum_{n=0}^{\infty} p_n x^n\right) y' + \left(\sum_{n=0}^{\infty} q_n x^n\right) y = 0$$

$$(3) \text{-----} \quad x^2 y'' + x\,[\underbrace{p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n + \dots}_{\text{complicates the calculation}}]\,y' + [\underbrace{q_0 + q_1 x + q_2 x^2 + \dots + q_n x^n + \dots}_{\text{complicate the calculation}}]\,y = 0$$

$\Rightarrow$ If all coefficients p_n and q_n are zero except

$$p_0 = \lim_{x\to 0} \frac{x\,Q(x)}{P(x)} \quad \text{and} \quad q_0 = \lim_{x\to 0} \frac{x^2\,R(x)}{P(x)}$$

Then, we arrive <u>Euler equation</u>:

$$x^2 y'' + p_0 x y' + q_0 y = 0 \quad \text{----} (4)$$

and so, the essential character of solutions of (3) is identical to (4).

But the solution of (4) is in terms of power series since the coefficients in (3) are Euler coefficient "analytic at ". Thus, the solution is:

$$y = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \right),$$

$$y = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \quad * \quad \text{--} (5)$$

$a_0 \neq 0$ since r is the exponent of the first term in the series, and a_0 is its coefficient.

- So we need to find
  - r for eq. (1) has solution of the form (5).
  - the recurrence relation a_n for the coefficient
  - the radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$

<u>Example</u>: Consider the DE $\quad * \; 2x^2 y'' - x y' + (1+x) y = 0$

The solution is $\boxed{5}$, so we need to find r, a_n and ρ.

(a) show that * has a regular singular point at $x = 0$.

- $P(x) = 2x^2$, $Q(x) = -x$, $R(x) = 1+x$
- $P(x) = 0 \Leftrightarrow x_0 = 0$ is a singular point.
- The coefficients P, Q, R are polynomials and

$$\lim_{x\to x_0} (x - x_0) \frac{Q(x)}{P(x)} = \lim_{x\to 0} x \frac{(-x)}{2x^2} = -\frac{1}{2} < \infty$$

$$\lim_{x\to x_0} (x - x_0)^2 \frac{R(x)}{P(x)} = \lim_{x\to 0} x^2 \frac{(1+x)}{2x^2} = \frac{1}{2} < \infty$$

Both limits are finite. Thus, $x_0 = 0$ is a regular singular point

(b) Determine the indicial equation, the recurrence relation and the roots of the indicial equation.

- $x\,p(x) = x\,\frac{Q(x)}{P(x)} = x\,\frac{(-x)}{2x^2} = \frac{-1}{2} = \sum_{n=0}^{\infty} p_n x^n \Leftrightarrow p_0 = -\frac{1}{2}$

$x^2 q(x) = x^2 \frac{R(x)}{P(x)} = x^2 \frac{(1+x)}{2x^2} = \frac{1}{2} + \frac{x}{2} = \sum_{n=0}^{\infty} q_n x^n \Leftrightarrow q_0 = \frac{1}{2},\ q_1 = \frac{1}{2}$

Note that $p_1 = p_2 = \cdots = q_2 = q_3 = \cdots = 0$

- Thus, the corresponding Euler Equation:

$$x^2 y'' + p_0 x y' + q_0 y = 0 \Leftrightarrow x^2 y'' + \frac{1}{2} x y' + \frac{1}{2} y = 0$$

$$\Leftrightarrow 2x^2 y'' + x y' + y = 0$$

$\Leftrightarrow$ If $y = x^r$, $y' = r x^{r-1}$, $y'' = r(r-1)x^{r-2}$

$$x^r[2r(r-1) - r + 1] = 0 \Leftrightarrow$$

$$2r^2 - 3r + 1 = 0 \Leftrightarrow$$

$$(2r-1)(r-1) = 0 \Leftrightarrow r_1 = 1 \checkmark$$

$$r_2 = \frac{1}{2} \checkmark$$

- To find a_n: Our solution is assumed to be

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad y'(x) = \sum_{n=0}^{\infty} a_n (r+n) x^{r+n-1}, \quad y''(x) = \sum_{n=0}^{\infty} a_n (r+n)(r+n-1) x^{r+n-2}$$

Substitute y, y', y'' in DE above:

$$\sum_{n=0}^{\infty} 2a_n(r+n)(r+n-1)x^{r+n} - \sum_{n=0}^{\infty} a_n(r+n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{n+r} + \underbrace{\sum_{n=0}^{\infty} a_n x^{n+r+1}}_{\sum_{n=1}^{\infty} a_{n-1} x^{n+r}} = 0$$

$$a_0[2r(r-1) - r + 1]x^r + \sum_{n=1}^{\infty}\left[2a_n(r+n)(r+n-1) - a_n(r+n) + a_n + a_{n-1}\right]x^{n+r} = 0$$

First: The coefficient of x^r:

$$a_0[2r(r-1) - r + 1] = 0 \Rightarrow 2r^2 - 3r + 1 = 0 \quad (\text{since } a_0 \neq 0)$$

$$\boxed{(2r-1)(r-1) = 0}$$ is called indicial equation that was obtained earlier when we examined the corresponding Euler Equation.

- The roots $r_1 = 1$ and $r_2 = \frac{1}{2}$ of the indicial equations are called the **exponents of the singularity for the regular singular point $x_0 = 0$.**

*

$\Rightarrow$ The exponents of the singularity determine the qualitative behavior of solution in neighborhood of the regular singular point $x_0 = 0$.

- Second the coefficient of x^{n+r} :

$$2 a_n (r+n)(r+n-1) - a_n(r+n) + a_n + a_{n-1} = 0 \quad , \quad n = 1,2,3,\ldots$$

$$a_n \left[2(r+n)(r+n-1) - (r+n) + 1 \right] + a_{n-1} = 0$$

$$a_n = \frac{-a_{n-1}}{2(r+n)(r+n-1) - (r+n) + 1} \quad , n = 1,2,3,\ldots$$

$$a_n = \frac{-a_{n-1}}{2(r+n)^2 - 3(r+n) + 1}$$

$$\boxed{a_n = \frac{-a_{n-1}}{\big(2(r+n) - 1\big)\big((r+n) - 1\big)} \quad , \quad n = 1,2,3,\ldots} \; *$$

recurrence relation

(c) Find the series solution $(x>0)$ corresponding to the largest root.

* Series Solution crossponding to the 1^{st} root : $(r_1 = 1)$ or the largest root :

$r_1 = 1 \Rightarrow$ the recerrence relation * becomes:

$$a_n = \frac{-a_{n-1}}{(2n+1)n} \quad , n = 1,2,3,\ldots$$

when $n = 1 \Rightarrow a_1 = \dfrac{-a_0}{(3)(1)}$

$n = 2 \Rightarrow a_2 = \dfrac{-a_1}{(5)(2)} = \dfrac{a_0}{(3)(5)(1)(2)}$

$n = 3 \Rightarrow a_3 = \dfrac{-a_2}{(7)(3)} = \dfrac{-a_0}{(3)(5)(7)(1)(2)(3)}$

$\vdots \qquad \vdots$

$$a_n = \frac{(-1)^n a_0}{(3)(5)(7)\cdots(2n+1)\, n!} \quad , n = 1,2,3,\ldots$$

• Thus, for $x>0$, one solution to our DE is (when $r=1$)

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = a_0 x + \sum_{n=1}^{\infty} \frac{(-1)^n a_0 \, x^{n+1}}{(3)(5)(7)\cdots(2n+1)\, n!}$$

$$\text{initial condition} \quad = a_0 x \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(3)(5)(7)\cdots(2n+1)\, n!} \right]$$

• If we omit a_0, the first solution is $y_1(x) = x[\ \_\_\_\ ]$

• To determine the radius of convergence, we use the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{(3)(5)(7)\cdots(2n+1)\, n!\, (-1)^{n+1} x^{n+1}}{(3)(5)(7)\cdots(2n+3)(n+1)!\, (-1)^n x^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{(2n+3)(n+1)} = 0 < 1$$

Thus, the radius of convergence ($\rho = \infty$) is infinit, and hence the series $y_1(x)$ converges for all x.

(d) Find the series solution corresponding to the smallest root ($x>0$) if $r_1 - r_2 \notin \mathbb{Z}$

* Series solution crossponding to the second root ($r_2 = \frac{1}{2}$) or the smallest root.

$r_2 = \frac{1}{2} \Rightarrow$ the recurrence relation * becomes

$$a_n = \frac{-a_{n-1}}{\left(2\left(\frac{1}{2}+n\right)-1\right)\left(\left(\frac{1}{2}+n\right)-1\right)} = \frac{-a_{n-1}}{n(2n-1)}, \quad n = 1, 2, 3, \ldots$$

when $n=1 \Rightarrow a_1 = \dfrac{-a_0}{(1)(1)}$

$n=2 \Rightarrow a_2 = \dfrac{-a_1}{(2)(3)} = \dfrac{a_0}{(1)(2)(1)(3)}$

$n=3 \Rightarrow a_3 = \dfrac{-a_2}{(3)(5)} = \dfrac{-a_0}{(1)(2)(3)\,(1)(3)(5)}$

$\vdots$

$$a_n = \frac{(-1)^n a_0}{(1)(3)(5)\cdots(2n-1)\, n!}, \quad n = 1, 2, 3, \ldots$$

Thus, for $x > 0$, a second solution to our DE is (when $r_2 = \frac{1}{2}$)

$$y_2(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = a_0 x^{\frac{1}{2}} + \sum_{n=1}^{\infty} \frac{(-1)^n a_0 x^{n+\frac{1}{2}}}{(1)(3)(5)\cdots(2n-1)\, n!}$$

$$= a_0 x^{\frac{1}{2}} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(1)(3)(5)\cdots(2n-1)\, n!} \right]$$

- If we omit a_0, then the second solution is

$$y_2(x) = x^{\frac{1}{2}} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(1)(3)(5)\cdots(2n-1)\, n!} \right]$$

- To determine the radius of convergence for this series solution:

$$\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_{n\to\infty} \left| \frac{(1)(3)(5)\cdots(2n-1)\, n!\, (-1)^{n+1} x^{n+1}}{(1)(3)(5)\cdots(2n-1)(2n+1)(n+1)!\, (-1)^n x^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{(2n+1)n} = 0 < 1$$

Thus, the radius of convergent is infinite ($\rho = \infty$), and hence the series solution $y_2(x)$ converges for all x.

- The two solutions are:

$$y_1(x) = x \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(3)(5)(7)\cdots(2n+1)\, n!} \right]$$

$$y_2(x) = x^{\frac{1}{2}} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(1)(3)(5)\cdots(2n-1)\, n!} \right]$$

Since the leading terms of y_1 and y_2 are x and $x^{\frac{1}{2}}$, respectively, it follows that y_1 and y_2 are linearly independent and hence form a fundamental set of solutions.

Thus, the general solution is $y(x) = c_1 y_1(x) + c_2 y_2(x)$, $x > 0$.

* In the more general case of a singular point at $x = x_0$, our series solution will have the form
$$y(x) = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

* If the roots r_1, r_2 of the indicial equation are equal or differ by an integer; then the second solution y_2 has a more complicated structure. (see Section 5.7)

* If the roots r_1, r_2 of the indicial equation are complex, then there are always two solutions with the above form. These solutions are complex valued, but we can obtain real-valued solutions from the real and imaginary parts of the complex solutions.