6.1 Definition of the Laplace Transform

* An improper integral over an unbounded interval is defined as a limit of integrals over finite intervals. That is

$$\int_a^{\infty} f(t)\,dt = \lim_{b\to\infty} \int_a^b f(t)\,dt \quad , \; b \in \mathbb{R}^+$$

→ The improper integral converges if $\int_a^b f(t)\,dt$ exists for each $b>a$ and $\lim_{b\to\infty} \int_a^b f(t)\,dt$ exists.

→ Otherwise, the improper integral diverge.

Example ① Let $f(t) = e^{ct}$, $t \geq 0$, $c \neq 0 \in \mathbb{R}$.

Then the improper integral is

$$\int_0^{\infty} e^{ct}\,dt = \lim_{b\to\infty} \int_0^b e^{ct}\,dt = \lim_{b\to\infty} \left.\frac{e^{ct}}{c}\right|_0^b = \lim_{b\to\infty} \left[\frac{e^{cb}}{c} - \frac{1}{c}\right]$$

→ The improper integral converges to $\frac{-1}{c}$ if $c<0$.

→ " " " diverges if $c \geq 0$.

→ If $c=0$, then $f(t) = 1$. Thus, the improper integral diverges.

Example ② Let $f(t) = \frac{1}{t}$, $t \geq 1$. Then the improper integral is $\int_1^{\infty} \frac{dt}{t} = \lim_{b\to\infty} \int_1^b \frac{dt}{t} = \lim_{b\to\infty} \ln b = \infty$. Thus, the improper integral diverges.

Example ③ Let $f(t) = \frac{1}{t^p}$, $t \geq 1$, $p \in \mathbb{R}$ "when $p=1$, see Example 2"

The improper integral is: $\int_1^{\infty} \frac{dt}{t^p} = \lim_{b\to\infty} \int_1^b \frac{dt}{t^p} = \lim_{b\to\infty} \left(\frac{b^{1-p} - 1}{1-p}\right)$

Note that $b^{1-p} \to 0$ as $b \to \infty$ if $p>1$, and so $\int_1^{\infty} \frac{dt}{t^p}$ converges to $\frac{1}{p-1}$

but $b^{1-p} \to \infty$ as $b \to \infty$ if $p<1$, and so " diverges.

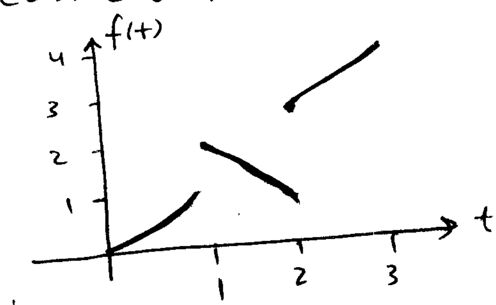
see example ②

* Many practical engineering problems involve mechanical or electrical systems acted by discontinuous or impulsive (صنغ) forcing terms.

* For such problems, the methods described in ch3 are difficult to apply
* In this chapter, we use laplace transform to convert a problem for an unknown function f into a simpler problem for F, solve for F, then recover f from its transform.

<u>Example ①</u> Consider the following piecewise defined function

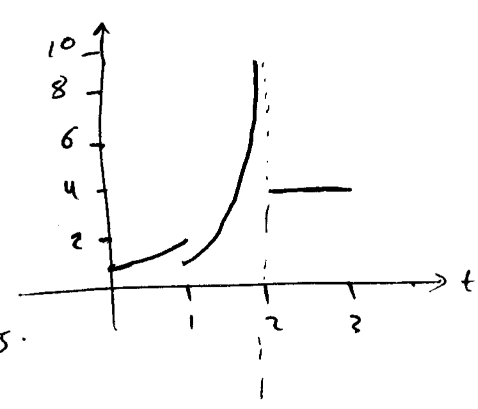
$$f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 3-t, & 1 < t \le 2 \\ t+1, & 2 < t \le 3 \end{cases}$$



We see $f(t)$ is piecewise continuous on $[0,3]$

<u>Example ②</u> Consider the following piecewise defined function

$$f(t) = \begin{cases} t^2+1, & 0 \le t \le 1 \\ (2-t)^{-1}, & 1 < t \le 2 \\ 4, & 2 < t \le 3 \end{cases}$$

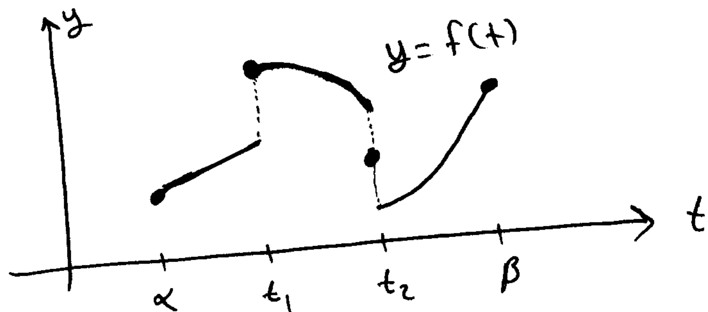


f is not piecewise continuous.

* A function f is <u>piecewise continuous</u> on $[\alpha,\beta]=I$ if I can be partitioned by a finite number of points $\alpha=t_0<t_1<\cdots<t_n=\beta$ so that

1) f is continuous on each open subinterval $t_{i-1}<t<t_i$.
2) f approches a finite limit as the endpoints of each subinterval are approached from within the subinterval. $\left|\lim_{t\to t_{i-1}^+} f(t)\right|<\infty$ $\quad i=1,2,\dots,n$

* That is, f is piecewise continuous on $\alpha\le t\le\beta$ if it is continuous there except for a finite number of jumb discontinuities. $\left|\lim_{t\to t_i^-} f(t)\right|<\infty$ $\quad i=2,3,\dots,n+1$



$$\int_\alpha^\beta f(t)\,dt=\int_\alpha^{t_1} f(t)\,dt+\int_{t_1}^{t_2} f(t)\,dt+\int_{t_2}^{\beta} f(t)\,dt \quad --- *$$

$\Rightarrow$ Note that the integral $*$ stays the same if $y=f(t)$ is not defined at the end points α and β.

$\Rightarrow$ If f is piecewise continuous on $a\le t\le b$, then $\int_a^b f(t)\,dt$ exists.

$\Rightarrow$ $\int_a^b f(t)\,dt$ exists for $t\ge a$

$\Rightarrow$ But being only piecewise continuous does not ensure convergence of the <u>improper integral</u> !!

<u>Th 6.1.1</u> If f is piecewice continuous for $t\ge a$,
$|f(t)|\le g(t)$ for $t\ge M$, $M>0$.
$\int_M^\infty g(t)\,dt$ converges, then $\int_a^\infty f(t)\,dt$ converges.

On the other hand, If $f(t)\ge g(t)\ge 0$ for $t\ge M$
If $\int_M^\infty g(t)\,dt$ diverges, then $\int_a^\infty f(t)\,dt$ diverges.

* An integral transform is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t)\, f(t)\, dt \quad \text{where}$$

- $K(s,t)$ is the kernel of the transformation (given).
- The relation above transforms the function f into F, which is called transform of f.
- We will study in this chapter Laplace transform, which is defined by

* $$L\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt \quad , s \in \mathbb{R}$$

$\Rightarrow$ Laplace transform uses kernel $K(s,t) = e^{-st}$ since the solution of linear DEs with constant coefficients are based on the exponential function.

The laplace transform F of f exists if f satisfies certain conditions!!

Th 6.1.2 Suppose that

1) f is piecewise continuous on $0 \le t \le b$ for any positive b.
2) $|f(t)| \le K e^{at}$ for $t \ge M$, where $K, a, M \in \mathbb{R}$ and $K, M > 0$

Then the laplace transform $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.

to guarantee * converges $e^{-st} e^{at} = e^{(a-s)t}$

Now we find the Laplace transforms of some important elementary functions:

Example ④ Let $f(t) = 1$, $t \ge 0$ Find $\mathcal{L}\{1\}$

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st}\, dt = \lim_{b \to \infty} \left. \frac{-e^{-st}}{s} \right|_0^b = \frac{1}{s} \quad , s > 0$$

Thus $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$

* A function f that satisfies the conditions of Th 6.1.2 is said to have exponential order as $t \to \infty$.

Example: Let $f(t) = e^{at}$, $t \geq 0$

$$\Rightarrow \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \frac{1}{s-a}, \quad \underset{\text{by Th 6.1.2}}{s > a} \quad (\text{by Example 4})$$

Thus $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

Example: Consider the piecewise continuous function

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ k, & t = 1 \\ 0, & t > 1 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}\, dt = -\frac{e^{-st}}{s}\Big|_0^1 = \frac{1-e^{-s}}{s}, \quad s > 0.$$

- Note that $\mathcal{L}\{f(t)\}$ does not depend on k " the point of discontinuity.

Example: Let $f(t) = \sin at$, $t \geq 0$ then

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st} \sin at\, dt, \quad s > 0$$

$$= \lim_{b \to \infty} \int_0^b e^{-st} \sin at\, dt \quad (\text{by parts})$$

$$= \lim_{b \to \infty} \left[-\frac{e^{-st}\cos at}{a}\Big|_0^b - \frac{s}{a}\int_0^b e^{-st} \cos at\, dt \right]$$

$$= \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st} \cos at\, dt \quad (\text{by parts})$$

$$= \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st} \sin at\, dt$$

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2} F(s)$$

$$F(s) = \frac{a}{s^2 + a^2}, \quad s > 0$$

Thus, $\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$

* If f_1 and f_2 are two functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$. Then for $s > \max\{a_1, a_2\}$

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\} \quad \text{----} *$$

Proof: $\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}\left(c_1 f_1(t) + c_2 f_2(t)\right) dt$

$$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt$$

$$= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$

* means that the Laplace transform is a linear operator.

Example: Find the Laplace transform of

$$f(t) = 5e^{-2t} - 3\sin 4t, \quad t \geq 0$$

$$\mathcal{L}\{f(t)\} = 5\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin 4t\}$$

$$= \frac{5}{s+2} - \frac{3(4)}{s^2+16}, \quad s > 0$$

$$= \frac{5}{s+2} - \frac{12}{s^2+16}$$

$$\mathcal{L}^{-1}\left(\frac{5}{s+2}\right) = 5\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = 5e^{-2t}$$

$$\mathcal{L}^{-1}\left(\frac{12}{s^2+16}\right) = 3\mathcal{L}^{-1}\left(\frac{4}{s^2+4^2}\right) = 3\sin 4t$$