6.2 Solution of IVP's

* We will use Laplace transform to solve IVP's for linear DE's with constant coefficients. we start in this section with homogeneous DE.

Th 6.2.1: Suppose f is continuous and f' is piecewise continuous on any interval $0 \le t \le b$. Suppose $\exists$ constants K, a, M s.t $|f(t)| \le K e^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$

and given by $\quad \mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$.

Proof: $\mathcal{L}\{f'(t)\} = \displaystyle\int_0^\infty e^{-st} f'(t)\,dt = \lim_{b\to\infty} \int_0^b e^{-st} f'(t)\,dt$

Since f' is piecewise continuous on $0 \le t \le b$, then f' may have points of discontinuity $t_1, t_2, \dots, t_n$. Thus,

$$\mathcal{L}\{f'(t)\} = \lim_{b\to\infty}\left[\int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{t_2} e^{-st} f'(t)\,dt + \dots + \int_{t_n}^{b} e^{-st} f'(t)\,dt\right]$$

$u = e^{-st}$ $\qquad dv = f'(t)\,dt$

$du = -s\,e^{-st}\,dt$ $\qquad v = f(t)$

$$= \lim_{b\to\infty}\left[e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \dots + e^{-st} f(t)\Big|_{t_n}^{b} + s\left(\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1} e^{-st} f(t)\,dt + \dots + \int_{t_n} e^{-st} f(t)\,dt\right)\right]$$

$$= \lim_{b\to\infty}\left[e^{-sb} f(b) - f(0) + s\int_0^b e^{-st} f(t)\,dt\right]$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s\,\mathcal{L}\{f(t)\} - f(0)$$

Similarly, it can be shown that

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$

$= s\left[s\,\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$

$= s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$

Quize Use laplace transform to find the solution of

$$y'' + 5y' + 6y = 0 \qquad y(0) = 2, \quad y'(0) = 3$$

$$\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{0\} = 0$$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + 5[s\mathcal{L}\{y\} - y(0)] + 6\mathcal{L}\{y\} = 0$$

$$(s^2+5s+6)\mathcal{L}\{y\} - (s+5)\,y(0) - y'(0) = 0$$

$$(s^2+5s+6)\mathcal{L}\{y\} - 2(s+5) - 3 = 0$$

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r_1 = -2, \; r_2 = -3$

$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$

Using the IC

$y(t) = 9e^{-2t} - 7e^{-3t}$

$$\mathcal{L}\{y\} = Y(s) = \frac{2s+13}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2} = \frac{-7}{s+3} + \frac{9}{s+2}$$

Thus, $\mathcal{L}\{y\} = Y(s) = \dfrac{-7}{s+3} + \dfrac{9}{s+2}$

$$y(t) = \mathcal{L}^{-1}\left(\frac{-7}{s+3}\right) + \mathcal{L}^{-1}\left(\frac{9}{s+2}\right) = -7e^{-3t} + 9e^{-2t}$$

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* It can be shown that if f is continuous with $\mathcal{L}\{f(t)\} = F(s)$ then f is the unique continuous function with $f(t) = \mathcal{L}^{-1}(F(s))$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Example use Laplace transform to solve the IVP

$$y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0 \quad \text{--- DE --- (1)}$$

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = 0$$

$$s^2 \mathcal{L}\{y\} - s\,y(0) - y'(0) - \left[s\mathcal{L}\{y\} - y(0)\right] - 2\mathcal{L}\{y\} = 0$$

$$(s^2 - s - 2)\mathcal{L}\{y\} - s + 1 = 0 \quad \text{...... "algebraic equation" (2)}$$

$$\mathcal{L}\{y\} = \frac{-1+s}{s^2 - s - 2} = \frac{s-1}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{s+1}$$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2, \; r_2 = -1$

$y = c_1 e^{2t} + c_2 e^{-t}$

using the IC

$y = \frac{1}{3} e^{+2t} + \frac{2}{3} e^{-t}$

$$s - 1 = A(s+1) + B(s-2) \iff A = \frac{1}{3}, \quad B = \frac{2}{3}$$

$$\mathcal{L}\{y\} = \frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}\right) = \mathcal{L}^{-1}\left(\frac{\frac{1}{3}}{s-2}\right) + \mathcal{L}^{-1}\left(\frac{\frac{2}{3}}{s+1}\right)$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$y(t) = \frac{1}{3} e^{2t} + \frac{2}{3} e^{-t}$$

inverse transform

هذه المشكلة فقط والباقي سهل

See the table.

- Note that using Laplace transform, it reduces the DE (1) to an algebraic equation (2).
- We now consider an example with nonhomogeneous DE (note that in this case we don't need to solve homogeneous first).
- Note also that for higher order DE more than 2^{nd} order, a numerical approximation is required (especially when the roots are irrational or complex)
- We will use table page 317.

Example: Use the laplace transform to solve the IVP

$$y'' + y = \sin 2t \;, \quad y(0) = 2 \;, \quad y'(0) = 1$$

$$L\{y''\} + L\{y\} = L\{\sin 2t\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) - L\{y\} = \frac{2}{s^2+4}$$

$$(s^2+1)\,L\{y\} - 2s - 1 = \frac{s}{s^2+4}$$

$$(s^2+1)\,L\{y\} = \frac{s}{s^2+4} + 1 + 2s = \frac{s + (1+2s)(s^2+4)}{s^2+4}$$

$$L\{y\} = \frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$$

$$2s^3+s^2+8s+6 = (as+b)(s^2+4) + (s^2+1)(cs+d)$$
$$= (a+c)s^3 + (b+d)s^2 + (4a+c)s + (4b+d)$$

$$a+c=2 \;, \quad b+d=1 \;, \quad 4a+c=8 \;, \quad 4b+d=6$$

$$\boxed{a=2} \;, \quad \boxed{b=\frac{5}{3}} \;, \quad \boxed{c=0} \;, \quad \boxed{d=-\frac{2}{3}}$$

$$L\{y\} = \frac{2s+\frac{5}{3}}{s^2+1} + \frac{-\frac{2}{3}}{s^2+4} = \frac{2s}{s^2+1} + \frac{5}{3}\,\frac{1}{s^2+1} - \frac{2}{3}\,\frac{1}{s^2+4}$$

$$y(t) = 2L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{5}{3}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{3}L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$y(t) = \underbrace{2\cos t + \frac{5}{3}\sin t}_{y_h(t)} - \underbrace{\frac{1}{3}\sin 2t}_{y_p(t)}$$

Example: Find the solution of $y^{(4)} - y = 0$

$y(0)=0,\; y'(0)=1,\; y''(0)=0,\; y'''(0)=0$ using laplace transform.

$$L\{y^{(4)}\} - L\{y\} = 0$$

$$s^4 L\{y\} - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0) - L\{y\} = 0$$

$$(s^4-1)\,L\{y\} - 0 - s^2 - 0 - 0 = 0 \iff (s^4-1)\,L\{y\} = s^2$$

$$L\{y\} = \frac{s^2}{s^4-1} = \frac{s^2}{(s^2-1)(s^2+1)} = \frac{as+b}{s^2-1} + \frac{cs+d}{s^2+1}$$

$$s^2 = (as+b)(s^2+1) + (cs+d)(s^2-1)$$

$$s^2 = (a+c)s^3 + (b+d)s^2 + (a-c)s + b-d$$

$$a+c=0, \quad \underline{b+d=1}, \quad a-c=0, \quad \underline{b-d=0}$$

$$\boxed{a=0} \quad \boxed{b=\tfrac{1}{2}} \quad \boxed{c=0} \quad \boxed{d=\tfrac{1}{2}}$$

$$L\{y\} = \frac{\frac{1}{2}}{s^2-1} + \frac{\frac{1}{2}}{s^2+1}$$

$$y(t) = \frac{1}{2} L^{-1}\left(\frac{1}{s^2-1}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{1}{2}\sinh t + \frac{1}{2}\sin t = \frac{\sinh t + \sin t}{2}$$

~~Examples~~

* If $\underline{f(t)=c}$, then $L\{c\} = F(s) = \lim_{b\to\infty} \int_0^b e^{-st} c \, dt = c \lim_{b\to\infty} \frac{-1}{s} e^{-st}\Big|_0^b$

$$= c \lim_{b\to\infty} \left[\frac{1}{s} - \frac{e^{-sb}}{s}\right] = \frac{c}{s}$$

$$\Rightarrow L\{2\} = \frac{2}{s}, \quad L\{-4\} = \frac{-4}{s}, \quad L^{-1}\left(\frac{\sqrt{3}}{s}\right) = \sqrt{3}$$

* If $\underline{f(t)=t}$, then $L\{t\} = F(s) = \lim_{b\to\infty} \int_0^b t e^{-st} dt$

$$= \lim_{b\to\infty} \left[\frac{-t}{s} e^{-st} - \frac{1}{s^2} e^{-st}\right]\Big|_0^b$$

$$= \lim_{b\to\infty} \left[\frac{-b}{s} e^{-sb} - \frac{e^{-sb}}{s^2} + 0 + \frac{1}{s^2}\right] = \frac{1}{s^2}, \quad s>0$$

* If $\underline{f(t)=t^n}$, then $L\{t^n\} = \frac{n!}{s^{n+1}}, \quad s>0$

$$\Rightarrow L\{t^2\} = \frac{2}{t^3}, \quad L\left\{\frac{t^4}{20}\right\} = \frac{1}{20}\frac{4!}{s^5} = \frac{24}{20\, s^5}$$

$$\Rightarrow L\{t^2+1\} = L\{t^2\} + L\{1\} = \frac{2!}{s^3} + \frac{1}{s}$$

Recall that $L\{e^{at}\} = \frac{1}{s-a}$, $s>a$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{\cos at\} = \frac{s}{s^2+a^2}$

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* Now $L\{\sinh at\} = \frac{a}{s^2-a^2}$ because $\sinh at = \frac{e^{at}-e^{-at}}{2}$

$L\{\sinh at\} = \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left(\frac{s+a-s+a}{s^2-a^2}\right)$

$= \frac{a}{s^2-a^2}$

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* $L\{\cosh at\} = \frac{s}{s^2-a^2}$ because $\cosh at = \frac{e^{at}+e^{-at}}{2}$

$L\{\cosh at\} = \frac{1}{2}\left[L\{e^{at}\} + L\{e^{-at}\}\right]$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\,\frac{s+a+s-a}{s^2-a^2} = \frac{s}{s^2-a^2}$

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* If $h(t) = 2\sin 3t - 10t^2 + 5e^{-3t}$, then

$L\{h(t)\} = 2\,\frac{3}{s^2+9} - 10\,\frac{2}{s^3} + 5\,\frac{1}{s+3} = \frac{6}{s^2+9} - \frac{20}{s^3} + \frac{5}{s+3}$

$f(t)$	$F(s)$	$f(t)$	$F(s)$	$f(t)$	$F(s)$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2-a^2}$	$U_c(t)\,f(t-c)$	$e^{-cs}F(s)$
e^{at}	$\frac{1}{s-a}$	$\cosh at$	$\frac{s}{s^2-a^2}$	$y^{(n)}$	$s^n L\{y\} - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{n-1}(0)$
$\sin at$	$\frac{a}{s^2+a^2}$	$U_c(t)$	$\frac{e^{-cs}}{s}$		
$\cos at$	$\frac{s}{s^2+a^2}$	$e^{at}f(t)$	$F(s-a)$		

see table 317

$$\mathcal{L}^{-1}\left(\frac{4s-10}{s^2-6s+10}\right) = \mathcal{L}^{-1}\left(\frac{4s-10}{(s^2-6s+9)+1}\right) = \mathcal{L}^{-1}\left(\frac{4s-12+2}{(s-3)^2+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{4(s-3)+2}{(s-3)^2+1}\right) = 4\mathcal{L}^{-1}\left(\frac{s-3}{(s-3)^2+1}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2+1}\right)$$

$$= 4e^{3t}\cos t + 2e^{3t}\sin t$$

<u>Quize</u> Use laplace transform to solve:

$$y'' - 8y' + 25y = 0 \quad , \quad y(0) = 0 \quad ; \quad y'(0) = 6$$

Take laplace to both sides

$$s^2 L\{y\} - s\,y(0) - y'(0) - 8[s\,L\{y\} - y(0)] + 25\,L\{y\} = 0$$

$$(s^2-8s+25)\,L\{y\} - (s-8)\,y(0) - y'(0) = 0$$

$$(s^2-8s+25)\,L\{y\} = 6 \iff L\{y\} = \frac{6}{s^2-8s+25}$$

$$y(t) = L^{-1}\left(\frac{6}{(s^2-8s+16)+9}\right) = 2L^{-1}\left(\frac{3}{(s-4)^2+9}\right) = 2e^{4t}\sin 3t$$

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* $L^{-1}\left(\frac{10}{s-5}\right) = 10\, L^{-1}\left(\frac{1}{s-5}\right) = 10\, e^{5t}$

• $L^{-1}\left(\frac{2}{[illegible]}\right) = 2$

• $L^{-1}\left(\frac{6}{s^4}\right) = t^3$

* $L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{3} L^{-1}\left(\frac{3}{s^2+3^2}\right) = \frac{1}{3}\sin 3t$

• $L^{-1}\left(\frac{8}{s^3}\right) = 4t^2$

* $L^{-1}\left(\frac{1}{s^2-5s+6}\right) = L^{-1}\left(\frac{1}{(s-2)(s-3)}\right) = L^{-1}\left(\frac{A}{s-2} + \frac{B}{s-3}\right)$

partial fraction $= L^{-1}\left(\frac{-1}{s-2}\right) + L^{-1}\left(\frac{1}{s-3}\right)$

$= -e^{2t} + e^{3t}$

* $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ or $L^{-1}\left(F(s-a)\right) = e^{at} f(t)$

* $L\{e^{3t}\cos 2t\} = \frac{s-3}{(s-3)^2+4}$

• $L^{-1}\left(\frac{4s+1}{s^2+9}\right) = L^{-1}\left(\frac{4s}{[illegible]}\right) + L^{-1}\left(\frac{1}{[illegible]}\right)$

$= 4\cos 3t + \frac{1}{3}\sin 3t$

* $L\{e^{t} t^2\} = \frac{2}{(s-1)^3}$

* $L^{-1}\left(\frac{1}{(s-1)^2+1}\right) = e^{t}\sin t$

• $L^{-1}\left(\frac{4s+1}{s^2-4}\right) = 4\cosh 3t + \frac{1}{3}\sinh 3t$

* $L^{-1}\left(\frac{s+3}{(s+3)^2-4}\right) = e^{-3t}\cosh 2t$

• $L^{-1}\left(\frac{-10}{(s+1)^3}\right) = -5 e^{-t} t^2$

* $L^{-1}\left(\frac{s}{(s-2)^2+9}\right) = L^{-1}\left(\frac{(s-2)+2}{(s-2)^2+9}\right) = L^{-1}\left(\frac{s-2}{(s-2)^2+9}\right) + \frac{2}{3} L^{-1}\left(\frac{3}{(s-2)^2+9}\right)$

$= e^{2t}\cos 3t + \frac{2}{3} e^{2t}\sin 3t$

* $L^{-1}\left(\frac{2s-1}{s^2+2s+5}\right) = L^{-1}\left(\frac{2s}{(s+1)^2+4}\right) - \frac{1}{2} L^{-1}\left(\frac{2}{(s+1)^2+4}\right)$

$= 2 L^{-1}\left(\frac{(s+1)-1}{(s+1)^2+4}\right) - \frac{1}{2} e^{-t}\sin 2t$

$= 2 e^{-t}\cos 2t - e^{-t}\sin 2t - \frac{1}{2} e^{-t}\sin 2t$

$= 2 e^{-t}\cos 2t - \frac{3}{2} e^{-t}\sin 2t.$

Th 6.3.2 • If $F(s) = L\{f(t)\}$ exists for $s > a \geq 0$, and if c is constant, then

$$L\{e^{ct} f(t)\} = F(s-c) \quad , \quad s > a+c$$

• Conversly, if $f(t) = L^{-1}(F(s))$, then

$$e^{ct} f(t) = L^{-1}(F(s-c))$$

Thus, multiply $f(t)$ by e^{at} results in translating $F(s)$ by a distance c in the positive direction of t-axis.

Proof: $L\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t)\, dt = \int_0^\infty e^{-(s-c)t} f(t)\, dt = F(s-c)$

Example: Find the inverse transform of $G(s) = \dfrac{s+1}{s^2+2s+5}$

$G(s) = \dfrac{s+1}{(s+1)^2+4}$. Hence, $f(t) = L^{-1}\{F(s)\} = L^{-1}\left(\dfrac{s}{s^2+4}\right)$

$$= \cos(2t)$$

It follows that:

$$L^{-1}(G(s)) = L^{-1}\{F(s+1)\} = e^{-t} f(t) = e^{-t} \cos 2t$$

$L\{\sin\sqrt{5}t\} = \dfrac{\sqrt{5}}{s^2+5} = F(s)$

$*\quad L\{e^{2t} \sin\sqrt{5}t\} = \dfrac{\sqrt{5}}{(s-2)^2+5}$

$$= F(s-2) \qquad s > 2$$