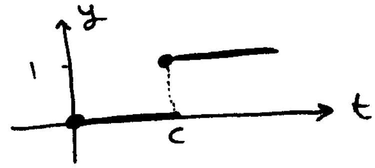
6.3 Step Functions

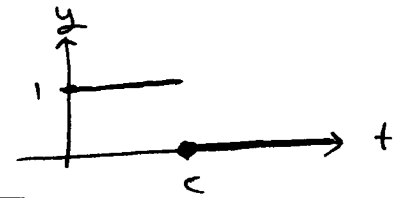
- Let $c \geq 0$. The <u>unit step function</u>, or <u>Heaviside function</u> is defined by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$



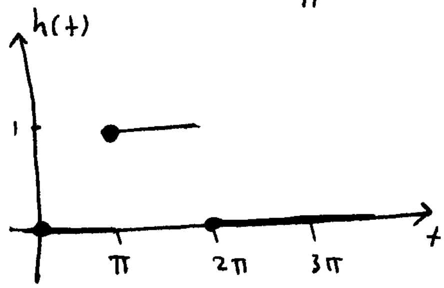
- A negative step can be represented by

$$y(t) = 1 - u_c(t) = \begin{cases} 1, & t < c \\ 0, & t \geq c \end{cases}$$



<u>Example</u>: sketch the graph of $h(t) = u_\pi(t) - u_{2\pi}(t)$, $t \geq 0$

$$h(t) = \begin{cases} 0, & 0 \leq t < \pi \\ 1, & \pi \leq t < 2\pi \\ 0, & 2\pi \leq t < \infty \end{cases}$$



$$u_\pi(t) = \begin{cases} 0, & t < \pi \\ 1, & t \geq \pi \end{cases}$$

$$u_{2\pi}(t) = \begin{cases} 0, & t < 2\pi \\ 1, & t \geq 2\pi \end{cases}$$

<u>Example</u> Express the following function in terms of $u_c(t)$:

$$f(t) = \begin{cases} 2, & 0 \leq t < 1 \\ -1, & 1 \leq t < 2 \\ 2, & 2 \leq t < 3 \\ -1, & 3 \leq t < 4 \\ 0, & 4 \leq t \end{cases}$$

$$f(t) = 2 - 3u_1(t) + 3u_2(t) - 3u_3(t) + u_4(t)$$

$t \geq 0$

<u>Example</u> Find the laplace transform of $u_c(t)$.

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

$$L\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_c^\infty e^{-st}\,dt$$

$$= \lim_{b\to\infty} \int_c^b e^{-st}\,dt = \lim_{b\to\infty} \left[\frac{-1}{s} \cdot e^{-st} \Big|_c^b \right]$$

$$= \lim_{b\to\infty} \left[\frac{-e^{-bs}}{s} + \frac{e^{-cs}}{s} \right]$$

$$= \frac{e^{-cs}}{s}.$$

Example: Find ① $\mathcal{L}\{u_2(t)\} = \frac{e^{-2s}}{s}$

② $\mathcal{L}^{-1}\left(\frac{e^{-3s}}{s}\right) = u_3(t)$

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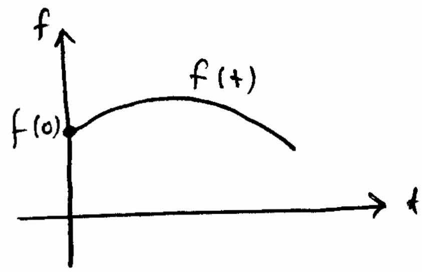
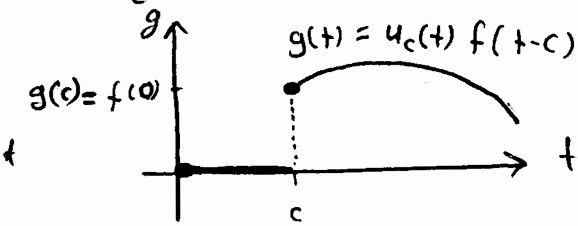
* Given the function $f(t)$ defined for $t \geq 0$.
The translation of f by a distance c in the positive t direction is given by $g(t)$, where

$$g(t) = u_c(t)\, f(t-c) = \begin{cases} 0 & 0 \leq t < c \\ f(t-c), & t \geq c \end{cases}$$

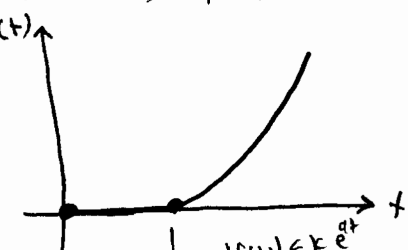




Example: Let $f(t) = t^2$, $t \geq 0$. Sketch the graph of $g(t) = f(t-1)\,u_1(t)$.

Recall that $u_1(t) = \begin{cases} 0, & t < 1 \\ 1, & t \geq 1 \end{cases}$

Thus, $g(t) = \begin{cases} 0, & 0 \leq t < 1 \\ (t-1)^2, & 1 \leq t \end{cases}$



$|f(t)| \leq K e^{at}$

Th 6.3.1 • If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if $c > 0$, then

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs}\, \mathcal{L}\{f(t)\} = e^{-cs} F(s)$$

• Conversely, if $f(t) = \mathcal{L}^{-1}(F(s))$, then

$$u_c(t)\, f(t-c) = \mathcal{L}^{-1}\left(e^{-cs} F(s)\right).$$

Thus, the translation of $f(t)$ a distance c in the positive t direction corresponds to a multiplication of $F(s)$ by e^{-cs}.

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Proof: $L\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt$

$= \int_c^\infty e^{-st} f(t-c)\,dt$

Let $u = t-c$
$du = dt$

$= \int_0^\infty e^{-s(u+c)} f(u)\,du$

$= e^{-cs} \int_0^\infty e^{-su} f(u)\,du = e^{-cs} F(s)$

Example (1) Find the laplace transform of $f(t) = \begin{cases} 0 & , 0 \le t < 1 \\ (t-1)^2 & , t \ge 1 \end{cases}$

Note that $f(t) = (t-1)^2 u_1(t)$

Thus, $L\{f(t)\} = L\{u_1(t)(t-1)^2\} = e^{-s} L\{t^2\} = \frac{2e^{-s}}{s^3}$

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(2) Find the laplace transform of

$$f(t) = \begin{cases} \sin t & , 0 \le t < \frac{\pi}{4} \\ \sin t + \cos(t - \frac{\pi}{4}) & , t \ge \frac{\pi}{4} \end{cases}$$

Note that $f(t) = \sin t + u_{\frac{\pi}{4}}(t) \cos(t - \frac{\pi}{4})$, $t \ge 0$

Thus, $L\{f(t)\} = L\{\sin t\} + L\{u_{\frac{\pi}{4}}(t) \cos(t - \frac{\pi}{4})\}$

$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} L\{\cos t\}$

$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} \frac{s}{s^2+1} = \frac{1 + s e^{-\frac{\pi}{4}s}}{s^2+1}$

Example Find the inverse transform of $F(s) = \frac{3 + e^{-7s}}{s^4}$

$f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{3 + e^{-7s}}{s^4}\right) = L^{-1}\left(\frac{3}{s^4}\right) + L^{-1}\left(\frac{e^{-7s}}{s^4}\right)$

$= \frac{1}{2} L^{-1}\left(\frac{3!}{s^4}\right) + \frac{1}{6} L^{-1}\left\{e^{-7s} \frac{3!}{s^4}\right\}$

$= \frac{t^3}{2} + \frac{1}{6} u_7(t)(t-7)^3 = \begin{cases} \frac{t^3}{3} & , 0 \le t < 7 \\ \frac{t^3}{3} + \frac{1}{6}(t-7)^3 & , 7 \le t \end{cases}$

* Recall that if $L\{f(t)\} = F(s)$ then

1 <u>First shifting</u> is

$$L\{e^{at} f(t)\} = F(s-a)$$

$$e^{at} f(t) = L^{-1}\left(F(s-a)\right)$$

- $L\{e^{2t}\sin 5t\} = \frac{5}{(s-2)^2+25}$
- $L^{-1}\left(\frac{3}{(s-1)^3}\right) = \frac{3}{2} L^{-1}\left(\frac{2}{(s-1)^3}\right) = \frac{3}{2} e^{t} t^2$

2 <u>Second shifting</u> is

$$L\{u_c(t) f(t-c)\} = e^{-cs} L\{f(t)\} = e^{-cs} F(s)$$

$$u_c(t) f(t-c) = L^{-1}\left(e^{-cs} F(s)\right)$$

<u>Example</u> ① $L\{u_2(t)(t-2)\} = e^{-2s} L\{t\} = e^{-2s}\frac{1}{s^2} = \frac{e^{-2s}}{s^2}$

② $L\{u_2(t)(t-1)\} = L\{u_2(t)(t-2) + u_2(t)\}$

$= L\{u_2(t)(t-2)\} + L\{u_2(t)\}$

$= \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s}$

③ Find $L\{f(t)\}$ where $f(t) = \begin{cases} 0 & , \ 0 \le t < 1 \\ t-1 & , \ 1 \le t \le 2 \\ 1 & \ \ t \ge 2 \end{cases}$

$f(t) = (t-1)\left(u_1(t) - u_2(t)\right) + u_2(t)$

$= (t-1)u_1(t) - u_2(t)[t-1-1]$

$= (t-1)u_1(t) - u_2(t)(t-2)$

$L\{f(t)\} = L\{u_1(t)(t-1)\} - L\{u_2(t)(t-2)\}$

$= e^{-s} L\{t\} - e^{-2s} L\{t\}$

$= \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$

④ $L\{u_2(t)t^2\} = L\{u_2(t)(t-2+2)^2\} = L\{u_2(t)\left((t-2)^2 + 4(t-2) + 4\right)\}$

$= L\{u_2(t)(t-2)^2\} + 4L\{u_2(t)(t-2)\} + 4L\{u_2(t)\}$

$= \frac{2e^{-2s}}{s^3} + 4\frac{e^{-2s}}{s^2} + 4\frac{e^{-2s}}{s}$