6.4 Differential Equations with Discontinuous Forcing Functions

In this section we study the solution of nonhomogenous IVP in which the forcing function "g(t)" is discontinuous.

Example: Find the solution of the IVP

$$2y'' + y' + 2y = g(t) \quad , \quad y(0) = 0, \; y'(0) = 0 \quad \text{--}(B)$$

$$\text{where } \; g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \leq t < 20 \\ 0, & 0 \leq t < 5 \text{ and } t \geq 20 \end{cases}$$

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \{u_5(t)\} - \mathcal{L}\{u_{20}(t)\}$$

$$2\left[s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)\right] + s\,\mathcal{L}\{y\} - y(0) + 2\,\mathcal{L}\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$(2s^2 + s + 2)\,\mathcal{L}\{y\} = \frac{e^{-5s} - e^{-20s}}{s} \qquad \text{"using the initial conditions"}$$

$$\mathcal{L}\{y\} = \left(e^{-5s} - e^{-20s}\right)H(s) \quad , \text{ where } \; H(s) = \frac{1}{s(2s^2+s+2)}$$

• If $h(t) = \mathcal{L}^{-1}\left(H(s)\right)$, then the solution is

$$y(t) = \phi(t) = u_5(t)\,h(t-5) - u_{20}(t)\,h(t-20) \qquad \text{by Th 6.3.1.}$$

• $h(t) = \mathcal{L}^{-1}\left(\dfrac{1}{s(2s^2+s+2)}\right)$

see (*) on the back:

$$\frac{1}{s(2s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{2s^2+s+2}$$

$$1 = (2A+B)s^2 + (A+C)s + 2A$$

$$A = \frac{1}{2} \; , \quad B = -1 \; , \quad C = \frac{-1}{2}$$

$$= \mathcal{L}^{-1}\left(\frac{\frac{1}{2}}{s} - \frac{s+\frac{1}{2}}{2s^2+s+2}\right)$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{s+\frac{1}{2}}{s^2+\frac{s}{2}+1}\right)$$

$$= \frac{1}{2}(1) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{s+\frac{1}{4}+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right)$$

⊛ Partial Fraction :

Factor in denominator (المقام)	Term in partial Fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^K$	$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_K}{(ax+b)^K}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^K$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \dfrac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \dfrac{A_Kx+B_K}{(ax^2+bx+c)^K}$

$$h(t) = \frac{1}{2} - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right) - \frac{1}{2\sqrt{15}}\mathcal{L}^{-1}\left(\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right)$$ *

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-\frac{t}{4}}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{2\sqrt{15}}e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

- Thus, the solution becomes $\phi(t) = u_5(t)\,h(t-5) - u_{20}(t)\,h(t-20)$. (2)
where $h(t)$ is as given above.

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*Laplace transform methods provide a much more convenient approach to solve discontinuous forcing functions.

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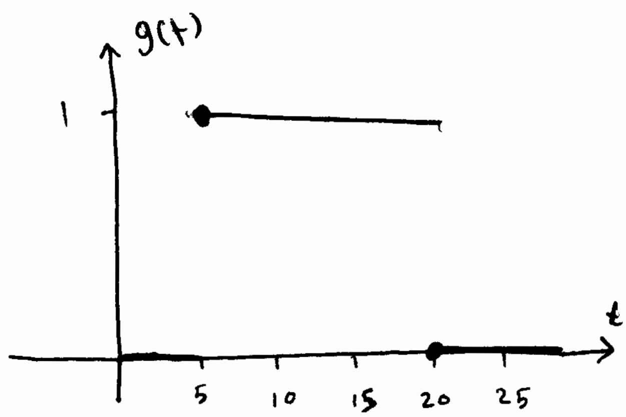
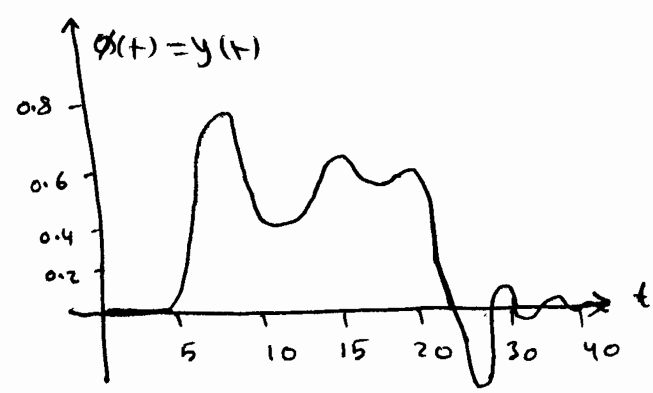
* The solution of the IVP (A) can be seen as a composite of 3 separate solutions to 3 separate IVPs:

$\boxed{1}$ $0 \le t < 5$: $2y_1'' + y_1' + 2y_1 = 0$, $y_1(0) = 0$, $y_1'(0) = 0$

$\boxed{2}$ $5 < t < 20$: $2y_2'' + y_2' + 2y_2 = 1$, $y_2(5) = 0$, $y_2'(5) = 0$
$\qquad y_2(5) = y_1(5)$, $y_2'(5) = y_1'(5)$

$\boxed{3}$ $t > 20$: $2y_3'' + y_3' + 2y_3 = 0$, $y_3(20) \cong 0.5$, $y_3'(20) \simeq 0.01$
$\qquad y_3(20) = y_2(20)$, $y_3'(20) = y_2'(20)$





$\boxed{1}$ $2r^2 + r + 2 = 0 \iff r_{1,2} = -\frac{1}{4} \pm \frac{\sqrt{15}}{4}i$

$\iff y_1(t) = c_1 e^{-\frac{t}{4}}\cos\left(\frac{\sqrt{15}}{4}t\right) + c_2 e^{-\frac{t}{4}}\sin\left(\frac{\sqrt{15}}{4}t\right)$

Using $y_1(0) = 0 \iff \boxed{c_1 = 0}$ and $y_1'(0) = 0 \iff \boxed{c_2 = 0}$

Hence $y_1(t) = 0$. The system is initially at rest, and since there is no external forcing, it remains at rest.

$y_2(5) = y_1(5) = 0$

$y_2'(5) = y_1'(5) = 0$ since $y_1'(t) = 0$

$\boxed{2}$ $y_2(t) = y_h(t) + y_p(t)$ since $2y'' + y' + 2y = 1$ $y_2(5) = 0$

$y_2'(5) = 0$

$y_2(t) = c_1 e^{-t/4} \cos\left(\frac{\sqrt{15}}{4} t\right) + c_2 e^{-t/4} \sin\left(\frac{\sqrt{15}}{4} t\right) + \frac{1}{2}$

Physically, the system responds with the sum of a constant (the response to the constant forcing function) and a damped oscillation over the time interval $(5, 20)$.

$\boxed{3}$ $y_3(t) = c_1 e^{-t/4} \cos\left(\frac{\sqrt{15}}{4} t\right) + c_2 e^{-t/4} \sin\left(\frac{\sqrt{15}}{4} t\right)$

Physically since there is no external forcing, the response is a damped oscillation about $y=0$ for $t > 20$.

What is the effect of discontinuity of the forcing function $g(t)$ on the solution $y = \phi(t)$ ②?

- It can be shown that ϕ and ϕ' are continuous at $t=5$ and $t=20$.

But ϕ'' has a jump of $\frac{1}{2}$ at $t=5$ and jump of $\frac{-1}{2}$ at $t=20$:

$\lim_{t \to 5^-} \phi''(t) = 0$ and $\lim_{t \to 5^+} \phi''(t) = \frac{1}{2}$ $\lim_{t \to 20^-} \phi''(t) \approx 0$ and $\lim_{t \to 20^+} \phi''(t) \approx -\frac{1}{2}$

- Thus, jump in the forcing term $g(t)$ at these points is balanced by a corresponding jump in the highest order term $2y''$ in the ODE.

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* Smoothness of solution in General ③

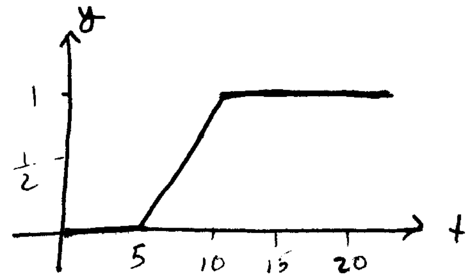
Consider a general 2^{nd} order linear equation: $y'' + p(t) y' + q(t) y = g(t)$ where p and q are continuous on some interval (a, b) but g is only piecewise continuous there.

- If $y = \psi(t)$ is a solution of ③, then ψ and ψ' are continuous on (a, b) but ψ'' has jump discontinuities at the same points as g.
- Similarly for higher order equations, where the highest derivative of the solution has jump discontinuities at the same points as the forcing function, but the solution itself and its lower derivatives are continuous over (a, b).

Example: Solve the IVP $y'' + 4y = g(t)$, $y(0) = 0$, $y'(0) = 0$

where $g(t) = \frac{t-5}{5} u_5(t) - \frac{t-10}{5} u_{10}(t) = \begin{cases} 0 & , 0 \le t < 5 \\ \frac{t-5}{5} & , 5 \le t < 10 \\ 1 & , t \ge 10 \end{cases}$

The graph of the forcing function $g(t)$ is know as ramp loading (منحدر تحميل)



$$\mathcal{L}\{y''\} + 4\,\mathcal{L}\{y\} = \frac{1}{5}\mathcal{L}\{u_5(t)(t-5)\} - \frac{1}{5}\mathcal{L}\{u_{10}(t)\,(t-10)\}$$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + 4\,\mathcal{L}\{y\} = \frac{1}{5}\frac{e^{-5s}}{s^2} - \frac{1}{5}\frac{e^{-10s}}{s^2}$$

$$(s^2+4)\,\mathcal{L}\{y\} = \frac{e^{-5s} - e^{-10s}}{5s^2} \iff \mathcal{L}\{y\} = \frac{e^{-5s} - e^{-10s}}{5}\,\frac{1}{s^2(s^2+4)}$$

$$\iff \mathcal{L}\{y\} = \frac{e^{-5s} - e^{-10s}}{5}\,H(s)$$

- If $h(t) = \mathcal{L}^{-1}(H(s))$, then the solution is

$$y(t) = \phi(t) = \frac{1}{5}\left[u_5(t)\,h(t-5) - u_{10}(t)\,h(t-10)\right] \quad \text{by Th 6.3.1.}$$

- $h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+4)}\right) = \mathcal{L}^{-1}\left(\frac{\frac{1}{4}}{s^2} - \frac{\frac{1}{4}}{s^2+4}\right)$

$$= \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{8}\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)$$

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

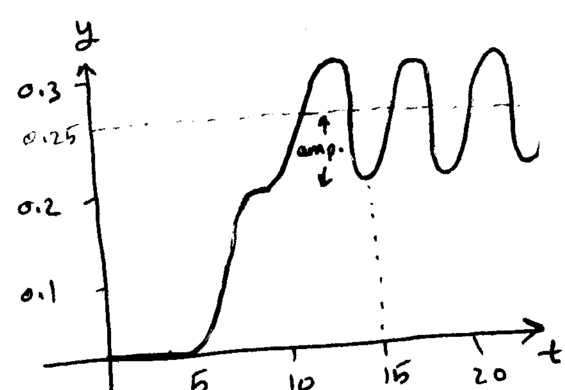
$$1 = (A+C)s^3 + (B+D)s^2 + 4As + 4B$$

$$A = 0, \quad B = \frac{1}{4}, \quad C = 0, \quad D = \frac{-1}{4}$$

$$h(t) = \frac{t}{4} - \frac{\sin 2t}{8}$$

- Thus, the solution to the IVP above is

$$* \begin{cases} \phi(t) = \frac{1}{5}\left[u_5(t)\,h(t-5) - u_{10}\,h(t-10)\right] \\ \text{where } h(t) = \frac{t}{4} - \frac{\sin(2t)}{8} \end{cases}$$



* The solution of the IVP above can be seen as a composite of 3 separate solutions to 3 separate IVPs :

1. $0 \le t < 5$: $y_1'' + 4y_1 = 0$, $y_1(0) = 0$, $y_1'(0) = 0$

2. $5 < t < 10$: $y_2'' + 4y_2 = \frac{t-5}{5}$, $y_2(5) = 0$, $y_2'(5) = 0$

3. $10 < t$: $y_3'' + 4y_3 = 1$, $y_3(10) = y_2(10)$, $y_3'(10) = y_2'(10)$

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1. $r^2 + 4 = 0 \Leftrightarrow r_{1,2} = \pm 2i \Leftrightarrow y_1(t) = c_1 \cos 2t + c_2 \sin 2t$

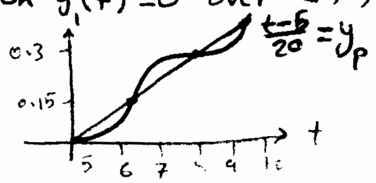
$y_1(0) = \boxed{0 = c_1}$, $y_1'(0) = 0 \Leftrightarrow \boxed{c_2 = 0}$

$\Leftrightarrow y_1(t) = 0$

$y_2(5) = y_1(5) = 0$

$y_2'(5) = y_1'(5) = 0$

The system is initially at rest, and since there is no external forcing, it remains at rest. Thus, the solution $y_1(t) = 0$ over $[0,5)$.



2. $y_2(t) = y_h(t) + y_p(t)$

$y_2(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{t}{20} - \frac{1}{4}$

The solution is an oscillation about the line $\frac{t-5}{20}$ over $(5,10)$.

3. $y_3(t) = y_h(t) + y_p(t)$

$y_3(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{4}$

The solution is an oscillation about $y_p = \frac{1}{4}$ for $\underline{\underline{t > 10}}$ (see the figure)

- Recall the solution *. To find the amplitude of the oscillation part for $t > 10$, we locate one of the Max or Min points by solving $y' = \phi(t) = 0$. Thus, the first max is $(10.642, 0.298)$. Hence, the amplitude of the oscillation is 0.0479.

- Note that in this example the forcing function g is continuous but g' is discontinu at $t = 5$ and $t = 10$.
- It follows that ϕ and its first two derivatives are continuous everywhere, but ϕ''' has discontinuities at $t = 5$ and $t = 10$ that match the discontinuities of g' at $t = 5$ and $t = 10$.