6.5 Impulse Functions

- In Some applications it is necessary to deal with impulsive nature, for example, voltages or forces of large magnitude that act over very short time intervals. Such problems lead to DEs' of the form

$$ay'' + by' + cy = g(t) \quad , \text{ where}$$

$$g(t) = \begin{cases} \text{Large} , & t_0 - \tau < t < t_0 + \tau \\ 0 , & \text{otherwise} \end{cases}$$

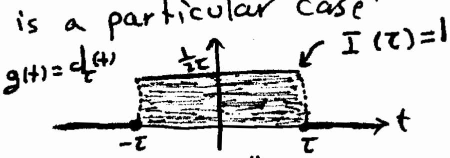
- To measure the strength of the forcing function $g(t)$, we define the integral $I(\tau)$ by

$$I(\tau) = \int_{-\infty}^{\infty} g(t)\, dt = \int_{t_0-\tau}^{t_0+\tau} g(t)\, dt .$$

$\Rightarrow$ In mechanical system, $I(\tau)$ is the total impulse of the force $g(t)$ over the time interval $(t_0-\tau, t_0+\tau)$.

$\Rightarrow$ For example: if y is the current in an electric circuit and $g(t)$ is the time derivative of the voltage, then $I(\tau)$ represents the total voltage impressed on the circuit during the interval $(t_0-\tau, t_0+\tau)$.

- Let $t_0 = 0$ and $g(t)$ be defined by " this is a particular case"



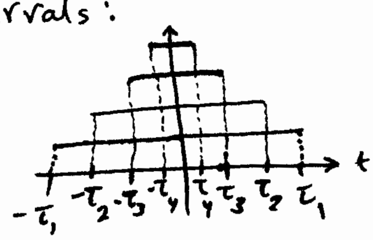
$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} & , -\tau < t < \tau \\ 0 & , \text{otherwise} \quad \text{" } t \le -\tau \text{ or } t \ge \tau \text{ "} \end{cases}$$

where τ is a small positive constant. It follows that :

$$\int_{-\infty}^{\infty} d_\tau(t)\,dt = I(\tau) = \int_{-\tau}^{\tau} \frac{dt}{2\tau} = \frac{1}{2\tau}(\tau + \tau) = 1 \quad \text{"independent of } \tau \text{"} \quad *^1$$

- We need to idealize the forcing function $g(t) = d_\tau(t)$ by making it to act over shorter and shorter time intervals :

$$\lim_{\tau \to 0} d_\tau(t) = 0 \quad , t \neq 0 \qquad *^2$$



Since $I(\tau) = 1 \quad \forall \tau \neq 0$, it follows that

$$_{\tau \to 0} I(\tau) = 1 \qquad *^{3}$$

- Using $*^{1}, *^{2}, *^{3}$, we defin an idealized unit impulse function δ to be a function with the following properities:

$$\delta(t-0) = \delta(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

- Note that the unit impulse function is an example of Dirac delta function: $\delta(t-c) = \begin{cases} 1 & \text{if } t = c \\ 0 & \text{if } t \neq c \end{cases}$

- Since $\delta(t)$ corresponds to aunit impulse at $t=0$, it follows that a unit impulse at an arbitrary point $t = t_0$ is

$$\delta(t-t_0) = \begin{cases} 1 & \text{if } t = t_0 \\ 0 & \text{if } t \neq t_0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-t_0)\, dt = 1.$$

- Note that $\delta(t) = \lim_{\tau \to 0} d_\tau(t)$, $t \neq 0$. Hence,

$$\delta(t-t_0) = \lim_{\tau \to 0} d_\tau(t-t_0), \quad t \neq t_0$$

" we will assume that $t_0 > 0$ " so $t_0 - \tau > 0$ since $\tau \to 0$

$$\mathcal{L}\{\delta(t-t_0)\} = \lim_{\tau \to 0} \mathcal{L}\{d_\tau(t-t_0)\} = \lim_{\tau \to 0} \int_0^{\infty} e^{-st} d_\tau(t-t_0)\, dt$$

$$= \lim_{\tau \to 0} \int_{t_0-\tau}^{t_0+\tau} e^{-st} d_\tau(t-t_0)\, dt = \lim_{\tau \to 0} \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} e^{-st}\, dt$$

$$= \lim_{\tau \to 0} \left. \frac{-e^{-st}}{2\tau s} \right|_{t_0-\tau}^{t_0+\tau} = \lim_{\tau \to 0} \frac{e^{-st_0}}{s\tau} \left(\frac{e^{s\tau} - e^{-s\tau}}{2} \right)$$

$$= e^{-st_0} \lim_{\tau \to 0} \frac{\sinh s\tau}{s\tau} = e^{-st_0} \lim_{\tau \to 0} \frac{s \cosh s\tau}{s} = e^{-st_0}$$

- Hence, $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$ for any $t_0 > 0$. Thus, $\mathcal{L}\{\delta(t)\} = 1$

- Now we can use this to define the integral of the product of the delta function and any continuous function f by:

$$\int_{-\infty}^{\infty} \delta(t-t_0)\, f(t)\,dt = \lim_{\tau\to 0}\int_{-\infty}^{\infty} d_\tau(t-t_0)\, f(t)\,dt = \lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)\,dt$$

<u>MVT</u>: If f is cont. on $[a,b]$ then at some point $c\in[a,b]$: $f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$

$$= \lim_{\tau\to 0}\frac{1}{2\tau}\left(2\tau\, f(t^*)\right) \quad \text{by MVT, where } t^*\in(t_0-\tau,\, t_0+\tau)$$

$$= \lim_{\tau\to 0} f(t^*) = f(t_0) \quad \text{since as } \tau\to 0 \Rightarrow t^*\to t_0.$$

* Now we can use delta function in solving IVP with impulsive forcing function.

<u>Example</u>: Find the solution of the IVP

$$2y'' + y' + 2y = \delta(t-5), \quad y(0)=0, \quad y'(0)=0$$

$$2\,L\{y''\} + L\{y'\} + 2\,L\{y\} = L\{\delta(t-5)\}$$

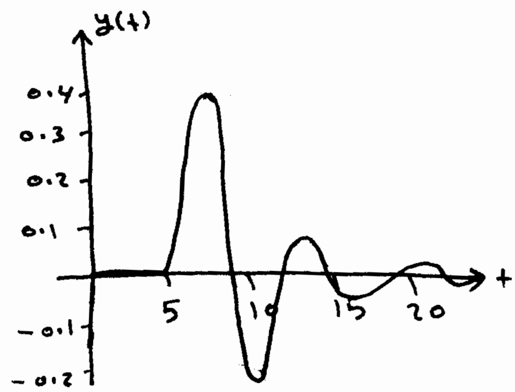
$$2\left[s^2 L\{y\} - s\,y(0) - y(0)\right] + s\,L\{y\} - y(0) + 2\,L\{y\} = e^{-5s}$$

$$(2s^2+s+2)\,L\{y\} = e^{-5s} \iff L\{y\} = \frac{e^{-5s}}{2s^2+s+2}$$

$$y(t) = L^{-1}\left(\frac{\frac{1}{2}e^{-5s}}{(s+\frac{1}{4})^2+\frac{15}{16}}\right) = \frac{1}{2}\,\frac{4}{\sqrt{15}}\,L^{-1}\left(\frac{\frac{\sqrt{15}}{4}e^{-5s}}{(s+\frac{1}{4})^2+\frac{15}{16}}\right)$$

$$y(t) = \frac{2}{\sqrt{15}}\,u_5(t)\,e^{-\frac{(t-5)}{4}}\sin\frac{\sqrt{15}}{4}(t-5)$$

$$= \begin{cases} 0, & t<5 \\ \frac{2}{\sqrt{15}}\,e^{-\frac{(t-5)}{4}}\sin\frac{\sqrt{15}}{4}(t-5), & t\geq 5 \end{cases}$$



- There is no external excitation until $t=5$. There is no response in the interval $0<t<5$.
- The impulse at $t=5$ produces a decaying oscillation that persists indefinitely
- The response is continuous at $t=5$ despite the singularity in the forcing function.
- The 1st derivative of the solution has a jump discontinuity at $t=5$ and the 2nd 〃 〃 〃 〃 〃 infinite 〃 at $t=5$.