6.6 The Convolution Integral

* Let $f(t) = 1$ and $g(t) = \sin t$

  $L\{f(t)\} = \frac{1}{s}$ and $L\{g(t)\} = \frac{1}{s^2+1}$

  Now $L\{f(t)\, g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$

  $L\{f(t)\}\, L\{g(t)\} = \frac{1}{s} \, \frac{1}{s^2+1} = \frac{1}{s(s^2+1)}$

  Thus for these function, it follows that $L\{f(t)\, g(t)\} \neq L\{f(t)\} L\{g\}$

* However, sometimes it is possible to write a Laplace transform $H(s) = F(s)\, G(s)$, where $F(s) = L\{f(t)\}$ and $G(s) = L\{g(t)\}$

  That is $H(s) = F(s)\, G(s) = L\{f(t)\}\, L\{g(t)\} = L\{f(t)\, g(t)\}$?

Th 6.6.1 Suppose $F(s) = L\{f(t)\}$ and $G(s) = L\{g(t)\}$ both exist for $s > a \geq 0$. Then $H(s) = F(s)\, G(s) = L\{h(t)\}$ for $s > a$

where $h(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$.

- The function $h(t)$ is called the convolution (التفاف) of f and g and the integrals above are called the convolution integrals.
- The equality of the two convolution integrals above can be seen by making change of variables $u = t - \tau$.
- The convolution integral defines a generalized product and can be written as $h(t) = (f * g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$.

Proof $F(s)\, G(s) = \int_0^\infty e^{-su} f(u)\, du \int_0^\infty e^{-s\tau} g(\tau)\, d\tau$

$$= \int_0^\infty g(\tau)\, d\tau \int_0^\infty e^{-s(\tau+u)} f(u)\, du$$

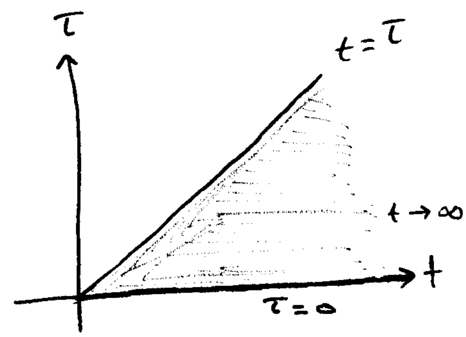
$$= \int_0^\infty g(\tau)\, d\tau \int_\tau^\infty e^{-st} f(t-\tau)\, dt \qquad \boxed{t = \tau + u}$$

$$F(s)\;G(s) = \int_0^{\infty}\int_{\tau}^{\infty} e^{-st} g(\tau)\, f(t-\tau)\, dt\, d\tau$$

Fubini Th. $$= \int_0^{\infty}\int_0^{t} e^{-st} f(t-\tau)\, g(\tau)\, d\tau\, dt$$

$$= \int_0^{\infty} e^{-st} \int_0^{t} f(t-\tau)\, g(\tau)\, d\tau\, dt$$

$$= L\{h(t)\}$$



③ $u_3(t) * t^2$

Example: Find the laplace transform of ① $h(t) = \int_0^t (t-\tau)\sin 2\tau\, d\tau$

$$L\{h(t)\} = L\{f(t)\}\; L\{g(t)\}$$
$$= L\{t\}\; L\{\sin 2t\} = \frac{1}{s^2}\;\frac{2}{s^2+4} = \frac{2}{s^2(s^2+4)}$$

② $r(t) = \int_0^t (t-\tau)^2 \cos\tau\, d\tau$

$$L\{r(t)\} = L\{f(t)\}\; L\{g(t)\}$$
$$= L\{t^2\}\; L\{\cos t\} = \frac{2}{s^3}\;\frac{s}{s^2+1} = \frac{2}{s^2(s^2+1)}$$

Example: Find the inverse laplase Transform of $H(s) = \frac{2}{s^2(s-2)}$

$$H(s) = 2\left(\frac{1}{s^2}\right)\left(\frac{1}{s-2}\right) = 2\,F(s)\,G(s)$$

Note that $t = f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^2}\right)$

$e^{2t} = g(t) = L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s-2}\right)$

$\frac{2}{(s-2)s^2} = \frac{A}{s-2} + \frac{Bs+C}{s^2}$ or

$A = \frac{1}{2}$ $B = -\frac{1}{2}$ $C = -1$

$L^{-1}\left[\frac{\frac{1}{2}}{s-2}\right] - L^{-1}\left(\frac{\frac{1}{2}s+1}{s^2}\right)$

$\frac{1}{2}e^{2t} - \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s^2}\right) = \frac{e^{2t}}{2} - \frac{1}{2} - t$

Thus, By Th 6.6.1 $\Rightarrow$

$$L^{-1}(H(s)) = h(t) = 2\int_0^t f(t-\tau)\, g(\tau)\, d\tau = 2\int_0^t (t-\tau)\, e^{2\tau}\, d\tau = \frac{e^{2t}-1}{2} - t$$

$$= 2t\int_0^t e^{2\tau} d\tau - 2\int_0^t \tau e^{2\tau} d\tau = t e^{2\tau}\Big|_0^t - \left[\tau e^{2\tau}\Big|_0^t - \int_0^t e^{2\tau} d\tau\right]$$

$$= t(e^{2t}-1) - \left[t e^{2t} - \tfrac{1}{2}(e^{2t}-1)\right] = t e^{2t} - t - t e^{2t} + \tfrac{1}{2}e^{2t} - \tfrac{1}{2} = \frac{e^{2t}-1}{2} - t.$$

Example: Solve the IVP $y'' + 4y = g(t)$, $y(0) = 3$, $y'(0) = -1$

$$L\{y''\} + 4\,L\{y\} = L\{g(t)\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) + 4\,L\{y\} = G(s) \iff (s^2+4)\,L\{y\} - 3s + 1 = G(s)$$

$$\iff L\{y\} = \overset{\Phi(s)}{\frac{3s-1}{s^2+4}} + \overset{\Psi(s)}{\frac{G(s)}{s^2+4}} = 3\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\left(\frac{2}{s^2+4}\right)G(s)$$

$$y(t) = 3\,L^{-1}\left(\frac{s}{s^2+4}\right) - \frac{1}{2}\,L^{-1}\left(\frac{2}{s^2+4}\right) + \frac{1}{2}\,L^{-1}\left(\frac{2}{s^2+4}\,G(s)\right)$$

$$y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)\,g(\tau)\,d\tau$$

Note that if $g(t)$ is given, then the convolution integral can be evaluated.

- $\phi(t) = L^{-1}\{\Phi(s)\} = 3\cos 2t - \frac{1}{2}\sin 2t$ solves the homogeneous IVP
  $$y'' + 4y = 0 , \quad y(0) = 3 , \quad y'(0) = -1$$
- $\psi(t) = L^{-1}\{\Psi(s)\} = \frac{1}{2}\int_0^t \sin 2(t-\tau)\,g(\tau)\,d\tau$ solves the nonhomogeneous IVP
  $$y'' + 4y = g(t) , \quad y(0) = 0 , \quad y'(0) = 0$$
- $$\Psi(s) = \frac{G(s)}{s^2+4} = \frac{1}{s^2+4}\,G(s) = H(s)\,G(s)$$
  - The function $H(s) = \dfrac{1}{s^2+4}$ is known as the **transfer function** and depends only on the system coefficients.
  - The function $G(s)$ depends only on the external excitation $g(t)$ applied to the system.

Example (Inpute - Output Problem) Consider the problem

$$a y'' + b y' + c y = g(t) , \quad y(0) = y_0 , \quad y'(0) = y'_0 , \quad a, b, c \in \mathbb{R} \text{ and}$$

g is given function.

- The coefficients a, b, c describe the properities of some physical system.
- $g(t)$ is the input to the system
- y_0 and y'_0 describe the initial state
- The solution $y(t)$ is the output at time t
- The IVP above is called the input-output problem.

$$a\,L\{y''\} + b\{y'\} + c\,L\{y\} = L\{g(t)\}$$

$$a\left[s^2 L\{y\} - s\,y(0) - y'(0)\right] + b\left[s L\{y\} - y(0)\right] + c\,L\{y\} = G(s)$$

$$(as^2 + bs + c)\,L\{y\} - (as+b)y_0 - a y_0' = G(s)$$

$$Y(s) = L\{y\} = \frac{(as+b)y_0 + a y_0'}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c} = \Phi(s) + \Psi(s)$$

$$y(t) = \phi(t) + \psi(t) \quad \text{, where } \phi(t) = L^{-1}\left(\Phi(s)\right) \text{ and } \psi(t) = L^{-1}\left(\Psi(s)\right)$$

- Note that $\phi(t)$ is the solution of the IVP

$$ay'' + by' + cy = 0 \quad , \quad y(0) = y_0 \, , \; y'(0) = y_0'$$

- Note also that $\psi(t)$ is the solution of the IVP

$$ay'' + by' + cy = g(t) \, , \quad y(0) = 0 \, , \; y'(0) = 0$$

- Once a, b, c are given, we can find $\phi(t) = L^{-1}\left(\Phi(s)\right)$
- To find $\psi(t)$, we write $\Psi(s) = \dfrac{1}{as^2+bs+c}\, G(s) = H(s)\, G(s)$

Hence, $\psi(t) = L^{-1}\left(\Psi(s)\right)$, where $H(s) = \dfrac{1}{as^2+bs+c}$ is the transfer function. Note that $H(s)$ depends only on the coefficients a, b, c.

- $G(s)$ depends on the external excitation $g(t)$ applied to the system.

$$\psi(t) = L^{-1}\left(\Psi(s)\right) = L^{-1}\left(H(s)\, G(s)\right) = \int_0^t h(t-\tau)\, g(\tau)\, d\tau \text{ , where } h(t) = L^{-1}\left(H(s)\right)$$

Example: Take $G(s) = 1$. Hence, $g(t) = L^{-1}\left(G(s)\right) = L^{-1}(1) = \delta(t)$

since $L\{\delta(t-c)\} = e^{-cs}$

and $\Psi(s) = H(s)$.

This means $h(t)$ is the solution of the IVP $ay'' + by' + cy = \delta(t)$, $y(0) = 0$, $y'(0) = 0$

Thus, $h(t)$ is the response of the system to a unit impulse applied at $t = 0$. $h(t)$ is also called the impulse response of the system.

- $\psi(t) = \int_0^t h(t-\tau)\, g(\tau)\, d\tau = \int_0^t h(t-\tau)\, \delta(\tau)\, d\tau$ is the convolution of the impulse response $h(t)$ and the forcing function $\delta(t)$.

$$\delta(\tau) = \begin{cases} 1 & \text{if } \tau = 0 \\ 0 & \text{if } \tau \neq 0 \end{cases}$$