$\boxed{7.5}$ Homogeneous Linear Systems with Constant Coefficients

We study the solution of homogeneous linear equation given by the system $\boxed{\vec{x}\,' = \underset{n\times n}{A}\,\vec{x}}$ * where A is $n\times n$ matrix of real numbers only with $|A| \neq 0$, so $\vec{x} = \vec{0}$ is eq. solution.

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The Question • is the eq. solution $\vec{x} = \vec{0}$ is asymptotically stable or unstable?
• what happens to the solution as $t \to \infty$.

As before, we assume exponential solution $\vec{x} = \vec{\xi} e^{rt}$ and substitute it in *

$$r\vec{\xi} e^{rt} = A\vec{\xi} e^{rt}$$

$$r\vec{\xi} = A\vec{\xi}$$

$$(A - rI)\vec{\xi} = \vec{0}$$

where • the exponent r and the vector $\vec{\xi}$ are to be determined
r: is the eigenvalue and
$\vec{\xi}$: is the coresspoding eigenvector.
• I is $n\times n$ identity matrix

Real Eigenvalues:

Example: Find the general solution of $\vec{x}\,' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{x}$ and $x_1' = x_1 + x_2$, $x_2' = 4x_1 + x_2$

describe the behavior of the solution.

Assume exponential solution $\vec{x} = \vec{\xi} e^{rt}$, and substitute it above to solve the algebraic equation $(A - rI)\vec{\xi} = 0$

$$\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This equation has non trivial solution iff $\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0 \iff$ characteristic equation

$(1-r)(1-r) - 4 = 0 \iff (1-r)^2 - 4 = 0 \iff \boxed{r^2 - 2r - 3 = 0} \iff r_1 = 3$ and $r_2 = -1$

• To find the eigenvector corresponding to the eigenvalue $r_1 = 3$:

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad -2\xi_1 + \xi_2 = 0 \iff \xi_2 = 2\xi_1$$

so the eigenvector $\vec{\xi}^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

- To find the eigenvector corresponding to the eigenvalue $r_2 = -1$:

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad 2\xi_1 + \xi_2 = 0 \iff \xi_2 = -2\xi_1$$

so the eigenvector $\vec{\xi}^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

- The corresponding solutions of the DE above are

$$\vec{X}^{(1)}(t) = \vec{\xi}^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$$

$$\vec{X}^{(2)}(t) = \vec{\xi}^{(2)} e^{r_2 t} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

- Note that $\vec{X}^{(1)}(t)$ and $\vec{X}^{(2)}(t)$ form a fundamental set of solutions since

$$W[\vec{X}^{(1)}(t), \vec{X}^{(2)}(t)](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -2e^{2t} - 2e^{2t} = -4e^{2t} \neq 0.$$

- The general solution is $\vec{X} = c_1 \vec{X}^{(1)}(t) + c_2 \vec{X}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

$\boxed{a}$ $\vec{X} = c_1 \vec{X}^{(1)}(t)$

$$\left.\begin{aligned} X_1 &= c_1 e^{3t} \\ X_2 &= 2c_1 e^{3t} \end{aligned}\right\} \boxed{X_2 = 2X_1}$$

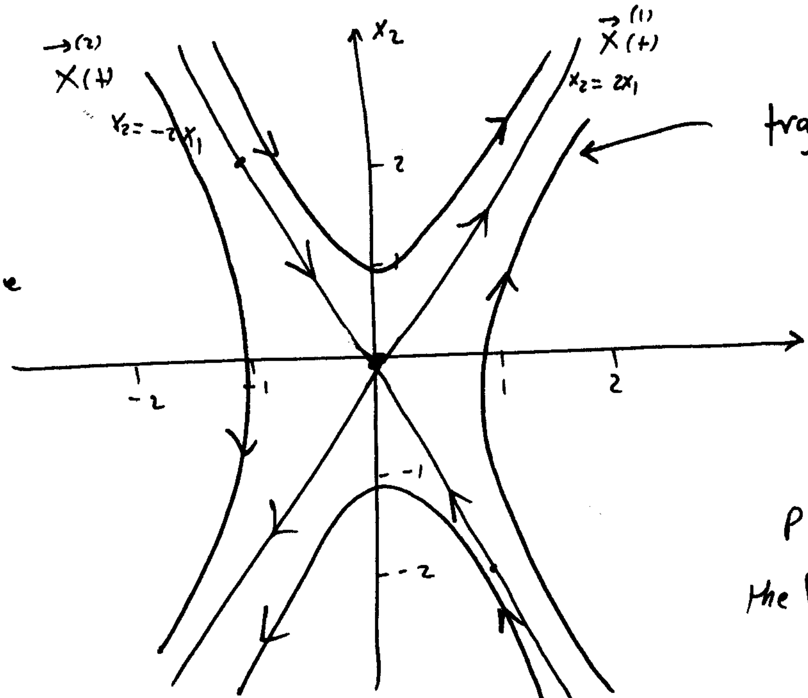
- The solution lies on the straight line $X_2 = 2X_1$ passes through origin in the direction of $\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- If $c_1 > 0$, then the particle moves in the 1^{st} quadrant
- If $c_1 < 0$, then the particle moves in the 3^{rd} quadrant
- The particle moves away from the origin in both cases as $t \uparrow$

$\boxed{b}$ $\vec{X} = c_2 \vec{X}^{(2)}(t)$

$$\left.\begin{aligned} X_1 &= c_2 e^{-t} \\ X_2 &= -2c_2 e^{-t} \end{aligned}\right\} \boxed{X_2 = -2X_1}$$

- The solution lies on the straight line $X_2 = -2X_1$ passes through origin in the direction of $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
- If $c_2 > 0$, then the particle moves in the 4^{th} quadrant.
- If $c_2 < 0$, then the particle moves in the 2^{nd} quadrant
- The particle moves toward the origin in both cases as $t \uparrow$.

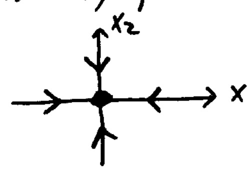
The origin is a saddle point which is unstable eq. point



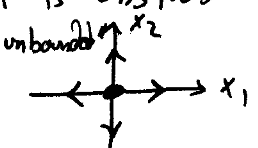
phase portrait in the $\boxed{x_1 x_2 - \text{plane}}$ = phase plane

- For large t, $\vec{X}^{(1)}(t)$ is dominant and $\vec{X}^{(2)}(t)$ becomes negligible. Thus, all solutions for which $c_1 \neq 0$ are asymptotic to the line $x_2 = 2x_1$ as $t \to \infty$
- For small t, $\vec{X}^{(2)}(t)$ is dominant and $\vec{X}^{(1)}(t)$ becomes negligible. Thus, all solutions for which $c_2 \neq 0$ are asymptotic to the line $x_2 = -2x_1$ as $t \to -\infty$.

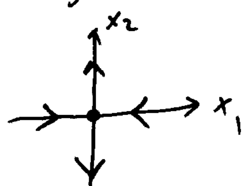
- If r_1 and r_2 are both negative, then the origin is asymptotical node stable eq. point of system $*$

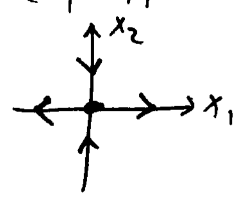


- If r_1 and r_2 are both positive, then the origin is unstable node. eq. point of system $*$ "the trajectories become unbounded"



- If $r_1 r_2 < 0$ "opposite signs", then the origin is saddle point which is unstable eq. point





- If $r_1 = 0$, then $|A| = 0$ which contradict our assumption $|A| \neq 0$

Note that in the above example if $\vec{X}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, then the solution becomes

$$\vec{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + 2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} \qquad \boxed{c_1 = 1}, \ \boxed{c_2 = 2}$$

__Example__ Find the general solution of $\vec{x}\,' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \vec{x}$ $\quad x_1'(t) = -3x_1 + \sqrt{2}x_2$, $x_2'(t) = \sqrt{2}x_1 - 2x_2$

and describe the behavior of the solution.

Assume exponential solution $\vec{x} = \vec{\xi} e^{rt}$, and substitute it above to solve the algebraic equation $(A - rI)\vec{\xi} = \vec{0}$

$$\begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This equation has non trivial solution iff $\begin{vmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{vmatrix} = 0 \Leftrightarrow$ characteristic equation

$$(-3-r)(-2-r) - 2 = 0 \Leftrightarrow (3+r)(r+2) - 2 = 0 \Leftrightarrow r^2 + 5r + 4 = 0$$

$$\Leftrightarrow (r+1)(r+4) = 0 \Leftrightarrow r_1 = -1 \text{ and } r_2 = -4.$$

- To find the eigenvector corresponding to the eigenvalue $r_1 = -1$:

$$\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad -2\xi_1 + \sqrt{2}\xi_2 = 0 \Leftrightarrow \xi_2 = \sqrt{2}\xi_1$$

so the eigenvector $\vec{\xi}^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

- To find the eigenvector coresspoding to the eigenvalue $r_2 = -4$:

$$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \xi_1 + \sqrt{2}\xi_2 = 0 \Leftrightarrow \xi_1 = -\sqrt{2}\xi_2$$

so the eigenvector $\vec{\xi}^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

- Thus, the fundamental solutions are

$$\vec{x}^{(1)}(t) = \vec{\xi}^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} \quad \text{and} \quad \vec{x}^{(2)}(t) = \vec{\xi}^{(2)} e^{r_2 t} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

- The general solution is $\vec{x} = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$

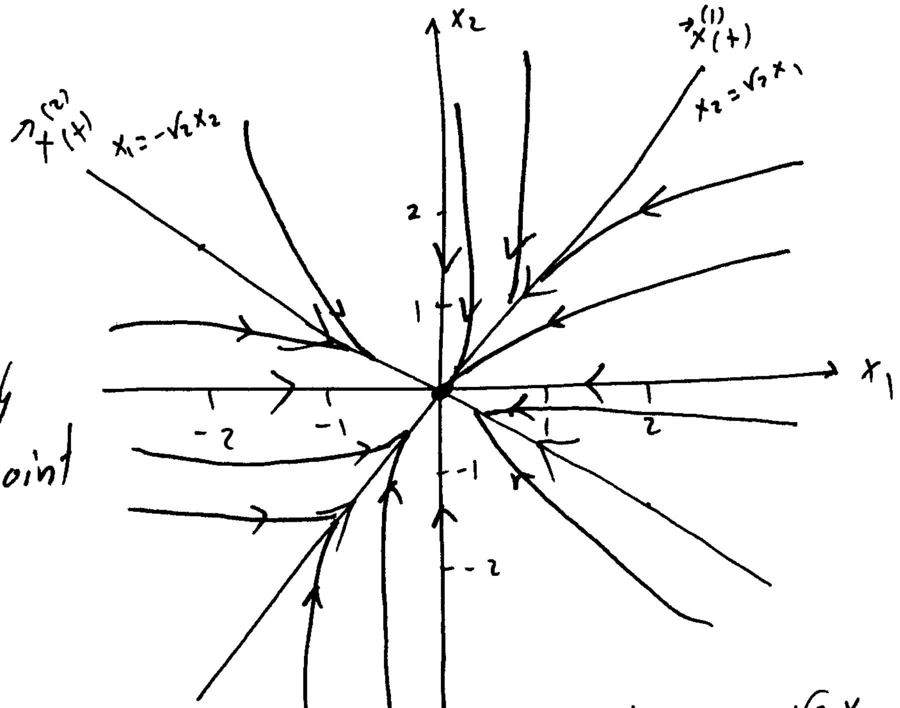
$\boxed{a}$ $\vec{x} = c_1 \vec{x}^{(1)}(t)$

$$\left.\begin{aligned} x_1 &= c_1 e^{-t} \\ x_2 &= \sqrt{2}c_1 e^{-t} \end{aligned}\right\rangle \boxed{x_2 = \sqrt{2}x_1}$$

$\boxed{b}$ $\vec{x} = c_2 \vec{x}^{(2)}(t)$

$$\left.\begin{aligned} x_1 &= -\sqrt{2}c_2 e^{-4t} \\ x_2 &= c_2 e^{-4t} \end{aligned}\right\rangle x_1 = -\sqrt{2}x_2 \quad \boxed{x_2 = \frac{-1}{\sqrt{2}} x_1}$$

Origin is a node. In this case the node is asymptotically stable eq. point



The solutions:

- $\vec{X}^{(1)}(t)$ approcches the origin alonge the line $x_2 = \sqrt{2}\, x_1$
- $\vec{X}^{(2)}(t)$ = = = = = = $x_1 = -\sqrt{2}\, x_2$
- As $t \to \infty$, $\vec{X}^{(1)}(t)$ is dominant and $\vec{X}^{(2)}(t)$ is negligible.

  $\Rightarrow$ If $c_1 = 0$, then the solution $\vec{x} \to \vec{0}$ tangent to $x_2 = \sqrt{2}\, x_1$

Example: Find the general solution of $\vec{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$ and describe the behavior of solution as $t \to \infty$.

Assume exponential solution $\vec{x} = \vec{\xi} e^{rt}$, and substitute it above to solve the algebraic equation $(A - rI)\vec{\xi} = 0$

$$\begin{pmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This equation has a non trivial solution iff $\begin{vmatrix} -r & 1 & 1 \\ 1 & -r & 1 \\ 1 & 1 & -r \end{vmatrix} = 0 \Leftrightarrow$ The characteristic equation is

$$-r \begin{vmatrix} -r & 1 \\ 1 & -r \end{vmatrix} - (1) \begin{vmatrix} 1 & 1 \\ 1 & -r \end{vmatrix} + (1) \begin{vmatrix} 1 & -r \\ 1 & 1 \end{vmatrix} = 0 \Leftrightarrow$$

$$-r(r^2-1) - (-r-1) + (1+r) = 0 \Leftrightarrow -r(r^2-1) + 2(r+1) = 0$$

$$\Leftrightarrow -r(r-1)(r+1) + 2(r+1) = 0 \Leftrightarrow (r+1)[2 - r(r-1)] = 0 \Leftrightarrow$$

$$-(r+1)[r^2 - r - 2] = 0 \Leftrightarrow -(r+1)(r+1)(r-2) = 0 \Leftrightarrow$$

$$r_1 = 2, \quad r_2 = r_3 = -1$$

- To find the eigenvector corresponding to the eigenvalue $r_1 = 2$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} -2 & 1 & 1 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{matrix} \qquad \begin{matrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{matrix} \qquad \begin{matrix} -2 & 1 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{matrix}$$

so $\boxed{\xi_3 = k} \Rightarrow -3\xi_2 + 3\xi_3 = 0 \Leftrightarrow \boxed{\xi_2 = \xi_3 = k} \Leftrightarrow -2\xi_1 + \xi_2 + \xi_3 = 0 \Leftrightarrow 2\xi_1 = 2k \Leftrightarrow \xi_1 = k$

so the eigenvector $\vec{\xi}^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

- Similarly, $\vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\vec{\xi}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and the general solution is

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) + c_3 \vec{x}^{(3)}(t) = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} e^{-t}$$

- $W(\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}) \neq 0$ so they form a fundamental set of solutions
- For large t and $c_1 \neq 0$, $\vec{x}$ become unbounded as $t \to \infty$
- For large t with $c_1 = 0$, $\vec{x} \to 0$ as $t \to \infty$