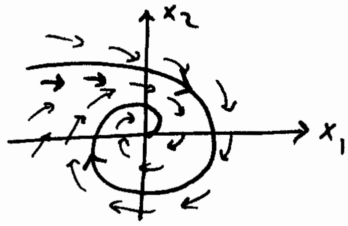
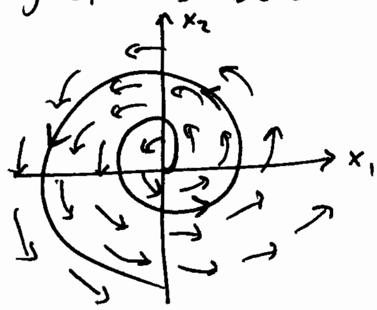
$\boxed{7.6}$ Complex Eigenvalues $(r = \lambda \pm \mu i)$

* In case where the eigenvalues are complex for the system $\vec{x}' = A\vec{x}$ then the origin is an spiral point. Moreover,

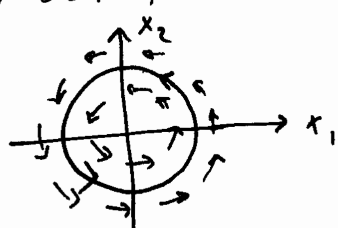
$\boxed{a}$ if $\lambda < 0$, then the origin is asymptotically stable eq. point because all trajectories approach it as t increases.



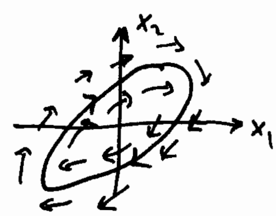
$\boxed{b}$ If $\lambda > 0$, then the origin is unstable eq. point because the direction of motion is away from the origin and the trajectories become unbounded.



$\boxed{c}$ If $\lambda = 0$, then the origin is stable eq. point but not asymptotically stable. In this case the origin is called a center, and the trajectories neither approach the origin nor become unbounded, but form a closed curve about the origin.



$a_{12} < 0$ and $a_{21} > 0$



$a_{12} > 0$ and $a_{21} < 0$

Example: Find the solution of the given system of equations in terms of real-valued functions and describe the behavior of the solution as $t \to \infty$. sketch a few trajectories and draw a direction field: ① $\vec{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \vec{x}$

Assume exponential solution $\vec{x} = \vec{\xi} e^{rt}$, and substitute it in the above system to solve the algebraic equation $(A - rI)\vec{\xi} = \vec{0}$

$\begin{pmatrix} 1-r & -1 \\ 5 & -3-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ This equation has a non trivial solution iff $\begin{vmatrix} 1-r & -1 \\ 5 & -3-r \end{vmatrix} = 0 \Leftrightarrow$ The characteristic equation is $(1-r)(-3-r) + 5 = 0 \Leftrightarrow r^2 + 2r + 2 = 0 \Leftrightarrow$

$r_{1,2} = -1 \pm i$ " $\lambda = -1$ " and " $\mu = 1$ ". $r_1 = -1 - i$ and $r_2 = -1 + i$

- To find the eigenvector coresponding to the eigenvalue $r_1 = -1 - i$:

$\begin{pmatrix} 2+i & -1 \\ 5 & -2+i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $(2+i)\xi_1 - \xi_2 = 0$

$\xi_2 = (2+i)\xi_1$

So the eigenvector is $\vec{\xi}^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix}$

- To find the eigenvector coresponding to the eigenvalue $r_2 = -1 + i$

$\begin{pmatrix} 2-i & -1 \\ 5 & -2-i \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $(2-i)\xi_1 - \xi_2 = 0 \Leftrightarrow \xi_2 = (2-i)\xi_1$

So the eigenvector is $\vec{\xi}^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

- The coresponding solutions of the system above are

$\vec{x}^{(1)}(t) = \vec{\xi}^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-(1+i)t}$ and $\vec{x}^{(2)}(t) = \vec{\xi}^{(2)} e^{r_2 t} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(-1+i)t}$

- To find a real-valued solutions, we find the real and imaginary parts of either $\vec{x}^{(1)}(t)$ or $\vec{x}^{(2)}(t)$:

$\vec{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t} e^{-it} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t} (\cos t - i \sin t)$

$$\vec{x}^{(1)}(t) = \begin{pmatrix} e^{-t}\cos t - i\, e^{-t}\sin t \\ 2e^{-t}\cos t + e^{-t}\sin t + i\left(e^{-t}\cos t - 2e^{-t}\sin t\right) \end{pmatrix}$$

$$= e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i\, e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}$$

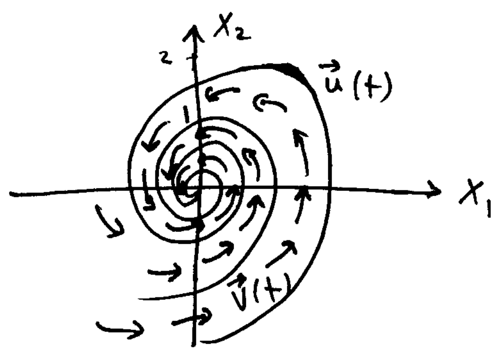
- Hence, $\vec{u}(t) = e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}$ and $\vec{v}(t) = e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}$ are real-valued solution.

- Thus, the general solution is

$$\vec{x}(t) = c_1\vec{u}(t) + c_2\vec{v}(t) = c_1 e^{-t}\begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}$$

- Since the eigenvalues are complex with negative real part "$\lambda < 0$", it follows that the origin is spiral eq. point that is asymptotically stable and all trajectories approach it as t increases.

$\vec{u}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{v}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



- Note that $W(\vec{u}(t), \vec{v}(t))(t) =$

$$\begin{vmatrix} e^{-t}\cos t & -e^{-t}\sin t \\ e^{-t}(2\cos t + \sin t) & e^{-t}(\cos t - 2\sin t) \end{vmatrix} = e^{-2t} \neq 0$$

Thus, $\vec{u}$ and $\vec{v}$ are linearly independent and so they form a fundamental set of solutions.

$\boxed{2}$ $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$ $\qquad$ $\vec{x} = \vec{0}$ is eq. point or eq. solution

Assume exponential solution $\vec{x}(t) = \vec{\xi} e^{rt}$, and substitute it above to solve the algebraic equation $(A - rI)\vec{\xi} = \vec{0}$

$$\begin{pmatrix} 2-r & -5 \\ 1 & -2-r \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This equation has a non trivial solution iff $\begin{vmatrix} 2-r & -5 \\ 1 & -2-r \end{vmatrix} = 0 \iff$ The characteristic equation is $r^2 + 1 = 0 \iff r_{1,2} = \pm i$ "$\lambda = 0$ and $\mu = 1$". $r_1 = i$ and $r_2 = -i$

- To find the eigenvector coresponding to the eigenvalue $r_1 = i$:

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \xi_1 - (2+i)\xi_2 = 0 \iff \xi_1 = (2+i)\xi_2$$

so the eigenvector $\vec{\xi}^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

- To find the eigenvector coresspon ding to the eigenvalue $r_2 = -i$

$$\begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \xi_1 + (-2+i)\xi_2 = 0 \iff \xi_1 = (2-i)\xi_2$$

so the eigenvector $\vec{\xi}^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix}$

- The coresspon ding solutions of the system above are

$$\vec{x}^{(1)}(t) = \vec{\xi}^{(1)} e^{r_1 t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} \quad \text{and} \quad \vec{x}^{(2)}(t) = \vec{\xi}^{(2)} e^{r_2 t} = \begin{pmatrix} 2-i \\ 1 \end{pmatrix} e^{-it}$$

- To find a real-valued solutions, we find the real and imaginary parts of either $\vec{x}^{(1)}(t)$ or $\vec{x}^{(2)}(t)$:

$$\vec{x}^{(1)}(t) = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}(\cos t + i \sin t) = \begin{pmatrix} 2\cos t - \sin t + i(\cos t + 2\sin t) \\ \cos t + i \sin t \end{pmatrix}$$

$$= \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

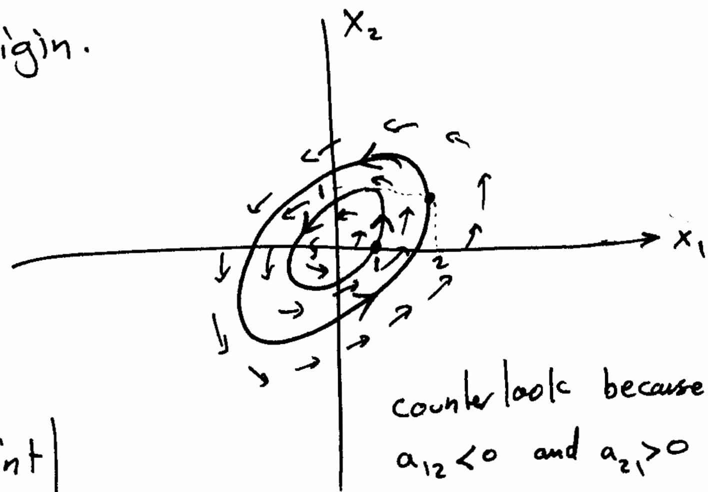
- Hence, $\vec{u}(t) = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$ and $\vec{v}(t) = \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$ are real-valued solutions.

- Thus, the general solution is

$$\vec{x}(t) = c_1 \vec{u}(t) + c_2 \vec{v}(t) = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

- Since the eigenvalues are complex with no real part "$\lambda = 0$", it follows that the origin is spiral eq. point that is stable but not asymptotically stable and called a center. The trajectories neither approach the origin nor become unbounded, but form a closed curve about the origin.

$$\vec{u}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \vec{v}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



counterclock because $a_{12} < 0$ and $a_{21} > 0$

- Note that $w(\vec{u}(t), \vec{v}(t))(t) =$

$$\begin{vmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{vmatrix} = -1 \neq 0$$

Thus, $\vec{u}$ and $\vec{v}$ are linearly independent and so they form a fundamental set of solutions.

Example: Consider the system $\vec{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \vec{x}$.

Describe how the solutions depend qualitatively on α. Find the critical values of α at which the qualitative behavior of the trajectories in the phase plane changes markedly
"portrait"

Assume exponential solution $\vec{x}(t) = \vec{\xi} e^{rt}$, and substitute it above to solve the algebraic equation $(A - rI)\vec{\xi} = \vec{0}$

$$\begin{pmatrix} -r & -5 \\ 1 & \alpha - r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This equation has a non trivial solution iff $\begin{vmatrix} -r & -5 \\ 1 & \alpha - r \end{vmatrix} = 0 \Leftrightarrow$ the characteristic equation is $r^2 - \alpha r + 5 = 0$

$$\Leftrightarrow \boxed{r_{1,2} = \frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 - 20}}$$

- The roots are complex when $-\sqrt{20} < \alpha < \sqrt{20}$, and real otherwise. Hence $\alpha = \pm\sqrt{20}$ are critical values where the eigenvalues changes from real "nodes" to complex "spirals", or vice versa.
- If $\alpha < -\sqrt{20}$, then both eigenvalues are real and negative, so all trajectories appraach the origin which is asymptotically stable node.
- If $\alpha > \sqrt{20}$, then both eigenvalues are real and positive, so the origin is unstable node. All trajectories become unbounded (except $\vec{x} = \vec{0}$)
- If $\alpha \in (-\sqrt{20}, \sqrt{20})$, then the eigenvalues are complex and the trajectories are spirals.
  - $\Rightarrow$ If $\alpha \in (-\sqrt{20}, 0)$, then the real part of the eigenvalue is negative, so the spirals are directed inward, and the origin is asymptotically stable.
  - $\Rightarrow$ If $\alpha \in (0, \sqrt{20})$, then the real part of the eigenvalue is positive, so the spirals are directed upward, and the origin is unstable.

$\Rightarrow \alpha = 0$ is also a critical value since the direction of the spirals changes from inward to outward.
The origin is a center and the trajectories are closed curves about the origin.

$\boxed{3}$ $\vec{x}\,' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \vec{x}$ ....

The characteristic equation is $r^2 - r + \frac{5}{2} = 0 \Leftrightarrow r_{1,2} = \underset{\lambda}{\frac{1}{2}} \pm \underset{\mu}{\frac{3}{2}} i$

- $r_1 = \frac{1}{2} + \frac{3}{2} i$ is an eigenvalue with eigenvector $\vec{\xi}^{(1)} = \begin{pmatrix} 5 \\ 3-3i \end{pmatrix}$
- $r_2 = \frac{1}{2} - \frac{3}{2} i$ " " " " " $\vec{\xi}^{(2)} = \begin{pmatrix} 5 \\ 3+3i \end{pmatrix}$
- $\vec{x}^{(1)}(t) = \vec{\xi} e^{r_1 t} = \begin{pmatrix} 5 \\ 3-3i \end{pmatrix} e^{(\frac{1}{2}+\frac{3}{2}i)t} = e^{t/2}\begin{pmatrix} 2\cos\frac{3}{2}t - \sin\frac{3}{2}t \\ \cos\frac{3}{2}t \end{pmatrix} + i e^{t/2}\begin{pmatrix} \cos\frac{3}{2}t + 2\sin\frac{3}{2}t \\ \sin\frac{3}{2}t \end{pmatrix}$
- The general solution is

$$\vec{x}(t) = c_1 e^{t/2}\begin{pmatrix} 2\cos\frac{3}{2}t - \sin\frac{3}{2}t \\ \cos\frac{3}{2}t \end{pmatrix} + c_2 e^{t/2}\begin{pmatrix} \cos\frac{3}{2}t + 2\sin\frac{3}{2}t \\ \sin\frac{3}{2}t \end{pmatrix}$$

The origin is spiral eq. point that is unstable and the trajectories become unbounded.