7.8 Repeated Eigenvalues

Consider the linear homogeneous system with constant coefficients

$$\vec{x}' = A\vec{x}$$

which has a repeated eigenvalue ρ with corresponding eigenvector ξ such that

$$(A - \rho I)\vec{\xi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Then one solution is $\vec{x}^{(1)}(t) = \vec{\xi} e^{\rho t}$
- The second solution is

$$\vec{x}^{(2)}(t) = \vec{\xi} t e^{\rho t} + \vec{\eta} e^{\rho t} \quad \text{where}$$

$\vec{\eta}$ is called a generalized eigenvector corresponding to the eigenvalue ρ and satisfies

$$(A - \rho I)\vec{\eta} = \vec{\xi}$$

Example: Find the general solution of the IVP "system"

$$x_1' = x_1 - 4x_2 \quad , \quad x_1(0) = 3 \quad , \quad x_2(0) = 2$$
$$x_2' = 4x_1 - 7x_2$$

$$\vec{x}' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} \vec{x} \quad , \quad \vec{x}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Assume exponential solution $\vec{x}(t) = \vec{\xi} e^{rt}$, and substitute it above to solve the algebraic equation $(A - rI)\vec{\xi} = \vec{0}$

$$\begin{pmatrix} 1-r & -4 \\ 4 & -7-r \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This equation has a non trivial solution iff $\begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = 0 \Leftrightarrow$

The characteristic equation is $r^2 + 6r + 9 = 0 \Leftrightarrow r_1 = r_2 = \rho = -3$

- To find the eigenvector corresponding to the single eigenvalue $\rho = -3$:

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad 4\xi_1 - 4\xi_2 = 0 \iff \xi_1 = \xi_2$$

so only eigenvector is $\vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

- One solution is $\vec{X}^{(1)}(t) = \vec{\xi}\, e^{\rho t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$.
- In order to generate a second linearly independent solution, we must look for a generalized eigenvector. This leads to the system of equations $(A - \rho I)\vec{\eta} = \vec{\xi}$

$$\begin{pmatrix} 1+3 & -4 \\ 4 & -7+3 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \iff \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$4\eta_1 - 4\eta_2 = 1 \iff 4\eta_1 = 4\eta_2 + 1 \iff \eta_1 = \eta_2 + \frac{1}{4}$$

Setting $\eta_2 = K$, where K is arbitrary constant, we obtain $\eta_1 = K + \frac{1}{4}$. Hence,

$$\vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} K + \frac{1}{4} \\ K \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} + K\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- A second solution is

$$\vec{X}^{(2)}(t) = \vec{\xi}\, t e^{\rho t} + \vec{\eta}\, e^{\rho t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} e^{-3t} + \underline{K\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} e^{-3t}$$

we ignore this term since it is a multiple of the first solution.

- The general solution is

$$\vec{X}(t) = c_1 \vec{X}^{(1)}(t) + c_2 \vec{X}^{(2)}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} e^{-3t} \right]$$

- Note that $W(\vec{X}^{(1)}(t), \vec{X}^{(2)}(t)) = \begin{vmatrix} e^{-3t} & te^{-3t} + \frac{1}{4} e^{-3t} \\ e^{-3t} & t e^{-3t} \end{vmatrix} = -\frac{1}{4} e^{-6t} \neq 0$

Thus, $\vec{X}^{(1)}(t)$ and $\vec{X}^{(2)}(t)$ are linearly independent and so form a fundamental set of solutions.

- To find c_1 and c_2, we impose the initial conditions:

$$\vec{X}(0) = c_1 + \frac{1}{4} c_2 = 3 \quad , \quad c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\boxed{c_1 = 2} \quad , \quad \boxed{c_2 = 4}$$

- Therefore, the general solution is :

$$\vec{X}(t) = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + 4 \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} e^{-3t}$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-3t} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} t e^{-3t}$$

- Note that the solution approaches origin as $t \to \infty$. and the solution becomes unbounded as $t \to -\infty$.

- The origin is called an improper node and it is asymptotically stable "since $\rho < 0$"