زهراء خليلي
وحسيب ظاهر

د. عبد الرحيم صو

MATH 331

د. عبد الرحيم موسى   حسين ظاهر

(1.1) Mathmatical Modelings and Direction Fields

* Differential Equations (Mathmatical models) Are relations with <u>derivatives</u>.

ex1: Formulate a differential equation describing the motion of an object falling in the atmosphere near the sea level.

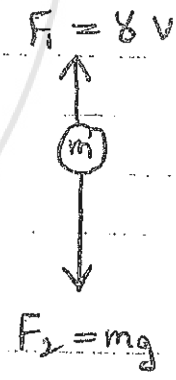
variables: time (t) $\Longrightarrow$ Independent variable
velocity (v) $\Longrightarrow$ dependent variable

Using 2nd newton's law:

$F = ma$, where: F: the net force
m: mass , a: acceleration

$F_{net} = F_2 - F_1$

$ma = mg - \gamma v$



where: g: gravitational acceleration
γ: drag coefficient
v: velocity

$$\boxed{m\frac{dv}{dt} = mg - \gamma v} \quad --- (1)$$

This is a <u>DE</u> (Differential Equation).

The solution for DE (1) is easy to find (next section) because it is a first order linear differential equation.

Now we will study the behaviour of the solution for the DE (1) without solving it. This is what we call the "Direction Field".

* To do that take $m = 10$ Kg, $\gamma = 2$ Kg/s, in this case:

$$\frac{dv}{dt} = 9.8 - 0.2V \quad ---(2)$$

↳ relative change

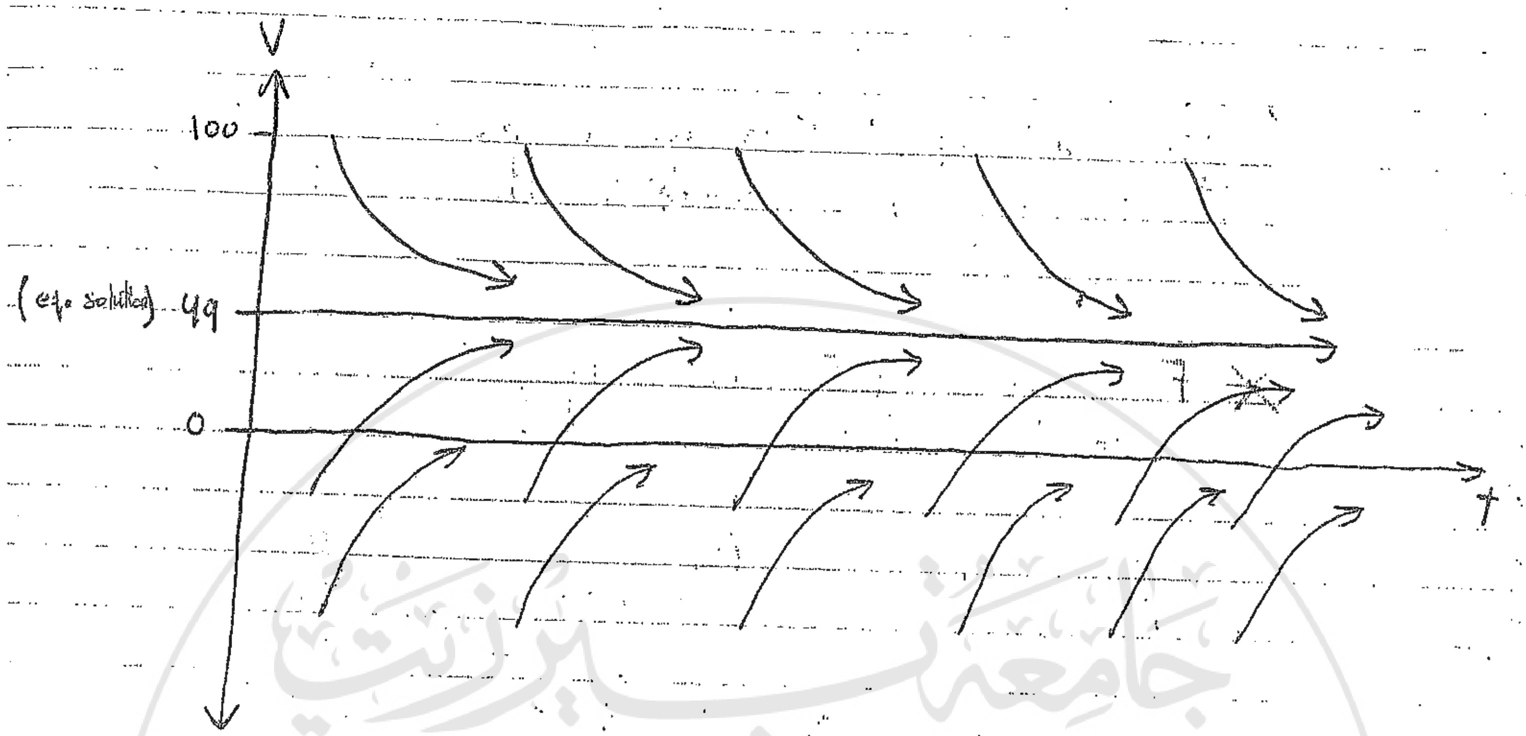
First: We find the equilibrium solution of the DE (1) by setting $\frac{dv}{dt} = 0$.

$$\frac{dv}{dt} = 9.8 - 0.2V$$

$$0 = 9.8 - 0.2V$$

$$\Longrightarrow \boxed{V = 49} \quad \text{(equilibrium Solution)}$$

Second: Choose values for V below 49, take $V=0$ for example, we get $\frac{dv}{dt} = 9.8 > 0$. Then choose values for V greater than 49, take $V=100$ for example, we get $\frac{dv}{dt} = -10.2 < 0$



* Does the behaviour of solution depends on the initial condition?

Answer: Clearly $\lim_{t \to \infty} V(t) = 49$
when ever you start.

ex2: (Mice and Owls): Consider a mouse population $p(t)$ increase proportionally to its present population size.

* Formulate a DE describing the growth in $p(t)$ in the abserve of owls.

$$\frac{dp}{dt} = rp \quad --- (3)$$

where: r: the growth rate, t: time in months.

Assume: 1) $r = 0.5$ per month.

2) Owls are present and they eat 15 mice per day, formulate a DE describing this situation.

* The Owls eat 15 mice daily $\Longrightarrow$ $15 * 30 = 450$ monthly

$$\Longrightarrow \frac{dp}{dt} = 0.5p - 450 \quad --- (4)$$

* Find the equilibrium solution of equation (4):

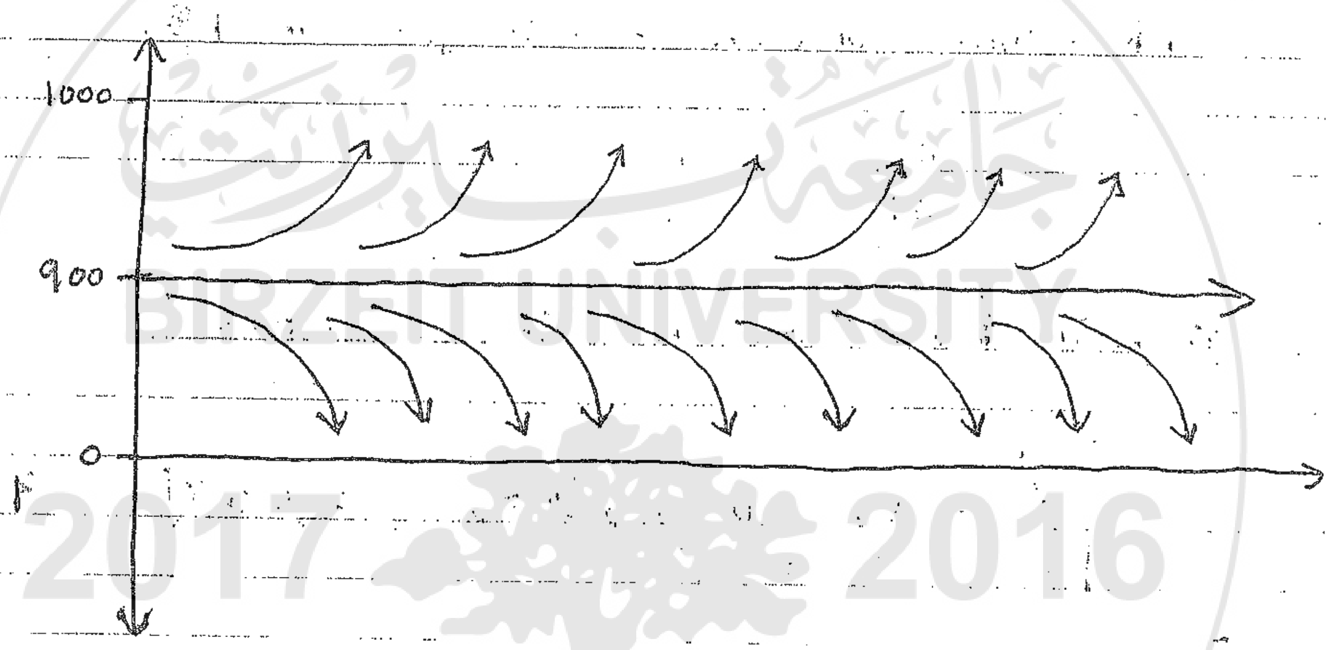
$$\frac{dp}{dt} = 0 = 0.5p - 450$$

$$\Longrightarrow \boxed{p = 900} \quad \text{(equilibrium solution)}$$

* Study the behaviour of the solution for the DE without solving it and draw the direction field.

let $p_0 = 0 \Longrightarrow \frac{dp}{dt} = -450 < 0$

$p_0 = 1000 \Longrightarrow \frac{dp}{dt} = 50 > 0$



* Does the behaviour of solution depend on the initial condition (P_0)?

Answer: Yes, because if $p_0 > 900$

$$\Longrightarrow \lim_{t \to \infty} P(t) = \infty$$

and if $p_0 < 900$

$$\Longrightarrow \lim_{t \to \infty} P(t) = 0$$ (Not $-\infty$ because P is population)

(1.2) Solutions for some DE's

* Recall that free fall example gives the DE:

$$\frac{dv}{dt} = 9.8 - 0.2\,v$$

The mice and owls example gives the DE:

$$\frac{dp}{dt} = 0.5p - 450$$

* Both DE's alone have the general form:

$$\boxed{y' = ay - b \;,\; y(0) = y_0 \;,\; a \neq 0} \quad \text{---} (1)$$

* A DE with an initial condition forms an initial value problem (IVP).

* The general solution of the (IVP) has the form:

$$\boxed{y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}} \quad \text{---} (2)$$

So this solution (equation (1)) represents a general solution for first order linear DE.

* We prove (1) by using methods of calculus:

$$\frac{dy}{dt} = ay - b$$

$$\frac{dy}{dt} = a\left(y - \frac{b}{a}\right)$$

$$\frac{dy}{y - \frac{b}{a}} = a\,dt$$

$$\int \frac{dy}{y - \frac{b}{a}} = \int a\,dt$$

$$\ln\left|y - \frac{b}{a}\right| = at + c$$

$$y - \frac{b}{a} = \mp e^{at+c}$$

$$y = \frac{b}{a} \mp e^{c} \cdot e^{at}$$

$$y = \frac{b}{a} \mp D \cdot e^{at}$$

to find the constant (D), we use the initial condition $y(0) = y_0$.

$$y(0) = \frac{b}{a} + D e^{a(0)}$$

$$y_0 = \frac{b}{a} + D \implies \boxed{D = y_0 - \frac{b}{a}}$$

$\implies$ The general solution is:

$$\boxed{y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}} \quad --- (2)$$

* $y' = ay - b$

$0 = ay - b$

$\implies \boxed{y = \frac{b}{a}}$ (equilibrium solution) ex 1: (free fall): find the solution for:

$$\frac{dv}{dt} = 9.8 - 0.2V \quad ; \quad V(0) = 0$$

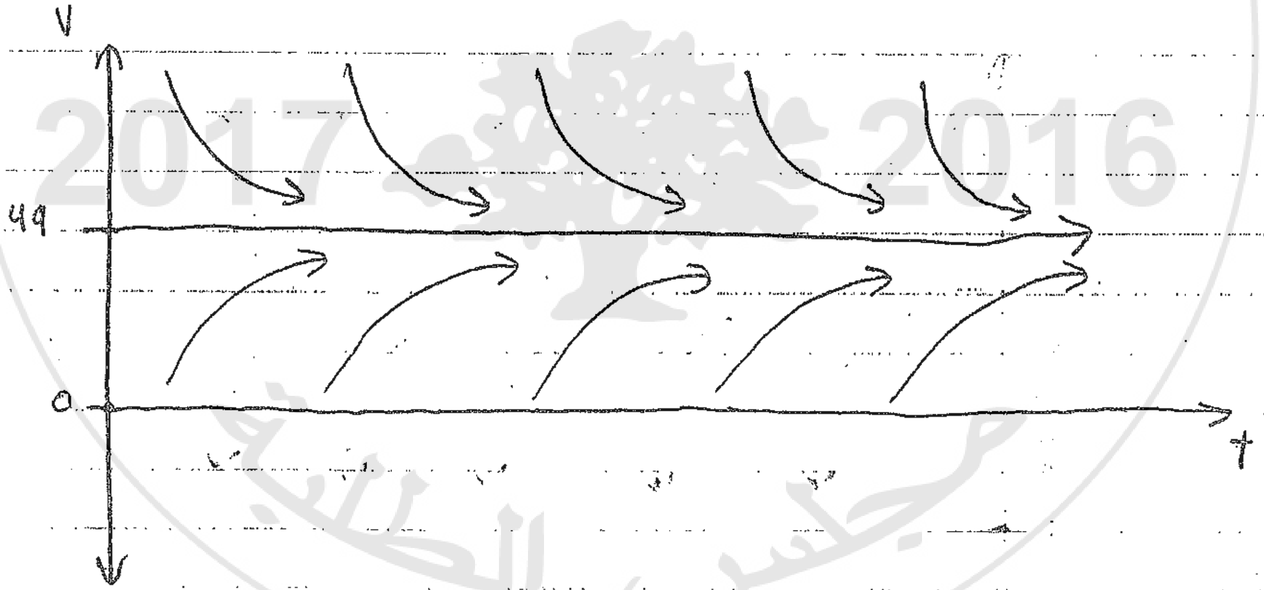
solution: $a = -0.2$ $\Longrightarrow$ $\frac{b}{a} = 49$

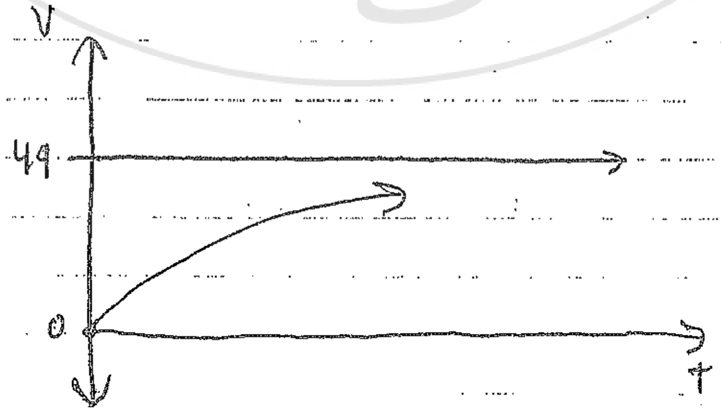
$b = -9.8$

$$V(t) = \frac{b}{a} + \left(V_0 - \frac{b}{a}\right) e^{at}$$

$$V(t) = 49 + (0 - 49)\, e^{-0.2t}$$

$$\boxed{V(t) = 49 - 49\, e^{-0.2t}}$$





(our solution | V=0 at t=0)

ex2: (mice and owls): find the solution for:

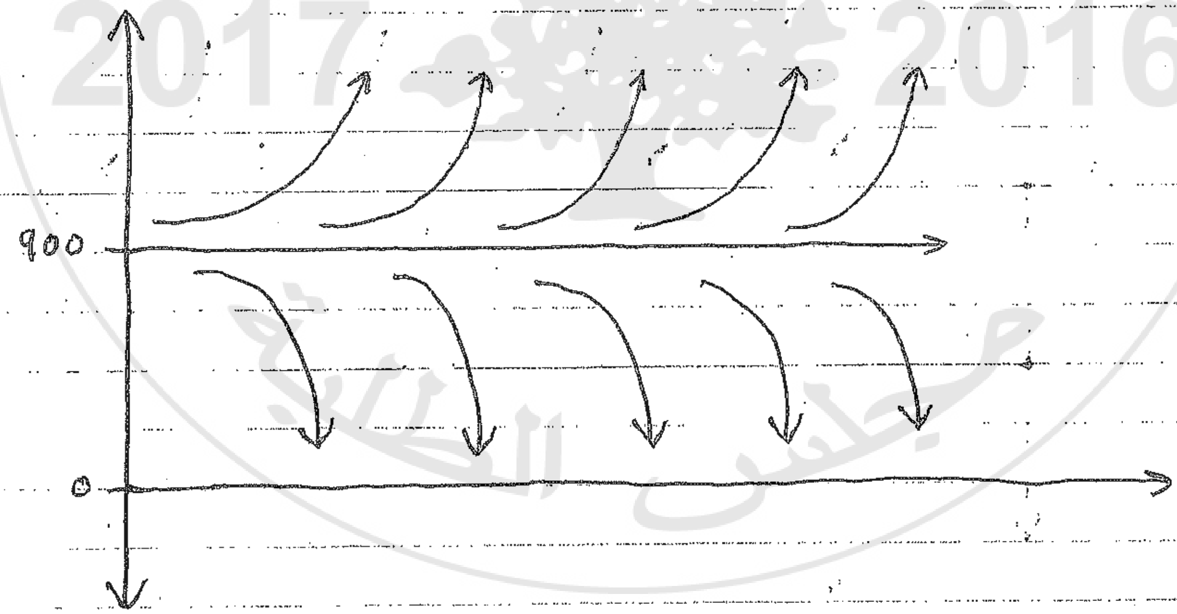
$$\frac{dp}{dt} = 0.5\,p - 450 \quad , \; p(0) = 850$$

solution: $a = 0.5$, $b = 450$ $\Rightarrow$ $\frac{b}{a} = 900$

$$p(t) = \frac{b}{a} + \left(p_0 - \frac{b}{a}\right) e^{at}$$

$$p(t) = 900 + (850 - 900)\, e^{0.5t}$$

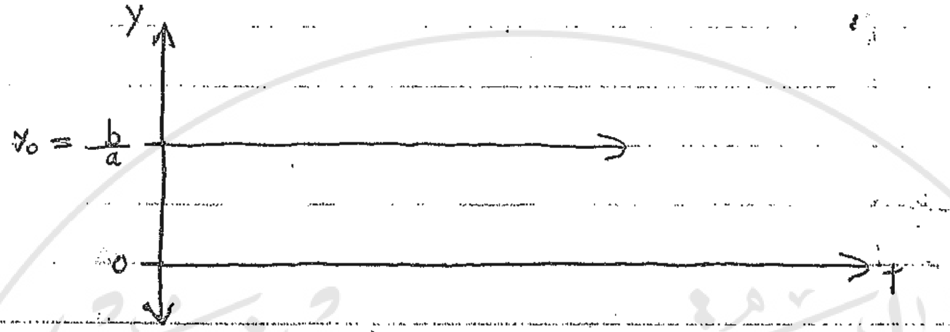
$$\boxed{p(t) = 900 - 50\, e^{0.5t}}$$



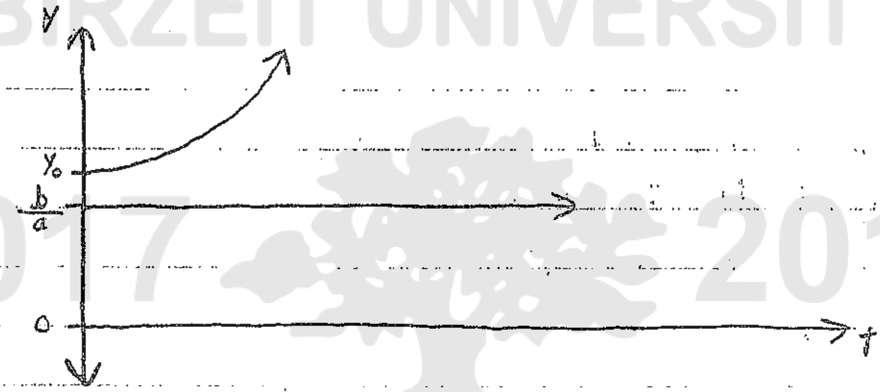
* $p_0 = 850 < 900 \Rightarrow \lim_{t\to\infty} p(t) = 0$ (Not $-\infty$ because p is population)

Notes about the behaviour of the solution $y(t)$:

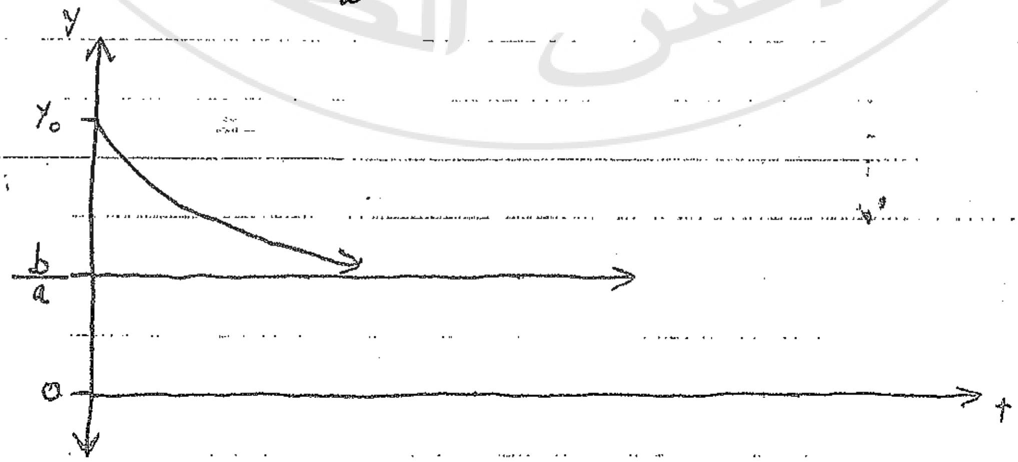
1) If $y_0 = \frac{b}{a}$, then the solution $y(t) = \frac{b}{a}$



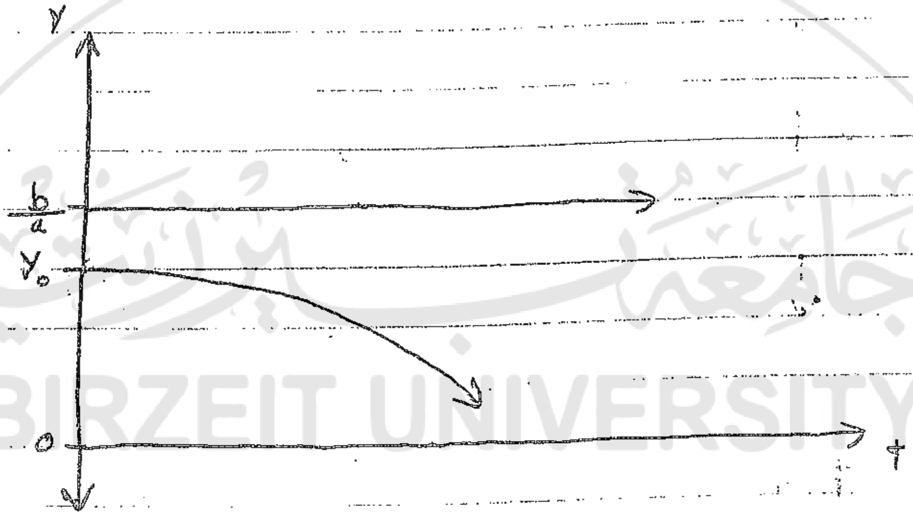
2) If $y_0 > \frac{b}{a}$ and $a > 0$, then the solution $y(t)$ increases exponentially without bound.



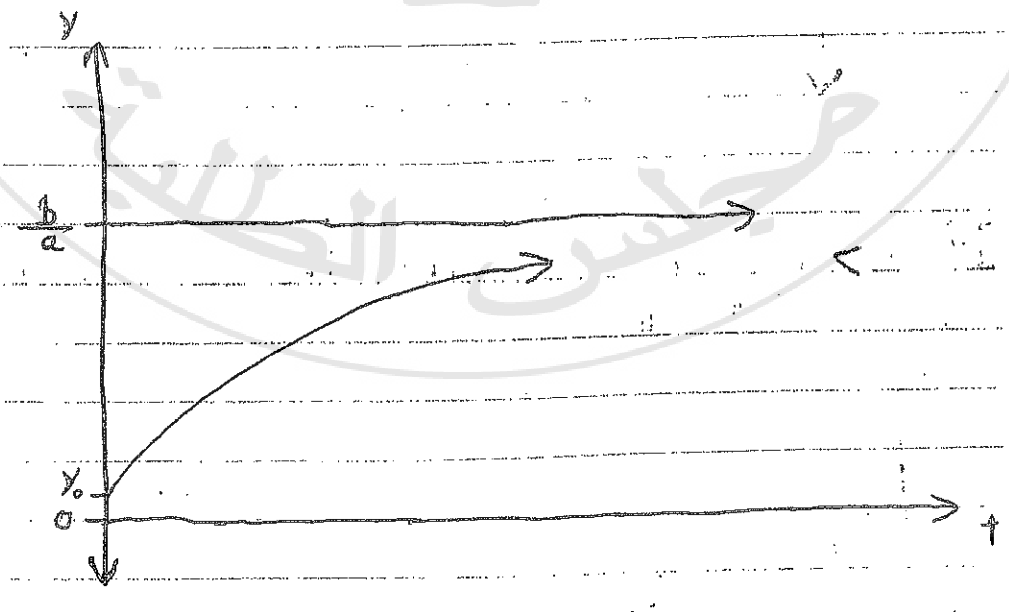
3) If $y_0 > \frac{b}{a}$ and $a < 0$, then the solution decreases asymptotically to $\frac{b}{a}$.



4) If $y_0 < \frac{b}{a}$ and $a > 0$; then the solution decreases ~~asymptotically~~ exponentially without bound.



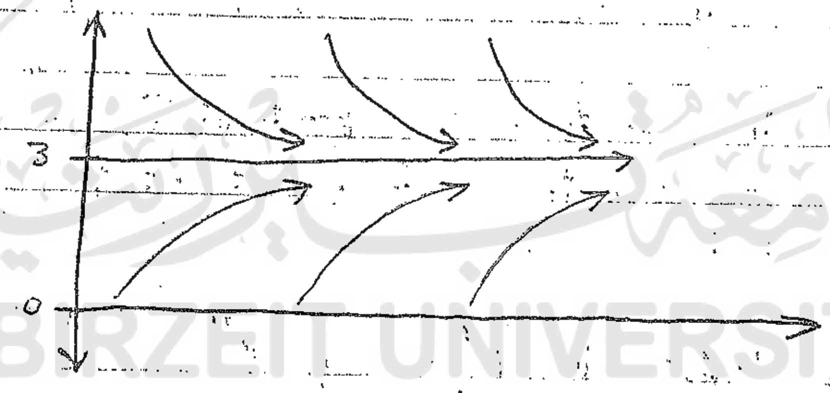
5) If $y_0 < \frac{b}{a}$ and $a < 0$, then the solution increases asymptotically to $\frac{b}{a}$.



ex8: Find the equilibrium solution of the following DE's:

1) $y' = -2y + 6$, $a = -2$, $b = -6$

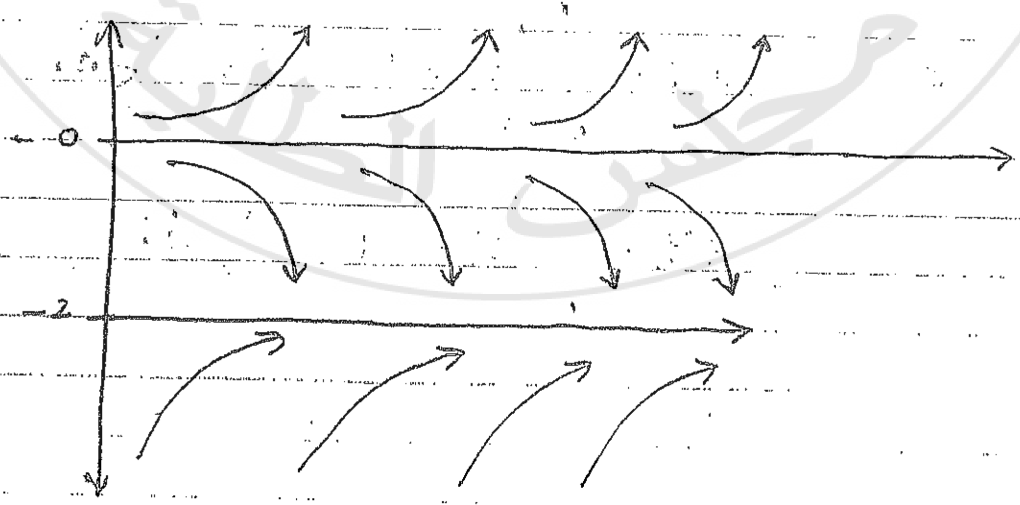
$y' = 0 = -2y + 6 \implies \boxed{y = 3}$ (equilibrium solution)



* behaviour: $\lim_{t \to \infty} y(t) = 3$

2) $y' = y(y+2)$

$y' = 0 = y(y+2) \implies y = \{0, -2\}$ (equilibrium solution)



when: $y_0 = 1 \implies y' = 3 > 0$

$y_0 = -1 \implies y' = -1 < 0$

$= -3 \implies y' = 3 > 0$

* behaviour: $\lim\limits_{t \to \infty} y(t) = \begin{cases} -2 & \text{if } y_0 < 0 \\ \infty & \text{if } y_0 > 0 \\ 0 & \text{if } y_0 = 0 \end{cases}$

ex41: Consider a falling object of mass $= 10$ Kg and drag coefficient $\gamma = 2$ Kg/sec, $\frac{dv}{dt} = 9.8 - 0.2V$.
Suppose this object is dropped from a height of 300m.

a) Find the velocity of the object at any time.
b) How long will the object take to hit the ground and how fast?

solution: a) $V(t) = \frac{b}{a} + \left(V_0 - \frac{b}{a}\right) e^{at}$

$$a = -0.2, \quad b = -9.8 \implies \frac{b}{a} = 49$$

$$\implies V(t) = 49 + (V_0 - 49)\, e^{-0.2t}$$

b) $$x(t) = \int V(t)\, dt$$

$$x(t) = \int \left(49 + (V_0 - 49)\, e^{-0.2t}\right) dt$$

$$x(t) = \int \left(49 - 49\, e^{-0.2t}\right) dt \qquad (V_0 = 0)$$

$$x(t) = 49t + 245\, e^{-0.2t} + C$$

$$x(0) = 0 = 49(0) + 245\, e^{-0.2(0)} + C$$

$$\implies \boxed{C = -245}$$

$$\Longrightarrow \quad x(t) = 49t + 245\, e^{-0.2t} - 245$$

$$x(T) = 300 = 49T + 245\, e^{-0.2T} - 245$$

Using computer, we approximate:

$$\Longrightarrow \boxed{T = 10.5 \text{ sec}} \quad \text{(falling time)}$$

$$V(T) = V(10.5) = 49 - 49\, e^{-0.2(10.5)}$$

$$\Longrightarrow \boxed{V(T) \approx 43.01 \text{ m/sec}}$$

(1.3) Classification of DE's:

DE

ODE | PDE | System of DE's

1) ODE: (Ordinary Differential Equation)

a) The unknown function depends on one independent variable.

b) Only ordinary derivatives appears in the equation.

ex: 1) $\frac{dv}{dt} = 9.8 - 0.2V \Rightarrow V(t)$ is unknown function.

2) $\frac{dp}{dt} = 0.5p - 450 \Rightarrow P(t)$ is unknown function.

2) PDE: (Partial Differential Equation)

a) The unknown function depends on two or more independent variables.

b) Partial derivatives appear in the equation.

ex: 1) $\alpha\, U_{xx}(t,x) = U_t(t,x)$ (heat equation)

$$\Rightarrow \alpha \frac{\partial^2 U}{\partial x^2}(t,x) = \frac{\partial U}{dt}(t,x)$$

$\Rightarrow$ the unknown function is $U(t,x)$.

2) $a\, U_{xx}(t,x) = U_{tt}(t,x)$ (wave equation)

$\Longrightarrow$ the unknown function is $U(t,x)$.

3) System of DE's :

For having one unknown function one DE is sufficient to solve. Two or more unknown functions require a system of DE's.

ex: Lotka-Volterra (predator-prey) equations:

$$\frac{dx}{dt} = ax - bxy \Longrightarrow x(t)$$

$$\frac{dy}{dt} = cy - dxy \Longrightarrow y(t)$$

$x(t)$: population size of one species.
$y(t)$: population size of second species.

*. The Order of DE :

The order of a given DE is the heighest derivative appears in the equation.

ex: What is the order of these equations:

1) $y' - y = 0$ $\Rightarrow$ first order linear ODE.

2) $2y'' - 5y' + 3t = 0$ $\Rightarrow$ second order linear ODE.

3) $\frac{d^6 y}{dt^6} + \frac{d^2 y}{dt^2} - 5 = e^{\sqrt{5}t}$ $\Rightarrow$ sixth order linear ODE.

4) $U_{xx} - U_{yy} - \cos t = 0$ $\Rightarrow$ second order linear PDE.

* Solution for DE :

The function $\phi(t)$ is a solution for the DE:

$$F(t, y, y', y'', y''', \dots, y^{(n)}) = 0$$

on the interval $t \in (\alpha, \beta)$, if:

$\phi, \phi', \phi'', \phi''', \dots \phi^{(n)}$ exist and satisfy the DE.

ex: Verify the following solutions for the DE:

$$y'' + y = 0$$

1) $y_1(t) = \sin t$

$y_1'(t) = \cos t$, $y_1''(t) = -\sin t$

$y_1'' + y_1' = -\sin t + \sin t = 0$

$\Longrightarrow$ y_1 is a solution for the DE.

2) $y_2(t) = -\cos t$

$y_2'(t) = +\sin t$, $y_2''(t) = \cos t$

$y_2'' + y_2 = \cos t + -\cos t = 0$

$\Longrightarrow$ y_2 is a solution for the DE.

3) $y_3(t) = 2\sin t$

$y_3'(t) = 2\cos t$, $y_3''(t) = -2\sin t$

$y_3'' + y_3 = -2\sin t + 2\sin t = 0$

$\Longrightarrow$ y_3 is a solution for the DE.

* Linear and Non-linear DE's:

In general we write the DE in the form:

$F(t, y, y', \ldots, y^{(n)}) = 0.$

F is linear, if it is linear in:

$y, y', y'', \ldots, y^{(n)}$.

F can be linear ODE or linear PDE.

The general form of linear ODE of order n is:

$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) = g(t)$$

ex: 1) $U_{xxx} - U \cdot U_{yy} - \sin t = 0$

$\Rightarrow$ Non-linear of order (3) (PDE).

2) $y'' - 5y' + t^2 = 0$

$\Rightarrow$ Linear of order (2) (ODE).

3) $\dfrac{d^2y}{dt^2} - e^y \dfrac{dy}{dt} = 5t$

$\Rightarrow$ Non-linear of order (2) (ODE).

4) $xy' + 2y = \sin$

ex 2: Classify the following DE's:

1) $\frac{1}{t}y' + y\cos t = t^2$

$\Longrightarrow$ First order linear ODE.

2) $(\sin t)y' = t$

$\Longrightarrow$ First order linear ODE.

3) $xy' + xy = 1 - y$

$xy' + y(x+1) = 1$

$\Longrightarrow$ First order linear ODE.

4) $ty' + \frac{1}{ty} = 10$

$\Longrightarrow$ First order non-linear ODE.

5) $(x+e^y)\,dy - dx = 0$

$\frac{dy}{dx} = \frac{1}{x+e^y}$

$\Longrightarrow$ First order non-linear ODE.

6) $\left(\frac{d^3y}{dx^3}\right)^4 + \left(\frac{d^2y}{dx^2}\right) + y^5 = x$

$\Longrightarrow$ Third order non-li ar ODE.

* There are three important questions in dealing with DE's:

1) Is there a solution? "Existence"

2) If there is a solution, is it unique? "uniqueness"

3) If there is a solution, how to find this solution?

(2.1) First Order DE.

* In general, a first order DE has the form:

$$\frac{dy}{dt} = f(t,y) \quad --- (1)$$

Note that:

1) The DE(1) will be linear if f is linear in y.
2) The DE(1) will be non-linear if f is not linear in y.

in case (1), the DE have two possible situations:

a) the coefficients are constants, that is DE(1) becomes:

$$y' = ay - b \quad , a \neq 0 \quad , a,b \in \mathbb{R}$$

in this case we can apply the method of calculus to get the solution:

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$$

b) the coefficients are variables, that is DE(1) becomes:

$$y' + P(t)\,y = g(t) \quad --- (2)$$

In this case, we can not use the method of calculus to find the solution of the DE(2), because it will not work.

But instead, we use the method of integrating factor.

The idea of this method is to multiply the DE (2) by a positive function $\mu(t)$ "integrating factor", so that the resulting ~~is easy to~~ equation is easy to ~~use~~ integrate.

* The solution of the DE (2) is given by:

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\, dt + C\right]$$

where the integrating factor $\mu(t)$ is given by:

$$\mu(t) = e^{\int P(t)\,dt}$$

proof: multiply the DE (2) by a positive function $\mu(t)$:

$$\mu(t)\frac{dy}{dt} + P(t)\,\mu(t)\,y = \mu(t)\,g(t) \quad ---(3)$$

recall that: $\frac{d}{dt}\left(\mu(t)\,y(t)\right) = \mu(t)\frac{dy(t)}{dt} + \frac{d\mu(t)}{dt}\,y(t)$

if $P(t)\,\mu(t)\,\cancel{y(t)} = \frac{d}{dt}\left(\mu(t)\right)\cancel{y(t)}$

$$\int \frac{d(\mu(t))}{\mu(t)} = \int P(t)\,dt$$

$$\ln|\mu(t)| = \int P(t)\,dt$$

$$\Longrightarrow \boxed{\mu(t) = e^{\int P(t)\,dt}} \qquad (\mu(t) > 0)$$

then integrating the DE (3) gives:

$$\int \left(\mu(t)\, y' + P(t)\, e^{\int P(t)\,dt}\, y \right) = \int \mu(t)\, g(t)\, dt$$

$$\mu(t)\, y(t) = \int \mu(t)\, g(t)\, dt$$

$$\Longrightarrow \boxed{y(t) = \frac{1}{\mu(t)} \int \mu(t)\, g(t)\, dt}$$

ex1: solve the IVP.

$$t y' - 2y - 5t^2 = 0 \quad , y(1) = 2 \quad , t > 0$$

$$y' - \frac{2}{t} y = 5t$$

$$\Longrightarrow P(t) = \frac{-2}{t} \quad , \quad g(t) = 5t$$

$$\mu(t) = e^{\int P(t)\,dt} = e^{\int \frac{-2}{t}\,dt} = e^{-2\ln|t|}$$
$$= e^{\ln|t|^{-2}} = |t|^{-2} = \frac{1}{t^2} \quad (t > 0)$$

The general solution is:

$$y(t) = \frac{1}{\mu(t)} \left[\int \mu(t)\, g(t)\, dt + C \right]$$

$$y(t) = \frac{1}{\frac{1}{t^2}} \left[\int \frac{1}{t^{\cancel{2}}} \cdot 5\cancel{t}\, dt + C \right]$$

$$y(t) = t^2 \left[\int \frac{5}{t}\, dt + C \right]$$

$$y(t) = t^2 \left[5\ln|t| + C \right]$$

$$y(t) = t^2 (5\ln t + C) \qquad (t>0)$$

to find C, we use the initial value condition:

$$y(1) = 2 \implies 2 = (1)^2 (5\ln(1) + C)$$
$$2 = (5(0) + C)$$

$$\implies \boxed{C = 2}$$

Hence, the solution becomes:

$$y(t) = t^2 (5\ln t + 2)$$

note that $\lim\limits_{t \to \infty} y(t) = \infty$

ex2: Solve the IVP.

$$x^2 y' - x\sin x + 2xy = 0 \quad , \quad y\left(\tfrac{\pi}{2}\right) = 1 \quad , \quad x > 0$$

$$y' - \frac{1}{x}\sin x + \frac{2}{x}y = 0$$

$$y' + \frac{2}{x}y = \frac{1}{x}\sin x$$

$$\Longrightarrow \quad P(x) = \frac{2}{x} \quad , \quad g(x) = \frac{1}{x}\sin x$$

$$\mu(t) = e^{\int P(t)\,dt} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2 \quad (x>0)$$

$$y(x) = \frac{1}{x^2}\left[\int x^2 \cdot \frac{1}{x}\sin x \, dx + C\right]$$

$$y(x) = \frac{1}{x^2}\left[\int x\sin x \, dx + C\right]$$

using tabular integration:

$$\begin{array}{ccc} x & + & \sin x \\ 1 & - & -\cos x \\ 0 & & -\sin x \end{array}$$

$$\Longrightarrow \quad \int x\sin x \, dx = -x\cos x + \sin x$$

$$\Longrightarrow \quad y(x) = \frac{1}{x^2}\left[\sin x - x\cos x + C\right]$$

to find C, we use the initial value conditions

$$y\left(\frac{\pi}{2}\right) = \frac{1}{\left(\frac{\pi}{2}\right)^2}\left[\sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2} + C\right]$$

$$1 = \frac{4}{\pi^2}\left[1 - 0 + C\right]$$

$$\frac{\pi^2}{4} = 1 + C$$

$$\Longrightarrow \boxed{C = \frac{\pi^2}{4} - 1}$$

Hence, the solution becomes:

$$y(x) = \frac{1}{x^2}\left[\sin x - x\cos x + \frac{\pi^2}{4} - 1\right]$$

(2.2) Separable Equations

Recall that a first orde DE has the general form:

$$\boxed{\frac{dy}{dt} = f(t, y)} \quad \text{---} (1)$$

The DE(1) is non-linear if f is not linear in y.

Also DE(1) can be written as a function:

$$M(t, y) - N(t, y)\frac{dy}{dt} = 0 \quad \text{---} (2)$$

If $M(t,y)$ becomes a function of ~~x~~ t only and $N(t,y)$ becomes a function of y only, then the DE(2) is called "Separable Equation".

ex1: Solve the DE:

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

$$\int (1-y^2)\,dy = \int x^2\,dx$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + c$$

$$\Rightarrow \boxed{y^3 - 3y + x^3 = D}$$

ex2: Solve the IVP.

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \quad , \quad y(0) = -1$$

$$\int 2(y-1)\,dy = \int (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$y(0) = -1 \Rightarrow (-1)^2 - 2(-1) = (0)^3 + 2(0)^2 + 2(0) + C$$

$$3 = C$$

$$\Rightarrow \boxed{C = 3}$$

$\Rightarrow$ The solution is:

$$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + 3}$$ (implicit solution)

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 4$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$

$$y = \mp\sqrt{x^3+2x^2+2x+4} + 1$$

but $y(0) = -1$, so the solution is:

$$\boxed{y = -\sqrt{x^3+2x^2+2x+4} + 1}$$ (explicit solution)

since it satisfies the initial condition.

ex3: Solve the IVP.

$$\frac{dy}{dx} = \frac{y \cos x}{1+3y^3} , \quad y(0)=1$$

$$\int \frac{(1+3y^3)}{y} dy = \int \cos x \, dx$$

$$\ln|y| + y^3 = \sin x + C$$

$$y(0)=1 \Longrightarrow \ln|1| + (1)^3 = \sin(0) + C$$

$$0 + 1 = 0 + C$$

$$\Longrightarrow \boxed{C=1}$$

the solution is:

$$\boxed{\ln|y| + y^3 = \sin x + 1} \quad (\text{implicit solution})$$

ex4: Solve the DE.

$$\frac{dy}{dx} = \frac{-(4x+3y)}{2x+y}$$

$$\frac{dy}{dx} = -\frac{\left(4+3\frac{y}{x}\right)}{2+\frac{y}{x}}$$

let $v = \frac{y}{x} \Rightarrow y = xv \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$

$$\Rightarrow x\frac{dv}{dx} + v = -\frac{(4+3v)}{2+v}$$

$$x\frac{dv}{dx} = -\left(\frac{4+3v}{2+v} + v\right)$$

$$x\frac{dv}{dx} = -\left(\frac{4+3v+2v+v^2}{2+v}\right)$$

$$x\frac{dv}{dx} = -\left(\frac{v^2+5v+4}{v+2}\right)$$

$$\frac{-(v+2)}{v^2+5v+4}\,dv = \frac{dx}{x}$$

$$-\int \frac{v+2}{(v+4)(v+1)}\,dv = \int \frac{dx}{x}$$

$$-\int \frac{v+2}{(v+4)(v+1)}\,dv = \ln|x| + C$$

$$-\int \frac{v+2}{(v+4)(v+1)}\,dv = -\int\left(\frac{A}{v+4}+\frac{B}{v+1}\right)dv$$

$$\Longrightarrow A = \frac{2}{3}, \quad B = \frac{1}{3},$$

$$\Longrightarrow -\int \frac{v+2}{(v+4)(v+1)}\,dv = -\int\left(\frac{\frac{2}{3}}{v+4}+\frac{\frac{1}{3}}{v+1}\right)dv$$

$$-\int \frac{v+2}{(v+4)(v+1)}\,dv = -\left(\frac{2}{3}\ln|v+4| + \frac{1}{3}\ln|v+1|\right)$$

$$\Longrightarrow -\left(\frac{2}{3}\ln|v+4| + \frac{1}{3}\ln|v+1|\right) = \ln|x| + C$$

$$2\ln|v+4| + \ln|v+1| = -3\ln|x| + D$$

$$\ln|v+4| + \ln|v+1| = \ln\frac{1}{|x|^3} + D$$

$$|v+4|\,|v+1| = \frac{1}{|x|^3}\cdot D$$

$$\Longrightarrow |x|^3 \left|\frac{y}{x}+4\right|\left|\frac{y}{x}+1\right| = D$$

$$|x|^{\cancel{3}}\,\frac{|y+4x|}{\cancel{|x|}}\,\frac{|y+x|}{\cancel{|x|}} = D$$

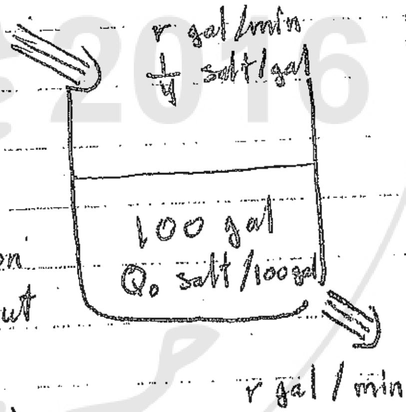
$$|x|\,|y+4x|\,|y+x| = D$$ (~~explicit~~ implicit solution)

(2.3) Modeling with First Order DE

ex1: At time $t=0$, a tank contains Q_0 pound of salt, dissolved in 100 gallons of water. Assume that water containing $\frac{1}{4}$ pound of salt/gallon is entering the tank at rate of r gallons/min and leaves at the same rate.

a) Set up the IVP that describes this process.
b) Find the amount of salt in the tank at any time t.
c) Find the limiting amount of salt (Q_L) in the tank after a very long time.
d) Assume $Q_0 = 2Q_L$, find the flow rate required if the time is not exceed 45 minutes, such that $Q(t) = 25.5$.

a) let $Q(t)$ be the amount of salt in the tank at any time, $Q(0) = Q_0$.



$$\frac{dQ}{dt} = \text{Rate in} - \text{Rate out}$$

$$= \text{concentration} * \text{flow in} - \text{concentration} * \text{flow out}$$

$$\frac{dQ}{dt} = \left(\frac{1}{4}\right)(r) - \left(\frac{Q}{100}\right)(r)$$

$$\Rightarrow \frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100} \quad , \quad Q(0) = Q_0$$

b) The previous equation is first linear ODE with constant coefficients, so:

$$Q(t) = \frac{b}{a} + \left(Q_0 - \frac{b}{a}\right) e^{at},$$

$$b = \frac{-r}{4}, \quad a = \frac{-r}{100} \implies \boxed{\frac{b}{a} = 25}$$

$$\implies Q(t) = 25 + (Q_0 - 25)\, e^{-rt/100}$$

c) $$\lim_{t \to \infty} Q(t) = 25 \implies \boxed{Q_L = 25}$$

d) Assuming $Q_0 = 2Q_L \implies Q_0 = 50$, $Q(45) = 25.5$

$$Q(t) = 25 + (50 - 25)\, e^{-rt/100}$$

$$Q(t) = 25 + 25 e^{-rt/100}$$

$$Q(45) = 25 + 25 e^{-r(45)/100}$$

$$25.5 = 25 + 25 e^{-45r/100}$$

$$0.5 = 25 e^{-45r/100}$$

$$0.02 = e^{-45r/100}$$

$$-2 \ln 10 = -45 \frac{r}{100}$$

$$\implies \boxed{r = \frac{200}{45} \ln 10}$$

ex21 Consider a pond that initially contains 10 millions gallons of fresh water. Water containing toxic waste flows into the pond at rate of 5 million gal/year and exits at the same rate. The concentration $C(t)$ of toxic waste in the incoming water varies periodically with time according to this equation:

$$C(t) = 2 + \sin 2t \quad g/gal$$

a) Construct a mathmatical model of this flow process

b) Find the amount of toxic waste in the pond at any time t.

a) $Q(0) = 0$ (fresh water)

let $Q(t)$ be the amount of toxic waste in the pond at any time t.

$$\frac{dQ}{dt} = (5 * 10^6)(2 + \sin 2t) - \frac{Q}{\cancel{10 * 10^6}\,2}(\cancel{5 * 10^6}\,1)$$

$$\frac{dQ}{dt} = (5 * 10^6)(2 + \sin 2t) - \frac{1}{2} Q \quad , Q(0) = 0$$

b) $\frac{dQ}{dt} + \frac{1}{2} Q = (5*10^6)(2+\sin 2t)$

$\Longrightarrow \quad P(t) = \frac{1}{2}, \quad g(t) = (5*10^6)(2+\sin 2t)$

$$\mu(t) = e^{\int P(t)dt} = e^{\int \frac{1}{2} dt} = e^{\frac{1}{2}t}$$

$$Q(t) = \frac{1}{\mu(t)} \left[\int \mu(t) g(t) dt + C \right]$$

$$Q(t) = \frac{1}{e^{\frac{1}{2}t}} \left[\int e^{\frac{1}{2}t} (5*10^6)(2+\sin 2t) dt + C \right]$$

$$Q(t) = \frac{1}{e^{\frac{1}{2}t}} \left[5*10^6 \int (2e^{\frac{1}{2}t} + e^{\frac{1}{2}t}\sin 2t) dt + C \right]$$

$$Q(t) = \frac{1}{e^{\frac{1}{2}t}} \left[5*10^6 \left(4e^{\frac{1}{2}t} + \frac{2}{17} e^{\frac{1}{2}t} (\sin 2t - 4\cos 2t) \right) + C \right]$$

$$Q(t) = \frac{5*10^6 \, e^{\frac{1}{2}t} \left(4 + \frac{2}{17}(\sin 2t - 4\cos 2t) \right) + C}{e^{\frac{1}{2}t}}$$

$$Q(t) = 5*10^6 \left(4 + \frac{2}{17}(\sin 2t - 4\cos 2t) \right) + \frac{C}{e^{\frac{1}{2}t}}$$

$$Q(0) = 0 = 5*10^6 \left(4 + \frac{2}{17} (\sin(0) - 4\cos(0)) \right) + \frac{C}{e^0}$$

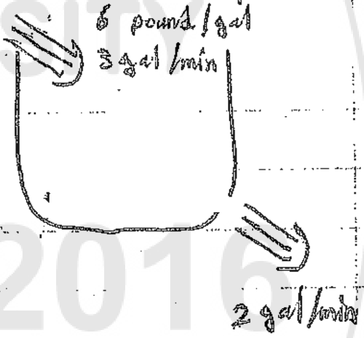
$$0 = 5*10^6 \left(4 - \frac{8}{17} \right) + C$$

$$\Longrightarrow \boxed{C = -\frac{3}{17} * 10^8}$$

so, the solution becomes:

$$\boxed{Q(t) = 5*10^{6}\left(4 + \frac{2}{17}(\sin 2t - 4\cos 2t)\right) + \frac{-3}{17}*10^{8}\, e^{-\frac{1}{2}t}}$$

ex3: At time $t=0$, a tank contains Q_0 pound of salt, dissolved in 50 gal of water. Assume that water containing 6 pound of salt / gallon is entering the tank at rate of 3 gal/min and leaves it at a rate of 2 gal/min. Set up the IVP that describes this salt process.



$$\frac{dQ}{dt} = \text{Rate in} - \text{Rate out}$$

$$\frac{dQ}{dt} = (6)(3) - \frac{Q}{50+(3-2)t}(2)$$

$$\boxed{\frac{dQ}{dt} = 18 - \frac{2Q}{50+t}}$$

(2.4) Difference Between Linear and Non-linear DE's

Recall that the first order ODE has the general form:

$$\frac{dy}{dt} = f(t, y) \quad \text{---} (1)$$

DE (1) is linear if f is linear in y.
DE (1) is non-linear if f is not linear in y.

Th 2.4.1 (First Order Linear ODE):

Consider the first order linear IVP:

$$\frac{dy}{dt} + P(t)y = g(t) \quad , \quad y(t_0) = y_0$$

If P(t) and g(t) are both continuous on some open interval I: $\alpha < t < \beta$ containing t_0, then there exist a unique solution $y = \phi(t)$, that satisfies the IVP.

proof: 1) Existence:

$$y(t) = \frac{1}{\mu(t)} \left[\int \mu(t)\, g(t)\, dt + C \right]$$

where: $\mu(t) = e^{\int P(t)dt}$

Note that Th 2.4.1 provides two important features: Existence and Uniqueness.

2) Uniqueness: Using the initial condition:

$$\mu(t) = e^{\int_{t_0}^{t} p(t)\,dt}$$

$$\Longrightarrow \quad y(t) = \frac{1}{\mu(t)} \left[\int_{t_0}^{t} \mu(t)\, g(t)\, dt + y_0 \right]$$

where: $y_0 = C$

ex: Solve the IVP.

$$x y' + 2y = 4x^2 \quad , \quad y(1) = 2$$

$$y' + \frac{2}{x} y = 4x$$

$$\Longrightarrow \quad P(x) = \frac{2}{x} \quad , \quad g(x) = 4x$$

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x|} = x^2$$

$$y(x) = \frac{1}{\mu(x)} \left[\int \mu(x)\, g(x)\, dx + C \right]$$

$$y(x) = \frac{1}{x^2} \left[\int x^2 \cdot 4x \, dx + C \right]$$

$$y(x) = \frac{1}{x^2} \left[\int 4x^3 \, dx + C \right]$$

$$y(x) = \frac{1}{x^2} \left(x^4 + C \right)$$

to find C, we use the initial condition:

$$y(1) = 2 \implies 2 = \frac{1}{(1)^2}\left((1)^4 + C\right),$$

$$\implies \boxed{C = 1}$$

Hence, the solution becomes:

$$\boxed{y(x) = \frac{1}{x^2}\left(x^4 + 1\right)}$$

ex 2: $xy' + 2y = 4x^2$, $y(1) = 2$

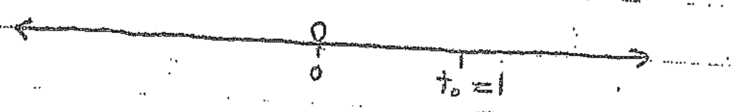
Find the largest interval in which the solution is certain to exist.

solution: Since the DE is first order linear ODE, we can apply Th 2.4.1 .

$P(x) = \frac{2}{x}$ is continuous every where expect at $x = 0$.

$g(x) = 4x$ is continuous every where.

$y(1) = 2 \implies t_0 = 1$



So the largest interval that contains the t_0 and gurantees the existence of solution is $(0, \infty)$.

ex3: Find the largest interval, where the solution can be defined for the IVP:

$$(t-3)y' + (\ln t)y = 2t \quad , \quad y(1) = 2$$

$$y' + \left(\frac{\ln t}{t-3}\right)y = \frac{2t}{t-3}$$

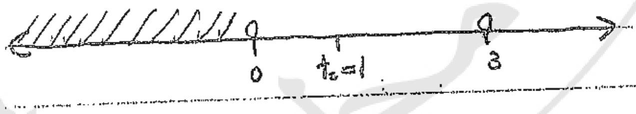
$$y(1) = 2 \Longrightarrow t_0 = 1$$

$$P(t) = \frac{\ln t}{t-3} \quad , \quad g(t) = \frac{2t}{t-3}$$

$P(t)$: $\ln t$

$t-3$

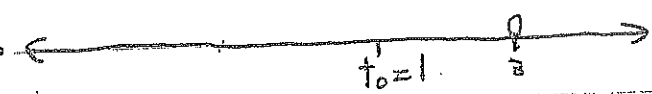
$P(t) = \frac{\ln t}{t-3}$



$g(t)$: $2t$

$t-3$

$g(t) = \frac{2t}{t-3}$



So, the largest interval that contains t_0 and the solution for the IVP can be definedin is $(0,3)$.

ex4: Find the largest interval, where the solution is certain to exist.

$$(\cos t)\, y' = \sin t\,(\cos t - y) \quad , \quad y(\pi) = 0$$

$$(\cos t)\, y' = \sin t \cos t - y \sin t$$

$$y' = \sin t - y \tan t \quad , \quad t \neq \frac{\pi}{2} + n\pi \; , \; n \in \mathbb{Z}$$

$$y(\pi) = 0 \implies t_0 = \pi$$

$P(t) = \tan t$

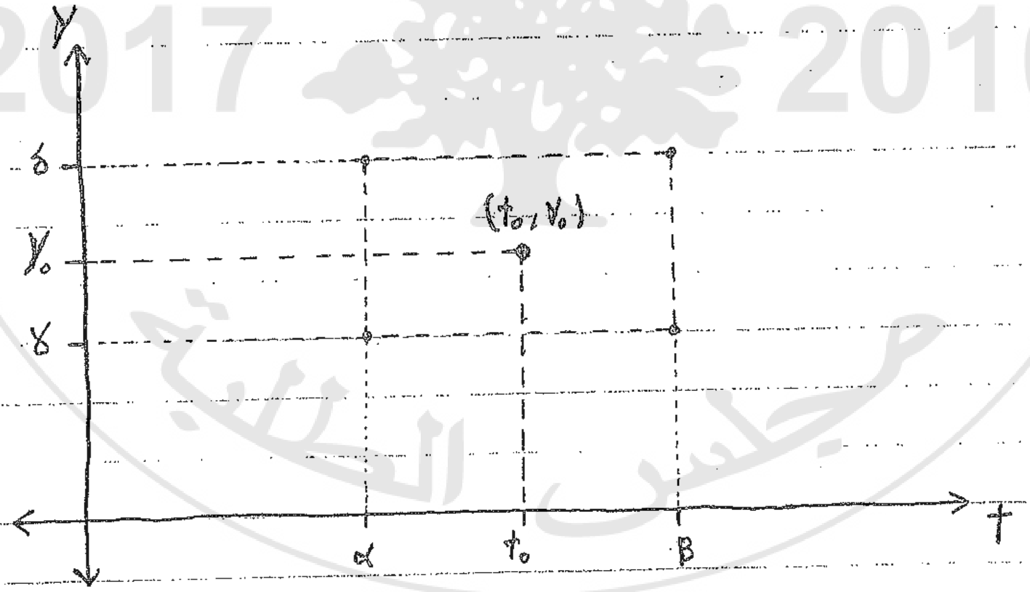
$\implies$ the largest interval is $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Th 2.4.2 (First Order More General ODE):

Consider the IVP:

$$\frac{dy}{dt} = f(t,y) \; ; \; y(t_0) = y_0$$

If $f(t,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on some open rectangle $(t,y) \in (\alpha, B) \times (\gamma, \delta)$ containing the initial point (t_0, y_0), then there exist a unique solution $y = \phi(t)$, that satisfies the IVP on some open rectangle.



ex1: Solve the IVP and find an interval in which the solution exists.

$$y' = y^2 \quad , \quad y(0) = 1$$

$f(t,y) = f(y) = y^2$ is continuous every where.

$\frac{\partial f}{\partial y} = 2y$ is also continuous every where.

Hence, by Th 2.4.2, there exist a unique solution defined on some open rectangle.

$$\frac{dy}{dt} = y^2$$

$$\int y^{-2}\, dy = \int dt$$

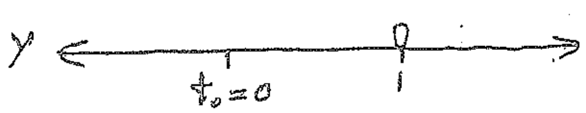
$$\frac{-1}{y} = t + c$$

$$y(0) = 1 \Longrightarrow \frac{-1}{(1)} = (0) + c$$

$$\Longrightarrow \boxed{c = -1}$$

$$\Longrightarrow \frac{-1}{y} = t - 1$$

$$\boxed{y = \frac{1}{1-t}}$$



$\Longrightarrow$ the interval is $(-\infty, 1)$.

ex21: Consider the IVP $y' = y^{\frac{1}{3}}$, $y(0) = 0$, $t \geq 0$

Apply Th 2.4.2 to this IVP.

$f(t,y) = y^{\frac{1}{3}}$ is continuous every where.

$\frac{\partial f}{\partial y} = \frac{1}{3} y^{\frac{-2}{3}} = \frac{1}{3\sqrt[3]{y^2}}$ is continuous every where expect at $y = 0$

So, Th 2.4.2 doesn't apply.

Solve the IVP.

$$\frac{dy}{dt} = y^{\frac{1}{3}}$$

$$\int y^{-\frac{1}{3}}\, dy = \int dt$$

$$\frac{3}{2} y^{\frac{2}{3}} = t + c$$

$$y(0) = 0 \implies \frac{3}{2} (0)^{\frac{2}{3}} = (0) + c$$

$$\implies \boxed{c = 0}$$

$$\implies \frac{3}{2} y^{\frac{2}{3}} = t$$

$$y^{\frac{2}{3}} = \frac{2}{3} t$$

$$y^2 = \left(\frac{2}{3} t\right)^3$$

$$y = \mp \sqrt{\frac{2}{3} t^3}$$

$\Longrightarrow \quad y_1 = +\sqrt{\left(\frac{2}{3}t\right)^3}$

$y_2 = -\sqrt{\left(\frac{2}{3}t\right)^3}$

$y_3 = 0$ (from the initial condition)

The reason for having more than one solution is because the condition of Th 2.4.2 fail " $\frac{\partial f}{\partial y}$ isn't continuous."

However, the continuity of $f(t,y)$ does ensure the existence of solution, but not uniqueness.

ex3: Apply Th 2.4.2 to the IVP:

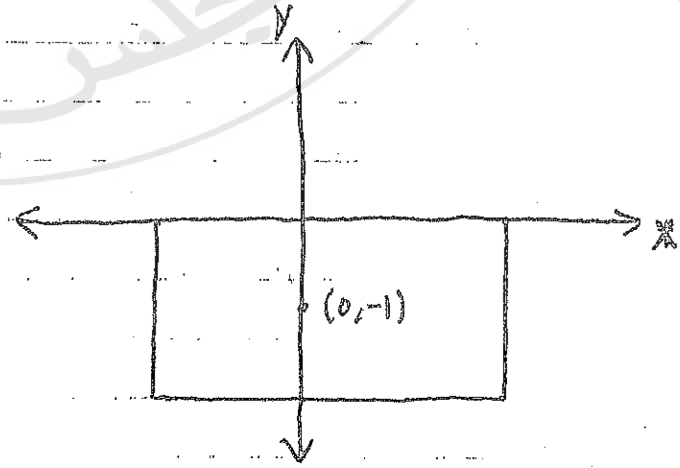
$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0) = -1$$

f is continuous every where expect $y=1$.

$$\frac{\partial f}{\partial y} = \frac{-2(3x^2+4x+2)}{(2(y-1))^2} = \frac{-(3x^2+4x+2)}{2(y-1)^2}$$

$\frac{\partial f}{\partial y}$ is continuous every where expect to $y=1$ also.

we can find open rectangle contains the point $(0,-1)$ and avoiding $y=1$.



note: if $y(0)=1$, then Th 2.4.2 doesn't apply
$\Longrightarrow$ we can't ensure uniqueness.

(2.5) Autonomous DE's and Population Dynamics

Recall that a first order ODE has the general form:

$$\frac{dy}{dt} = f(t, y) \quad --- (1)$$

In the DE (1), when the function $f(t,y)$ becomes $f(y)$ only (the independent variable t is hidden), then the DE (1) is called "Autonomous Equation".

ex1: $\frac{dy}{dt} = ay - b$

$f(t,y) = ay - b = f(y) \Longrightarrow$ Autonomous Equation.

However, the solution of this DE is:

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$$

ex2: $y' + 3y = 6$

$f(t,y) = 6 - 3y = f(y) \Longrightarrow$ Autonomous Equation.

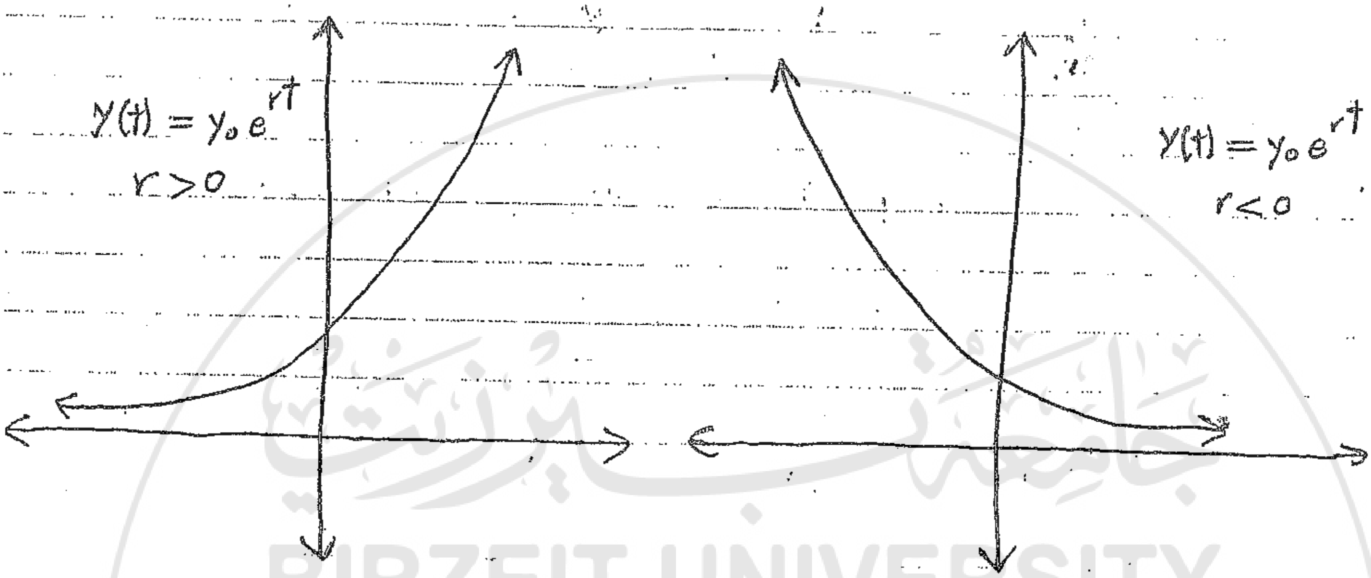
ex3: In case of population growth or decay:

$$\frac{dy}{dt} = ry \quad , \quad y(0) = y_0$$

$y(t)$: the population at any time t.

$\Longrightarrow$ the solution is: $y(t) = y_0 e^{rt}$

$f(t,y) = ry = f(y) \implies$ Autonomous Equation.



ex4: A population doubles in two years, when will it triple?

growth $\implies r > 0$

$P(t) = P_0 e^{rt}$, $P(2) = 2P_0$

$P(2) = P_0 e^{r(2)}$

$2\cancel{P_0} = \cancel{P_0} e^{2r}$

$\ln 2 = 2r \implies \boxed{r = \frac{\ln 2}{2}}$

$\implies$ the solution is: $P(t) = P_0 e^{\frac{\ln 2}{2} t}$

$P(t) = 3P_0$

$\cancel{P_0} e^{\frac{\ln 2}{2} t} = 3\cancel{P_0}$

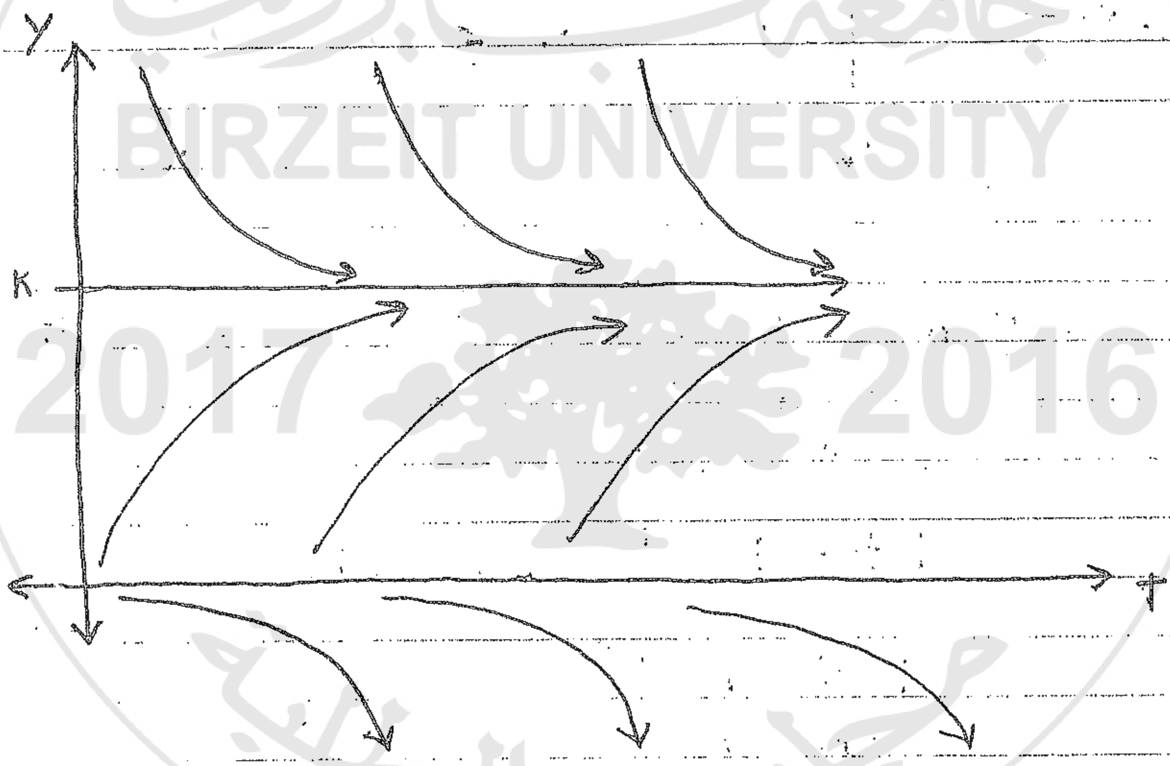
$\frac{\ln 2}{2} t = \ln 3 \implies \boxed{t = 2 \frac{\ln 3}{\ln 2}}$

ex5: Draw the logistic equation:

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right) y \ , \ r > 0 \ , \ K > 0$$

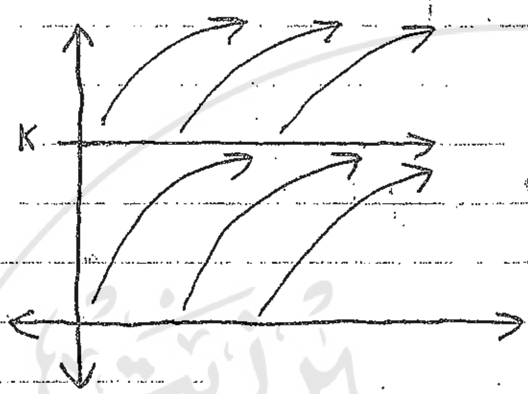
critical points (equilibrium solutions):

$$y = K \ , \ y = 0$$

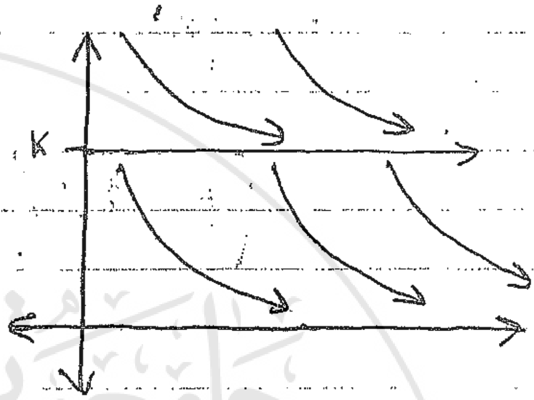


note: 1) $y = K$ is an asymptotically stable point.
2) $y = 0$ is an unstable point.

note: $y = K$ will be a semi-stable point, if the graph was like:



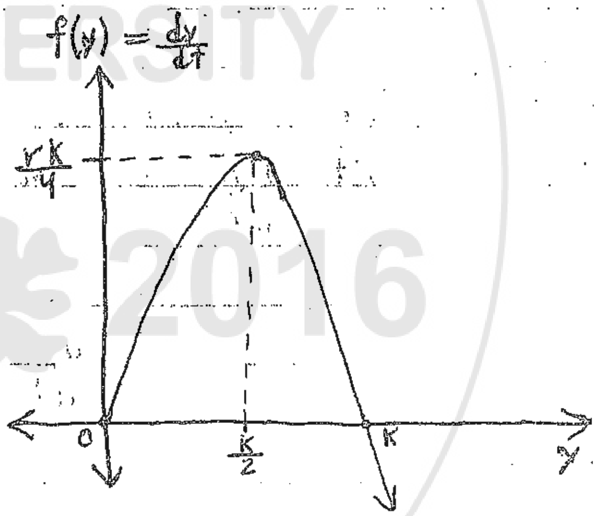
or



$$f(y) = \frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

$$0 = r\left(1 - \frac{y}{K}\right)y$$

$$\Longrightarrow \; y = 0 \;,\; y = K$$



* Increasing and Decreasing:

$f(y) > 0$ on $0 < y < K$, that is $\frac{dy}{dt} > 0$, so the the solution $y(t)$ is increasing on $(0, K)$.

$f(y) < 0$ on $y > K$, that is $\frac{dy}{dt} < 0$, so the solution $y(t)$ is decreasing on $y > K$.

To find the max of $f(y)$:

$$f'(y) = 0$$

$$r - \frac{2ry}{K} = 0 \implies \boxed{y = \frac{K}{2}}$$

$$\implies \left(\frac{K}{2}, f\left(\frac{K}{2}\right)\right) = \left(\frac{K}{2}, \frac{rK}{4}\right)$$

* Concave up and Concave down:

The graph of the solution $y(t)$ is concave up (convex), if $\frac{d^2y}{dt^2} > 0$ and concave down, if $\frac{d^2y}{dt^2} < 0$

$$\frac{dy}{dt} = f(y)$$

$$\frac{d^2y}{dt^2} = f'(y)\,\frac{dy}{dt}$$

$\frac{d^2y}{dt^2} > 0$ when both $f'(y)$ and $\frac{dy}{dt}$ have the same sign.

So, $y(t)$ is concave up and increasing on $(0, \frac{K}{2})$.
(Since $f'(y)$ is positive ($f(y)$ is increasing) and $\frac{dy}{dt}$ is also positive)

$y(t)$ is concave up but decreasing on (K, ∞).
(Since $f'(y)$ is negative ($f(y)$ is decreasing) and $\frac{dy}{dt}$ is also negative)

$\frac{d^2y}{dt^2} < 0$ when $f'(y)$ and $\frac{dy}{dt}$ have opposite sign.

So, $y(t)$ is concave down and increasing on $(\frac{K}{2}, K)$.
(Since $f'(y)$ is negative ($f(y)$ is decreasing) and $\frac{dy}{dt}$ is positive)

$\left(\frac{K}{2}, f\left(\frac{K}{2}\right)\right) = \left(\frac{K}{2}, \frac{rK}{4}\right)$ is an inflection point for $y(t)$.

note: Given the DE $f(y) = \frac{dy}{dt}$, if y_1 is a critical point (equilibrium solution), then y_1 is :-

1) Asymptotically stable, if $f'(y_1) < 0$.
2) Unstable, if $f'(y_1) > 0$.
3) Semi-stable, if $f'(y_1) = 0$.

ex6: Find the general solution of the autonomous DE:

$$\frac{dy}{dt} = r\left(1-\frac{y}{K}\right)y \quad , \quad y_0 \in (0,K)$$

$$\int \frac{dy}{\left(1-\frac{y}{K}\right)y} = \int r\,dt$$

$$\int \frac{dy}{\left(1-\frac{y}{K}\right)y} = rt + C$$

$$\frac{1}{\left(1-\frac{y}{K}\right)y} = \frac{A}{\left(1-\frac{y}{K}\right)} + \frac{B}{y}$$

$$\Longrightarrow A = \frac{1}{K} \quad , \quad B = 1$$

$$\Longrightarrow \int \frac{dy}{\left(1-\frac{y}{K}\right)y} = \int \left(\frac{\frac{1}{K}}{1-\frac{y}{K}} + \frac{1}{y}\right) dy$$

$$= -\ln\left|1-\frac{y}{K}\right| + \ln|y|$$

Since $y_0 \in (0,K) \Longrightarrow y > 0$

$$\therefore \ln y - \ln\left(1-\frac{y}{K}\right) = rt + C$$

$$\ln\left(\frac{y}{1-\frac{y}{K}}\right) = rt + C$$

$$\frac{y}{1-\frac{y}{K}} = e^{rt}\cdot e^{c}$$

$$\frac{y}{1-\frac{y}{K}} = D e^{rt}$$

$$y(0) = y_0 \Longrightarrow \boxed{D = \frac{y_0}{1-\frac{y_0}{K}}}$$

$$\Longrightarrow \frac{y}{1-\frac{y}{K}} = \frac{y_0}{1-\frac{y_0}{K}} e^{rt}$$

$$\frac{Ky}{K-y} = \frac{Ky_0}{K-y_0} e^{rt}$$

$$\frac{K-y}{Ky} = \frac{K-y_0}{Ky_0} e^{-rt}$$

$$\frac{1}{y} - \frac{1}{K} = \frac{K-y_0}{Ky_0} e^{-rt}$$

$$\frac{1}{y} = \frac{K-y_0}{Ky_0} e^{-rt} + \frac{1}{K}$$

$$\frac{1}{y} = \frac{(K-y_0)e^{-rt} + y_0}{Ky_0}$$

$$\Longrightarrow \boxed{y = \frac{Ky_0}{y_0 + (K-y_0)e^{-rt}}}$$

(2.6) Exact Equation

Solve the DE:

$$2x + y^2 + 2xyy' = 0$$

This DE is not linear and not separable, how to solve it?

Th 2.6.1:

Suppose that a DE has the form:

$$M(x,y) + N(x,y)\, y' = 0$$

where M, N, M_y, N_x are all continuous on the region:

$R: (x,y) \in (\alpha, \beta) \times (\gamma, \delta)$

The DE is exact, iff:

$$M_y(x,y) = N_x(x,y) \quad \text{for all } (x,y) \in R$$

That is, there exist a function $\psi(x,y)$, such that:

$$\psi_x(x,y) = M(x,y) \quad \text{and}$$

$$\psi_y(x,y) = N(x,y), \text{ iff}$$

M and N satisfying:

$$M_y(x,y) = N_x(x,y)$$

So, to solve the DE:

$$2x + y^2 + 2xyy' = 0$$

$$\left.\begin{aligned} M(x,y) &= 2x + y^2 \implies M_y = 2y \\ N(x,y) &= 2xy \implies N_x = 2y \end{aligned}\right\} \text{exact}$$

Thus, there exist a function $\psi(x,y)$, such that:

$$\psi_x = M(x,y) = 2x + y^2 \quad , \quad \psi_y = N(x,y) = 2xy$$

$$\psi(x,y) = \int \psi_x \, dx$$

$$= \int (2x + y^2)\, dx$$

$$= x^2 + xy^2 + g(y)$$

But $\psi_y = N(x,y)$

$$\implies 0 + \cancel{2xy} + g'(y) = \cancel{2xy}$$

$$g'(y) = 0 \implies \boxed{g(y) = C}$$

$$\implies \boxed{\psi(x,y) = x^2 + xy^2 + C}$$

The solution is given implicitly by:

$$x^2 + xy^2 = C$$

Since:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\psi_x + \psi_y \frac{dy}{dx} = 0$$

$$\frac{d\left(\psi(x,y)\right)}{dx} = 0$$

$$\Longrightarrow \boxed{\psi(x,y) = C}$$

ex1: Solve the DE:

$$\frac{dy}{dx} = \frac{-(x+4y)}{4x-y}$$

$$(x+4y) + (4x-y)\frac{dy}{dx} = 0$$

$$\left.\begin{array}{l} M(x,y) = x+4y \Longrightarrow M_y = 4 \\ N(x,y) = 4x-y \Longrightarrow N_x = 4 \end{array}\right\} \text{exact}$$

Thus, $\exists$ a function $\psi(x,y)$, such that:

$$\psi_x = M(x,y) \quad , \quad \psi_y = N(x,y)$$

$$\psi(x,y) = \int \psi_y \, dy = \int (4x - y) \, dy$$

$$\psi(x,y) = 4xy - \frac{y^2}{2} + g(x)$$

Now using $\psi_x = M(x,y)$, we can find $g(x)$:

$$\psi_x = M(x,y)$$

$$\cancel{4y} - 0 + g'(x) = x + \cancel{4y}$$

$$g'(x) = x \Longrightarrow \boxed{g(x) = \frac{x^2}{2}} + C$$

$$\Longrightarrow \boxed{\psi(x,y) = 4xy - \frac{y^2}{2} + \frac{x^2}{2} + C}$$

The solution is given implicitly by:

$$4xy - \frac{y^2}{2} + \frac{x^2}{2} = C$$

or $\quad x^2 + 4xy - y^2 = C$

ex2: Solve the DE:

$$(3xy + y^2) + (xy + x^2)\frac{dy}{dx} = 0 \quad --- (1)$$

$$\left.\begin{array}{l} M(x,y) = 3xy + y^2 \Longrightarrow M_y = 3x + 2y \\ N(x,y) = xy + x^2 \Longrightarrow N_x = y + 2x \end{array}\right\} \text{not exact}$$

So, we may change the DE into an exact equation by multiplying it with a suitable function which we call the "Integrating Factor (I)".

The question is how to find (I)?

~~Remember that:~~

Remember that:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad --- (2)$$

if equation (2) is not exact, we multiply it by the integrating factor $I(x,y)$.

$$\Longrightarrow I(x,y)\,M(x,y) + I(x,y)\,N(x,y)\frac{dy}{dx} = 0 \quad --- (3)$$

for equation (3) to be exact, we must have:

$$(IM)_y = (IN)_x$$

$$I_y M + I M_y = I_x N + I N_x$$

$$(I_y M - I_x N) + (M_y - N_x) I = 0 \quad --- (4)$$

equation (4) is PDE which difficult to solve.

So, we need the integrating factor (I) to be either:

1) $I(x)$ only.
2) $I(y)$ only.
3) $I(v)$, where $v = xy$.

Summary: to find I:

1) If $\frac{M_y - N_x}{N} = f(x)$, then:

$$I(x) = e^{\int f(x)\,dx}$$

proof: If $I = I(x)$, then equation (4) becomes:

$$(0 - I_x N) + (M_y - N_x) I = 0$$

$$I_x N = (M_y - N_x) I$$

$$\frac{I_x}{I} = \frac{M_y - N_x}{N}$$

$$\int \frac{I_x}{I} dx = \int f(x)\,dx$$

$$\ln I = \int f(x)\,dx$$

$$\Longrightarrow I = e^{\int f(x)\,dx}$$

2) If $\frac{M_y - N_x}{M} = g(y)$, then:

$$I(y) = e^{-\int g(y)\,dy}$$

3) If $\frac{M_y - N_x}{yN - xM} = h(v)$, then:

$$I(v) = e^{\int h(v)\,dv}$$, where $v = xy$.

Now, back to the example:

$$(3xy + y^2) + (xy + x^2)\frac{dy}{dx} = 0 \quad \text{---} (1)$$

$$\left.\begin{array}{l} M(x,y) = 3xy + y^2 \Longrightarrow M_y = 3x + 2y \\ N(x,y) = xy + x^2 \Longrightarrow N_x = y + 2x \end{array}\right] \text{not exact}$$

Now, let us try applying the second point:

$$\frac{M_y - N_x}{M} = \frac{(3x + 2y) - (y + 2x)}{3xy + y^2}$$

$$= \frac{x + y}{y(3x + y)} \neq g(y)$$

(not a function of y)

So, let us try the first point instead:

$$\frac{M_y - N_x}{N} = \frac{(3x+2y)-(y+2x)}{xy+x^2},$$

$$= \frac{\cancel{x+y}}{x\,(\cancel{x+y})}$$

$$= \frac{1}{x} = f(x).$$

(a function of x only)

So, $I(x) = e^{\int f(x)\,dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x|$

Now, we multiply the DE(1) by the integrating factor (I):

$$IM + IN\frac{dy}{dx} = 0$$

$$(3x^2y + xy^2) + (x^2y + x^3)\frac{dy}{dx} = 0 \quad ---(5)$$

In the new DE(5):

$$\left.\begin{aligned} M(x,y) &= 3x^2y + xy^2 \implies M_y = 3x^2 + 2xy \\ N(x,y) &= x^2y + x^3 \implies N_x = 2xy + 3x^2 \end{aligned}\right\} \text{exact}$$

Thus, $\exists\ \psi(x,y)$, such that:

$$\psi_x = M(x,y)\ ,\ \psi_y = N(x,y)$$

$$\Psi(x,y) = \int \Psi_y \, dy$$

$$= \int (x^2 y + x^3) \, dy$$

$$\Psi(x,y) = \frac{x^2 y^2}{2} + x^3 y + g(x)$$

Now, using $\Psi_x = M(x,y)$, we can find $g(x)$:

$$\Psi_x = M(x,y)$$

$$\cancel{xy^2} + \cancel{3x^2y} + g'(x) = \cancel{3x^2y} + \cancel{xy^2}$$

$$g'(x) = 0 \implies \boxed{g(x) = C}$$

The solution is given implicitly by:

$$\frac{x^2 y^2}{2} + x^3 y = C$$

or $\quad x^2 y^2 + 2x^3 y = C$

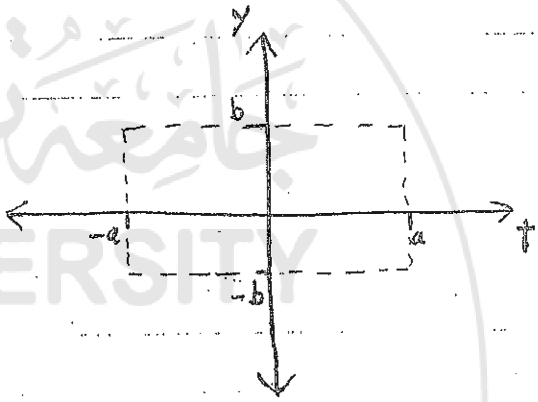
(2.8) The Existence and Uniqueness Theorem

In this section, we will disscuss the proof of Th 2.4.2 " The Fundamental Existence and Uniqueness Theorem ", which is equivalent to the following Theorem:

Th 2.8.1:

If $f(t,y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on:

R: $|t| \leq a$, $|y| \leq b$



then, there is some interval $|t| \leq h \leq a$, in which there is a unique solution $y = \phi(t)$ that satisfies the IVP (1):

$$\frac{dy}{dt} = f(t,y) \quad , \; y(0) = 0 \quad --- (1)$$

Note that the IVP (1) is exactly the same as the IVP in Th 2.4. expect the initial condition, which states here at the origin.

But this is not a problem since we can transform any IVP to an equivalent one that starts at origin.

ex 1) Transform the IVP $y' = y^3 - t^2$, $y(2) = -5$ to an equivalent one with initial condition starts at the origin.

$y(2) = -5 \Longrightarrow t_0 = 2 , y_0 = -5$

let $s = t - 2 \Longrightarrow s_0 = t_0 - 2 = 0 \Longrightarrow t = s+2$

let $z = y + 5 \Longrightarrow z_0 = y_0 + 5 = 0 \Longrightarrow y = z - 5$

$$z(s_0) = z_0 \Longrightarrow z(0) = 0$$

$$z' = y'$$

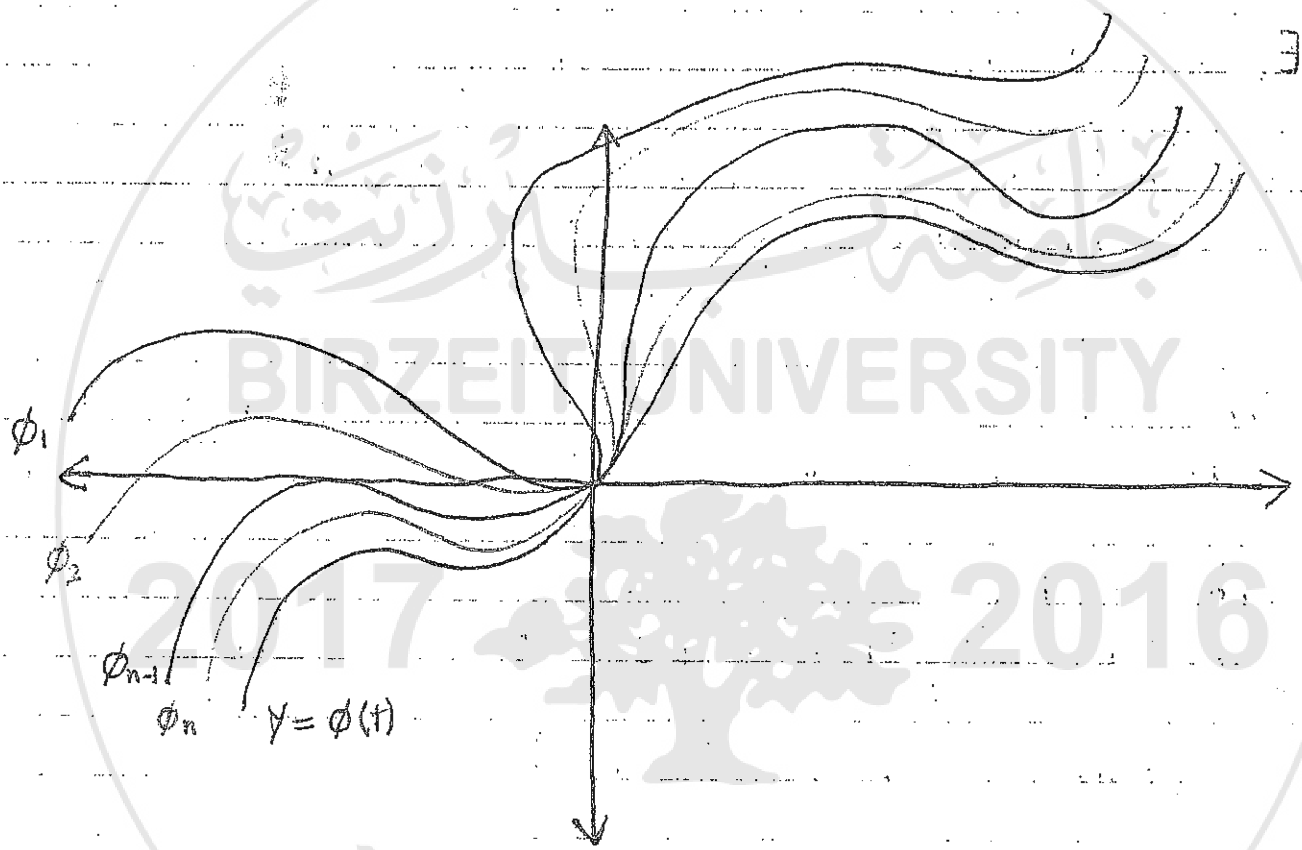
So, the IVP becomes:

$$z' = (z-5)^3 - (s+2)^2 \quad , \quad z(0) = 0$$

Recall that the IVP (1):

$$\frac{dy}{dt} = f(t,y) \quad , \; y(0)=0$$

$\Longrightarrow \; y = \phi(t)$ is the solution.



So, $\lim\limits_{n\to\infty} \phi_n = \phi(t)$ is the solution.

proof: (the existence):

Suppose that $y = \phi(t)$ is a solution for the IVP (1)

$$y = \phi(t) = \int_0^t f(s, \phi(s))\, ds \quad --- (2)$$

Equation (2) is called "Integral Equation".

Note that the integral equation is exactly the same as equation (1), therefore, any solution of one equation is a solution for the second equation.

We will use "The Method of Successive Approximation" (also called "Picard's Iteration").

The method works as follows, we can choose any value for ϕ_0 to start with, but we take the simplest choice ($\phi_0 = 0$):

$$\Longrightarrow \phi_1(t) = \int_0^t f(s, \phi_0(s))\, ds$$

$$\Longrightarrow \phi_2(t) = \int_0^t f(s, \phi_1(s))\, ds$$

$$\Longrightarrow \phi_3(t) = \int_0^t f(s, \phi_2(s))\, ds$$

$$\vdots$$

$$\Longrightarrow \phi_n(t) = \int_0^t f(s, \phi_{n-1}(s))\, ds$$

if $\lim_{n \to \infty} \phi_n(t) = \phi(t)$, then $\phi(t)$ will be the solution and we are done.

ex1: Use Picard's "The Method of Successive Approximation" to solve the IVP.

$y' = 2t(1+y)$, $y(0) = 0$

$f(t, y) = 2t(1+y)$

We choose the simplest choice to start with, which is $\phi_0 = 0$.

$$\phi_1 = \int_0^t f(s, \phi_0(s))\, ds$$

$$= \int_0^t 2s\,(1+0)\, ds$$

$$= \int_0^t 2s\, ds$$

$$= s^2 \Big]_0^t$$

$$\phi_1 = t^2$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\, ds$$

$$= \int_0^t f(s, s^2)\, ds$$

$$= \int_0^t 2s(1+s^2)\, ds$$

$$= s^2 + \frac{s^4}{2} \Big]_0^t$$

$$\phi_2(t) = t^2 + \frac{t^4}{2}$$

$$\phi_3(t) = \int_0^t f(s, \phi_2(s))\, ds$$

$$= \int_0^t f\left(s, s^2 + \frac{s^4}{2}\right) ds$$

$$= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2}\right) ds$$

$$= s^2 + \frac{s^4}{2} + \frac{s^6}{6} \Big]_0^t$$

$$\phi_3(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6}$$

$$\phi_4(t) = \int_0^t f(s, \phi_3(s))$$

$$= \int_0^t f\left(s, s^2 + \frac{s^4}{2} + \frac{s^6}{6}\right) ds$$

$$= \int_0^t 2s\left(1 + s^2 + \frac{s^4}{2} + \frac{s^6}{6}\right) ds$$

$$= s^2 + \frac{s^4}{2} + \frac{s^6}{6} + \frac{s^8}{24}\Big]_0^t$$

$$\phi_4(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \frac{t^8}{24}$$

$$\Longrightarrow \phi_n(t) = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \frac{t^8}{24} + \dots + \frac{t^{2n}}{n!}$$

Note that $\phi_n(t)$ is nth partial sum of the infinite series:

$$\sum_{K=1}^{\infty} \frac{t^{2K}}{K!}$$

The sequence $\phi_n(t)$ converges, iff $\sum_{K=1}^{\infty} \frac{t^{2K}}{K!}$ converges.

But we can use ratio test to check whether the infinite series converges or not:

$$\lim_{K\to\infty}\left|\frac{a_{K+1}}{a_K}\right| = \lim_{K\to\infty}\left|\frac{t^{2(K+1)}}{(K+1)!}\cdot\frac{K!}{t^{2K}}\right|$$

$$= \lim_{K\to\infty}\frac{t^2}{K+1} = 0 < 1 \Longrightarrow \text{The infinite series converge}$$

Hence, $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!}$

$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots + \frac{x^{2k}}{k!}$

$e^{x^2}-1 = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots + \frac{x^{2k}}{k!}$

$e^{x^2}-1 = \sum_{k=1}^{\infty} \frac{x^{2k}}{k!}$

using the method of integrating factor, the solution is: $y(t) = e^{t^2}-1$

So, $\phi(t) = e^{t^2}-1$ (the solution).

ex2: Use the method of successive approximation to solve the IVP.

$y' = 2(y+1)$, $y(0) = 0$

$Q_0 = 0$

$Q_1(t) = \int_0^t f(s,0)\,ds = \int_0^t 2\,ds = 2t$

$Q_2(t) = \int_0^t f(s,2s)\,ds = \int_0^t 2(2s+1)\,ds = 2t + 2t^2$

$\phi_3(t) = \int_0^t f(s, 2s+2s^2) = \int_0^t 2(2s+2s^2+1)\,ds = 2t + 2t^2 + \frac{4}{3}t^3$

$$\phi_4(t) = \int_0^t f\left(s, 2s + 2s^2 + \frac{4}{3}s^3\right) ds = \int_0^t 2\left(2s + 2s^2 + \frac{4}{3}s^3 + 1\right) ds$$

$$\phi_4(t) = 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4$$

$$\Longrightarrow \phi_n(t) = 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4 + \dots + \frac{2^n t^n}{n!}$$

Note that $\phi_n(t)$ is the nth partial sum of the infinite series: $\sum_{K=1}^{\infty} \frac{2^K t^K}{K!}$

So, $\phi_n(t)$ converges, iff $\sum_{K=1}^{\infty} \frac{2^K t^K}{K!}$ converges.

Applying ratio test:

$$\lim_{K \to \infty} \left| \frac{a_{K+1}}{a_K} \right| = \lim_{K \to \infty} \left| \frac{2^{K+1} t^{K+1}}{(K+1)!} \cdot \frac{K!}{2^K t^K} \right|$$

$$\lim_{K \to \infty} \left| \frac{2t}{K+1} \right| = 0 < 1 \Longrightarrow \text{the infinite series converges.}$$

Hence, $\phi(t) = \lim_{n \to \infty} \phi_n(t)$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \dots \frac{2^n x^n}{n!}$$

$$e^{2x} - 1 = 2x + 2x^2 + \frac{4x^3}{3} + \dots + \frac{2^n x^n}{n!}$$

$$e^{2x} - 1 = \sum_{K=1}^{\infty} \frac{2^K x^K}{K!}, \quad \text{So } \phi(t) = e^{2t} - 1 \quad (\text{the solution})$$

using the method of integrating factor the solution is:
$$y(t) = e^{2t} - 1$$

(3.1 - 3.4) Second Order Linear Hamogeneos Equations with Constant Coeffecients

The tite has the form:

$$a y'' + b y' + c y = 0 \quad , a, b, c \in \mathbb{R} , a \neq 0$$

To solve this DE we assume experimental solution of the form:

$$y = e^{rt} \quad , r \in \mathbb{R}$$

$$\Longrightarrow y' = r e^{rt} \quad , y'' = r^2 e^{rt}$$

Substitute y, y', y'' in the DE:

$$a (r^2 e^{rt}) + b (r e^{rt}) + c (e^{rt}) = 0$$

$$e^{rt} (a r^2 + b r + c) = 0$$

$$\Longrightarrow a r^2 + b r + c = 0$$

This equation is called "Characteristic Equation".

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

So, we have two solutions for the D.E.

We have three possible cases for the roots r_1 and r_2:

1) $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$ (real different)

The first solution will be $y_1(t) = e^{r_1 t}$
The second solution will be $y_2(t) = e^{r_2 t}$

The general solution is:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$\Longrightarrow \boxed{y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$$

2) $r_1 = r_2 = r$, $r_1 \cancel{\neq} r_2 \in \mathbb{R}$ (repeated)

The first solution will be $y_1(t) = e^{rt}$
The second solution will be $y_2(t) = t e^{rt}$

The general solution is:

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

$$\Longrightarrow \boxed{y(t) = c_1 e^{rt} + c_2 t e^{rt}}$$

3) r_1 and r_2 are conjugate complex numbers.

$r_1 = \lambda + i\mu$
$r_2 = \lambda - i\mu$

$r_{1,2} = \lambda \mp i\mu$

The first solution will be $y_1(t) = e^{\lambda t}\cos(\mu t)$
The ~~first~~ second solution will be $y_2(t) = e^{\lambda t}\sin(\mu t)$

The general solution is:

$y(t) = c_1 y_1(t) + c_2 y_2(t)$

$$\Longrightarrow \boxed{y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 \sin(\mu t) \cdot e^{\lambda t}}$$

ex1: a) Solve the IVP:

$y'' + 5y' + 6y = 0 \quad , \; y(0) = 2 \; , \; y'(0) = 3$

$r^2 + 5r + 6 = 0$ (Characteristic Equation)
$(r+3)(r+2) = 0$
$\Longrightarrow r_1 = -3 \; , \; r_2 = -2$ (real different)

$y_1 = e^{-3t} \neq y_2 = e^{-2t}$

The general solution is:

$y(t) = c_1 e^{-3t} + c_2 e^{-2t}$

To find c_1 and c_2, we use the initial value conditions.

$$y(0) = 2 \Longrightarrow 2 = c_1 e^{-3(0)} + c_2 e^{-2(0)}$$

$$\Longrightarrow \boxed{c_1 + c_2 = 2} \quad --- (1)$$

$$y'(t) = -3c_1 e^{-3t} - 2c_2 e^{-2t}$$

$$y'(0) = 3 \Longrightarrow 3 = -3c_1 e^{-3(0)} - 2c_2 e^{-2(0)}$$

$$\Longrightarrow \boxed{-3c_1 - 2c_2 = 3} \quad --- (2)$$

After solving equations (1) and (2):

$$c_1 = -7 \quad , c_2 = 9$$

$$\Longrightarrow \boxed{y(t) = -7e^{-3t} + 9e^{-2t}}$$

b) Find the max value obtained by the solution.

$$y'(t) = 0$$

$$21e^{-3t} - 18e^{-2t} = 0$$

$$21e^{-3t} = 18e^{-2t}$$

$$21 = 18e^{t} \Longrightarrow \boxed{t = \ln\left(\frac{21}{18}\right)}$$

The max value is:

$$y\left(\ln\left(\tfrac{7}{6}\right)\right) = -7e^{-3\ln\frac{7}{6}} + 9e^{-2\ln\frac{7}{6}}$$

ex2: Solve the DE:

$$y'' - 4y' + 4y = 0$$

$$r^2 - 4r + 4 = 0 \quad \text{(Characteristic Equation)}$$

$$(r-2)(r-2) = 0$$

$$\Longrightarrow r_1 = r_2 = r = 2 \quad \text{(repeated)}$$

$$y_1 = e^{2t}, \quad y_2 = te^{2t}$$

The general solution is:

$$\Longrightarrow \boxed{y(t) = c_1 e^{2t} + c_2 t e^{2t}}$$

ex3: Solve the DE:

$$y'' + 2y' + 2y = 0$$

$$r^2 + 2r + 2 = 0 \quad \text{(Characteristic Equation)}$$

$$r_{1,2} = -1 \mp i \quad \text{(conjugate complex numbers)}$$

$$y_1 = e^{-t}\cos t, \quad y_2 = e^{-t}\sin t$$

The general solution is:

$$y(t) = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$$

note: we found $r_{1,2}$ using: $r_{1,2} = \dfrac{-b \mp \sqrt{b^2 - 4ac}}{2a}$

ex4: Solve the DE:

$$y'' + 9y = 0$$

$$r^2 + 9 = 0 \quad \text{(characteristic equation)}$$

$$\Longrightarrow r_{1,2} = \mp 3i$$

$$y_1 = \cos 3t \quad , \quad y_2 = \sin 3t$$

The general solution is:

$$y(t) = c_1 \cos 3t + c_2 \sin 3t$$

ex5: Solve the IVP:

$$2y'' + 3y' = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 3$$

$$2r^2 + 3r = 0 \quad \text{(characteristic equation)}$$

$$r(2r + 3) = 0$$

$$\Longrightarrow r_1 = 0 \quad , \quad r_2 = \frac{-3}{2}$$

$$y_1 = e^0 = 1 \quad , \quad y_2 = e^{\frac{-3}{2}t}$$

The general solution is:

$$y(t) = c_1 + c_2 e^{\frac{-3}{2}t}$$

To find c_1, c_2 we use the initial value conditions:

$y(0)=1 \Rightarrow 1 = c_1 + c_2 e^{\frac{-3}{2}(0)}$

$\Rightarrow \boxed{c_1 + c_2 = 1}$ ---(1)

$y'(0) = 3 \Rightarrow 3 = c_2 \frac{-3}{2} e^{\frac{-3}{2}(0)}$

$\cancel{3} = -\frac{\cancel{3}}{2} c_2$

$\Rightarrow \boxed{c_2 = -2}$ ---(2)

$c_1 + c_2 = 1,\ c_2 = -2 \Rightarrow \boxed{c_1 = 3}$

So, $y(t) = 3 - 2e^{\frac{-3}{2}t}$

* Higher Order DE's:

ex1: Solve the DE:

$$y^{(4)} - y = 0$$

$r^4 - 1$ (charcteristic equation)

$$(r^2-1)(r^2+1) = 0$$

$r^2 - 1 = 0 \Longrightarrow r_{1,2} = \mp 1$

or

$r^2 + 1 = 0 \Longrightarrow r_{3,4} = \mp i$

$y_1 = e^t$, $y_3 = \cos t$

$y_2 = e^{-t}$, $y_4 = \sin t$

The general solution is:

$$y(t) = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4$$

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

ex2: Solve the DE:

$$y^{iv} + 2y'' + y = 0$$

$$r^4 + 2r^2 + 1 = 0$$

$$(r^2+1)^2 = 0$$

$$(r^2+1)(r^2+1) = 0$$

$$\Longrightarrow \quad r_{1,2} = \mp i \quad , \quad r_{3,4} = \mp i$$

$$\lambda = 0 \quad , \quad \mu = 1$$

$y_1 = \cos t \quad , \quad y_3 = t\cos t$
$y_2 = \sin t \quad , \quad y_4 = t\sin t$

The general solution is:

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t\cos t + c_4 t\sin t$$

$$y(t) = (c_1 \cos t + c_2 \sin t) + t\,(c_3 \cos t + c_4 \sin t)$$

ex3: Solve the DE:

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

$$r^4 + r^3 - 7r^2 - r + 6 = 0$$

try $r = 1 \Rightarrow (1)^4 + (1)^3 - 7(1)^2 - (1) + 6 = 0$

So, $r = 1$ is a root $\Rightarrow (r-1)$ is a factor.

try $r = -1 \Rightarrow (-1)^4 + (-1)^3 - 7(-1)^2 - (-1) + 6 = 0$

So, $r = -1$ is a root $\Rightarrow (r+1)$ is a factor.

$(r+1)(r-1)$ is also a factor, since $(r+1)$ & $(r-1)$ are factors.

$$\begin{array}{r} r^2 + r - 6 \\ r^2 - 1 \;\big)\; \overline{r^4 + r^3 - 7r^2 - r + 6} \\ -\; r^4 \qquad\;\; - r^2 \qquad\qquad \\ \hline + r^3 - 6r^2 - r + 6 \\ r^3 \qquad\;\; - r \qquad \\ \hline -6r^2 + 6 \\ -6r^2 + 6 \\ \hline 0 \end{array}$$

So, the equation becomes:

$$(r^2 - 1)(r^2 + r - 6) = 0$$

$$(r-1)(r+1)(r+3)(r-2) = 0$$

$$\Rightarrow r_1 = 1, \; r_2 = -1, \; r_3 = -3, \; r_4 = 2$$

The general solution is:

$$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 e^{-3t} + c_4 e^{2t}$$

Important Note: There is another way to solve second order (Section 2.9) linear DE's, but it can be used only if there is a missing dependent variable y or a missing independent variable t.

ex1: Solve the following DE: (Q36, page 134)

$$t^2 y'' + 2t y' - 1 = 0 \quad , t > 0 \qquad \text{"missing } y\text{"}$$

$$y'' + \frac{2}{t} y' = \frac{1}{t^2}$$

let $u = y' \Rightarrow u' = y''$

the equation becomes:

$$u' + \frac{2}{t} u = \frac{1}{t^2} \quad , t > 0$$

$$P(t) = \frac{2}{t} \quad , \quad g(t) = \frac{1}{t^2}$$

$$\mu(t) = e^{\int P(t)\,dt} = e^{\int \frac{2}{t} dt} = e^{2\ln|t|} = t^2$$

$$u(t) = \frac{1}{\mu(t)} \left[\int \mu(t) g(t)\,dt + C_1 \right]$$

$$= \frac{1}{t^2} \left[\int \cancel{t^2} \cdot \frac{1}{\cancel{t^2}}\,dt + C_1 \right]$$

$$= \frac{1}{t^2} \left[t + C_1 \right]$$

$$u(t) = \frac{1}{t} + \frac{C_1}{t^2}$$

$$y'(t) = u(t) = \frac{1}{t} + c_1 t^{-2}$$

$$y(t) = \int \left(\frac{1}{t} + c_1 t^{-2} \right) dt$$

$$= \ln|t| + -c_1 t^{-1} + c_2$$

$$y(t) = \ln t - \frac{c_1}{t} + c_2 \qquad (t > 0)$$

ex2: Solve the ~~DE:~~ IVP: (Q40, page 134)

$$y'' + y' = e^{-t}, \quad y(0) = 1, \quad y'(0) = 2$$

let $u = y' \Longrightarrow u' = y''$

$$\Longrightarrow u' + u = e^{-t}$$

$$P(t) = 1, \quad g(t) = e^{-t}$$

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int 1\,dt} = e^{t}$$

$$u(t) = \frac{1}{e^t} \left[\int \frac{1}{\cancel{e^t}} \cdot \cancel{e^t}\, dt + c_1 \right]$$

$$= \frac{1}{e^t} \left[t + c_1 \right]$$

$$u(t) = t e^{-t} + c_1 e^{-t}$$

$$y'(t) = u(t) = t e^{-t} + c_1 e^{-t}$$

$$y(t) = \int (t e^{-t} + c_1 e^{-t})\, dt$$

$$\begin{array}{ll} t & \searrow^{+} \; e^{-t} \\ 1 & \searrow^{-} \; -e^{-t} \\ 0 & \quad\; e^{-t} \end{array}$$

$$= -t e^{-t} - e^{-t} - c_1 e^{-t} + c_2$$

$$y(t) = -e^{-t}(t + c_1 + 1) + c_2$$

$$y(0) = 1 \implies 1 = -e^{-(0)}(0 + c_1 + 1) + c_2$$

$$1 = -c_1 - 1 + c_2$$

$$\implies \boxed{c_2 - c_1 = 2} \text{ --- (1)}$$

$$y'(0) = 2 \implies 2 = (0)\, e^{-(0)} + c_1 e^{-(0)}$$

$$\implies \boxed{c_1 = 2} \text{ --- (2)}$$

$$c_2 - c_1 = 2 \; ; \; c_1 = 2 \implies \boxed{c_2 = 4}$$

So, $y(t) = -e^{-t}(t + 2 + 1) + 4$

$$\implies y(t) = -e^{-t}(t+3) + 4$$

ex 3: Solve the DE: (Q42, page 135)

$$y y'' + (y')^2 = 0$$ "missing t"

let $v = y' \Longrightarrow v' = y''$

$$y v' + v^2 = 0$$

$$v' = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt}$$

$$v' = \frac{dv}{dy} \cdot y'$$

$$v' = \frac{dv}{dy} \cdot v$$

$$\Longrightarrow y \left(\frac{dv}{dy} \cdot \cancel{v} \right) + v^{\cancel{2}} = 0$$

$$y \frac{dv}{dy} + v = 0$$

$$y \frac{dv}{dy} = -v$$

$$\int \frac{dv}{v} = -\int \frac{dy}{y}$$

$$\ln|v| = -\ln|y| + c_1$$

$$\ln|v| + \ln|y| = c_1$$

$$\ln|vy| = c_1$$

$$|vy| = e^{c_1}$$

$$vy = K \quad , \text{ where } K = \mp e^{c_1}$$

$$y' y = K$$

$$\frac{dy}{dt} \cdot y = K$$

$$\int y \, dy = \int K \, dt$$

$$\frac{y^2}{2} = Kt + c_2$$

$$\Longrightarrow y^2 = 2Kt + 2c_2$$

ex4: Solve the IVP: (Q49, page 135)

$$y'' - 3y^2 = 0 \quad , y(0) = 2 , \; y'(0) = 4$$

let $v = y' \Longrightarrow v' = y''$

$$v' - 3y^2 = 0$$

$$v' = \frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt}$$

$$v' = \frac{dv}{dy} \cdot y'$$

$$v' = \frac{dv}{dy} \cdot v$$

$$\Longrightarrow \quad \frac{dv}{dy} \cdot v - 3y^2 = 0$$

$$v\frac{dv}{dy} = 3y^2$$

$$\int v\,dv = \int 3y^2\,dy$$

$$\frac{v^2}{2} = y^3 + c_1$$

$$v^2 = 2y^3 + 2c_1$$

$$(y')^2 = 2y^3 + 2c_1$$

$$y(0) = 2 \; ; \; y'(0) = 4 \Longrightarrow (4)^2 = 2(2)^3 + 2c_1$$

$$\Longrightarrow \boxed{c_1 = 0}$$

So, $y' = \mp \sqrt{2y^3}$

But since $y(0)=2$ and $y'(0)=4 \Longrightarrow y' = +\sqrt{2y^3}$

$$\frac{dy}{dt} = \sqrt{2}\, y^{\frac{3}{2}}$$

$$\int \frac{dy}{y^{\frac{3}{2}}} = \int \sqrt{2}\, dt$$

$$-2y^{-\frac{1}{2}} = \sqrt{2}\,t + c_2$$

$$y = \left(\frac{\sqrt{2}t + c_2}{-2}\right)^{-2}$$

$$y = \frac{4}{(\sqrt{2}t + c_2)^2}$$

$$y(0) = 2 \Longrightarrow 2 = \frac{4}{(\sqrt{2}(0) + c_2)^2}$$

$$\Longrightarrow \boxed{c_2 = \mp\sqrt{2}}$$

Thus, $y(t) = \dfrac{4}{(\sqrt{2}t - \sqrt{2})^2}$

$$= \frac{4}{(\sqrt{2}(t-1))^2}$$

$$= \frac{4}{2(t-1)^2}$$

$$\Longrightarrow y(t) = \frac{2}{(t-1)^2}$$

(3.2) Solutions of Linear Hamogeneos Equations ; the Wronskian

Th 3.2.1: Consider the IVP :

$$y'' + p(t)\,y' + q(t)\,y = g(t) \quad , \; y(t_0) = y_0 \; ; \; y'(t_0) = y_0'$$

where $P(t)$; $q(t)$; $g(t)$ are continuous on an open interval I contains t_0 , then the IVP has a unique solution $y = \phi(t)$, that satisfies it.

Th 3.2.1 ensures :

1) The existence of solution.
2) The uniqueness.

ex1: Given the IVP:

$$y'' - y = 0 \quad , \; y(0) = 3 \quad , \; y'(0) = 1$$

1) Show that the IVP has a unique solution on $\mathbb{R}$ without solving the IVP.
2) Solve the IVP.

1) $P(t) = 0$, $q(t) = -1$, $g(t) = 0$

$P(t)$, $q(t)$, $g(t)$ are continuous on $\mathbb{R}$.

So, by Th 3.2.1 , there exists a unique solution on $\mathbb{R}$.

2) Solve the IVP :

$$y'' - y = 0 \quad , \quad y(0) = 3 \ , y'(0) = 1$$

$r^2 - 1 = 0$ (characteristic equation)

$\Longrightarrow \quad r_1 = 1 \ , r_2 = -1$

The general solution is:

$$y(t) = c_1 e^{t} + c_2 e^{-t}$$

$y(0) = 3 \Longrightarrow 3 = c_1 e^{0} + c_2 e^{0}$

$$\Longrightarrow \boxed{c_1 + c_2 = 3} \text{ --- (1)}$$

$y'(0) = 1 \Longrightarrow 1 = c_1 e^{0} + - c_2 e^{0}$

$$\Longrightarrow \boxed{c_1 - c_2 = 1} \text{ --- (2)}$$

After solving (1) and (2) :

$c_1 = 2 \quad , \quad c_2 = 1$

$\Longrightarrow y(t) = 2e^{t} + e^{-t}$ (The unique solution) ex2: Consider the IVP:

$$y'' + P(t)y' + q(t)y = 0 \quad , y(t_0) = 0 \quad , y'(t_0) = 0$$

where $P(t)$, $q(t)$ are continuous on an open interval, that contains t_0, find the unique solution of the IVP above.

The unique solution is $y = 0$, since it satisfies the DE and the initial conditions.

note: in ex1, the solution $y \equiv 0$ satisfies the DE, but it doesn't satisfy the initial condition, so it isn't a solution.

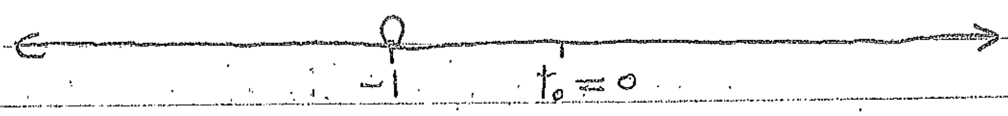
ex3: Find the largest interval in which the solution of the IVP is valid:

$$(t+1)y'' - (\cos t)y' = 1 - 3y \quad , y(0) = 1 , y'(0) = 0$$

$$y'' - \frac{\cos t}{t+1} y' + \frac{3}{t+1} y = \frac{1}{t+1}$$

$$P(t) = -\frac{\cos t}{t+1} \quad , q(t) = \frac{3}{t+1} \quad , g(t) = \frac{1}{t+1}$$

So, $P(t)$, $q(t)$, $g(t)$ are continuous on $\mathbb{R} / \{-1\}$.



So, the largest interval is $(-1, \infty)$.

Th 3.3.2 (Principle of Superposition):

If $y_1(t)$ and $y_2(t)$ are solutions for the DE:

$$L[y] = y'' + P(t)y' + q(t)y = 0$$

then the linear combination:

$$c_1 y_1(t) + c_2 y_2(t)$$

is also a solution for the DE above.

where $L[y]$ is just a differential operator to facilitate computation.

proof of the theorem:

y_1 is a solution $\Longrightarrow$ $L[y_1] = 0$
y_2 is a solution $\Longrightarrow$ $L[y_2] = 0$

we need to show that $L[c_1 y_1 + c_2 y_2] = 0$.

$$L[c_1 y_1 + c_2 y_2] = (c_1 y_1 + c_2 y_2)'' + P(t)(c_1 y_1 + c_2 y_2)' + q(t)(c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1'' + c_2 y_2'' + P(t)(c_1 y_1' + c_2 y_2') + q(t)(c_1 y_1 + c_2 y_2)$$

$$= c_1 (y_1'' + P(t) y_1' + q(t) y_1) + c_2 (y_2'' + P(t) y_2' + q(t) y_2)$$

$$L[c_1y_1 + c_2y_2] = c_1 L[y_1] + c_2 L[y_2]$$
$$= c_1(0) + c_2(0)$$
$$= 0$$

So, $c_1y_1 + c_2y_2$ is also a solution for the DE.

The Wronskian Determinants:

Suppose that y_1, y_2 are solutions for the IVP:

$$L[y] = y'' + P(t)y' + q(t)y = 0 \;,\; y(t_0) = y_0 \;,\; y'(t_0) = y_0'$$

by Th 3.2.2:

$$y(t) = c_1y_1 + c_2y_2$$

is also a solution.

To find c_1, c_2 we use the initial conditions:

$$y(t_0) = y_0 \Longrightarrow \boxed{y_0 = c_1y_1(t_0) + c_2y_2(t_0)} \quad \text{---} (1)$$

$$y'(t_0) = y_0' \Longrightarrow \boxed{y_0' = c_1y_1'(t_0) + c_2y_2'(t_0)} \quad \text{---} (2)$$

We can use cramer's rule to find c_1, c_2 :

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}} \quad , \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

We denote $y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0)$ the Wronskian and we write:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

$$= y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0)$$

So :

$$c_1 = \frac{y_0\, y_2'(t_0) - y_2(t_0)\, y_0'}{W(y_1, y_2)(t_0)}$$

$$c_2 = \frac{y_1(t_0)\, y_0' - y_0\, y_1'(t_0)}{W(y_1, y_2)(t_0)}$$

Now, for c_1, c_2 to make sense, we must have $W(y_1, y_2)(t_0) \neq 0$.

note: y_1 and y_2 are linearly independent, iff $W(y_1, y_2)(t) \neq 0$ for at least one $t \in I$.

ex: Given that $y_1 = 1$, $y_2 = x$, $y_3 = x^2$, are y_1, y_2, y_3 linearly independent?

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

we take the derivative $(n-1)$ times, where n is the number of functions.

$$W(1, x, x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 2 \neq 0$$

So, y_1, y_2, y_3 are linearly independent.

note: If y_1, y_2, y_3 are linearly dependent, then $W(y_1, y_2, y_3) = 0$, but the opposite isn't true.

Th 3.2.3 :

Suppose that y_1, y_2 are solutions of:

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0 \quad --- (1)$$

then there exist c_1, c_2, such that:

$$y(t) = c_1 y_1 + c_2 y_2$$

satisfies equation (1), iff $W(y_1, y_2)(t_0) \neq 0$, $t_0 \in I$.

ex: Given the DE:

$$y'' + 5y' + 6y = 0$$

$$r^2 + 5r + 6 = 0 \quad \text{(characteristic equation)}$$

$$(r+2)(r+3) = 0$$

$$\Longrightarrow r_1 = -2, \; r_2 = -3$$

$$\Longrightarrow y_1 = e^{-2t}, \; y_2 = e^{-3t}$$

Any linear combination of y_1 and y_2 can be used to construct solutions for the DE, and this becomes:

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix}$$

$$W(y_1, y_2)(t) = -3e^{-5t} + 2e^{-5t}$$
$$= -e^{-5t} \neq 0$$

In this sense, we call $c_1y_1 + c_2y_2$ the general solution and y_1, y_2 are the fundamental set of solution

Th 3.2.4: Suppose $y_1(t)$, $y_2(t)$ are solutions to the DE on an open interval I:

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

then, the family of solutions:

$$y(t) = c_1y_1 + c_2y_2$$

includes all possible solutions, iff there exist $t_0 \in I$, such that $W(y_1, y_2)(t_0) \neq 0$

Th 3.2.4 tells us the following:

1) The family of solutions $y(t)$ is called the general solution

2) $y_1(t)$ and $y_2(t)$ are called fundamental set of solution

y_1 & y_2 are fundamental set of solutions, implies that:

1) y_1 & y_2 are solutions.
2) y_1 & y_2 are linearly independent.

ex1: Consider the DE:

$$y'' - y = 0$$

with two solutions $y_1 = e^t$, $y_2 = e^{-t}$, show that y_1 & y_2 form a fundamental set of solutions.

Since y_1 & y_2 are solutions, it follows that we need only to show that y_1 & y_2 are linearly independent.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix}$$

$$= -1 - 1$$

$$= -2 \neq 0$$

So, they form a fundamental set of solutions.

ex2: Let $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$ be two solutions of the DE:

$$y'' + p(t)\, y' + q(t)\, y = 0$$

show that if $r_1 \neq r_2$, then y_1 & y_2 form a fundamental set of solutions.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix}$$

$$= r_2\, e^{(r_1+r_2)t} - r_1\, e^{(r_1+r_2)t}$$

$$= (r_2 - r_1)\, e^{(r_1+r_2)t} \neq 0$$

since $r_2 \neq r_1$ and $e^{(r_1+r_2)t} > 0$.

$W(y_1, y_2)(t) \neq 0 \implies y_1$ & y_2 are linearly independent.

So, y_1 & y_2 form a fundamental set of solutions

ex3: Show that $y_1 = t^{\frac{1}{2}}$, $y_2 = t^{-1}$ form a fundamental set of solutions for the DE:

$$2t^2 y'' + 3t y' - y = 0 \ , \ t > 0$$

$y_1 = t^{\frac{1}{2}}$, $y_1' = \frac{1}{2} t^{\frac{-1}{2}}$, $y_1'' = \frac{-1}{4} t^{\frac{-3}{2}}$

subsituting y_1 , y_1' , y_1'' in the DE:

$$2t^2 \left(\frac{-1}{4} t^{\frac{-3}{2}}\right) + 3t \left(\frac{1}{2} t^{\frac{-1}{2}}\right) - t^{\frac{1}{2}}$$

$$\frac{-1}{2} t^{\frac{1}{2}} + \frac{3}{2} t^{\frac{1}{2}} - t^{\frac{1}{2}} = 0$$

So, y_1 is a solution.

$y_2 = t^{-1}$, $y_2' = -t^{-2}$, $y_2'' = 2t^{-3}$

subsituting y_2 , y_2' , y_2'' in the DE:

$$2t^2 \left(2t^{-3}\right) + 3t \left(-t^{-2}\right) - t^{-1}$$

$$4t^{-1} - 3t^{-1} - t^{-1} = 0$$

So, y_2 is a solution.

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2} t^{\frac{-1}{2}} & -t^{-2} \end{vmatrix}$$

$$= -t^{\frac{-3}{2}} - \frac{1}{2} t^{\frac{-3}{2}}$$

$$= -\frac{3}{2} t^{\frac{-3}{2}}$$

$$= \frac{-3}{2\sqrt[3]{t^2}} < 0 \quad (\text{since } t > 0)$$

Hence, y_1 & y_2 are two solutions for the DE and they are linearly independent. Thus, y_1 & y_2 are fundamental set of solutions.

Th 3.3.2 (Abel's Theorem):

Suppose that y_1 & y_2 are solutions of the DE:

$$L[y] = y'' + p(t)\, y' + q(t)\, y = 0.$$

when p & q are continuous on some open interval I

then the wronskian of the solutions y_1 & y_2 is

given by:

$$W(y_1, y_2) = C\, e^{-\int p(t)\, dt}$$

where C is constant that depends on y_1 & y_2.

Furthermore:

$W(y_1, y_2) \neq 0$, when $C \neq 0$, $\forall t \in I$

$W(y_1, y_2) = 0$, only when $C = 0$, $\forall t \in I$

proof: since y_1 & y_2 are solutions, then:

$$y_1'' + P(t)\, y_1' + q(t)\, y_1 = 0 \quad \text{---}(1)$$

$$y_2'' + P(t)\, y_2' + q(t)\, y_2 = 0 \quad \text{---}(2)$$

multiply equation (1) by $-y_2$ and equation (2) by y_1, then add them to get:

$$(y_1 y_2'' - y_1'' y_2) + P(t)\,(y_1 y_2' - y_2 y_1') = 0$$

$$W'(t) + P(t)\, W(t) = 0$$

$$\frac{W'(t)}{W(t)} = -P(t)$$

$$\int \frac{dW}{W(t)} = -\int P(t)\, dt$$

$$\ln |W(t)| = -\int P(t)\, dt + D$$

$$|W(t)| = e^{-\int P(t)\, dt + D}$$

$$W(t) = \mp e^{D} \cdot e^{-\int P(t)\, dt}$$

$$W(t) = C\, e^{-\int P(t)\, dt}$$

ex: Consider the DE:

$$2t^2y'' + 3ty' - y = 0 \quad , t>0$$

with two solutions $y_1 = t^{\frac{1}{2}}$, $y_2 = t^{-1}$.

Find $W(y_1, y_2)$ and compare it with Abel's Th.

$$W(y_1, y_2) = \begin{vmatrix} t^{\frac{1}{2}} & t^{-1} \\ \frac{1}{2}t^{-\frac{1}{2}} & -t^{-2} \end{vmatrix} = -t^{-\frac{3}{2}} - \frac{1}{2}t^{-\frac{3}{2}}$$

$$= -\frac{3}{2}t^{-\frac{3}{2}}$$

$$= \frac{-3}{2\sqrt{t^3}} < 0 \quad (\text{since } t>$$

$$\neq 0$$

Abel's Th:

First divide by $2t^2$:

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0 \quad , t>0$$

$$P(t) = \frac{3}{2t} \Longrightarrow W = C\, e^{-\int p(t)\,dt}$$

$$= C\, e^{-\int \frac{3}{2t}dt}$$

$$= C\, e^{-\frac{3}{2}\ln t} \qquad (t>0)$$

$$W = C e^{\ln t^{\frac{-3}{2}}}$$

$$= C t^{\frac{-3}{2}}$$

$$= \frac{C}{\sqrt{t^3}}$$

Note that the wronskian computed by Abel's Th is more general (can be computed from every y_1 & y_2).

While $C = \frac{-3}{2}$ is a special case for $y_1 = t^{\frac{1}{2}}$ and $y_2 = t^{-1}$.

Def: f_1 & f_2 are linearly <u>independent</u>, if whenever

$$c_1 f_1 + c_2 f_2 = 0 \text{ implies that } c_1 = c_2 = 0.$$

Def: f_1 & f_2 are linearly <u>dependent</u>, if $\exists$ c_1 & c_2, not all zeros, such that:

$$c_1 f_1 + c_2 f_2 = 0$$

notes: 1) If y_1 & y_2 are linearly dependent, then $W(y_1, y_2) = 0$
, but the opposite isn't true.

2) y_1 & y_2 are linearly independent, iff $W(y_1, y_2) \neq 0$

ex: Are the functions $f(t) = \sin 2t$ & $g(t) = \sin t \cos t$ dependent?

$c_1 \sin 2t + c_2 \sin t \cos t = 0$, then

$c_1 = 1$ & $c_2 = -2$

$\Longrightarrow \sin 2t = 2 \sin t \cos t$

So, the functions are dependent (by Def.).

(3.3) Complex Roots for the Characteristic Equation

Recall that a taylor series expansion for the smooth function $f(x)$ about $x=a$ is given by:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

ex1: take $a=0$ and $x \in \mathbb{R}$, then:

1) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$

* Euler's Formula:

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \dots$$

$$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$e^{ix} = \cos x + i \sin x$$

This formula is called "Euler's Formula".

ex2: Use Euler's Formula to write $e^{2+\frac{\pi}{2}i}$ as a complete number $a+bi$.

$$
\begin{aligned}
e^{2+\frac{\pi}{2}i} &= e^{2}\cdot e^{\frac{\pi}{2}i} \\
&= e^{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\
&= e^{2}\left(0 + i\cdot 1\right) \\
&= e^{2}\cdot i \qquad (a=0\ ,\ b=e^{2})
\end{aligned}
$$

ex3: Find the general solution for the following DE:

$$y'' + y' + y = 0$$

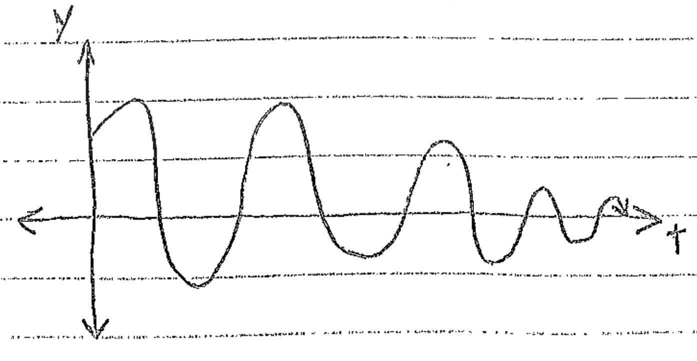
$$r^2 + r + 1 = 0 \qquad \text{(characteristic equation)}$$

$$r_{1,2} = \frac{-1 \mp \sqrt{3}\,i}{2}$$

$$\lambda = \frac{-1}{2}\ ,\ \mu = \frac{\sqrt{3}}{2}$$

the general solution is:

$$
\begin{aligned}
y(t) &= c_1 y_1 + c_2 y_2 \\
&= c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t) \\
&= c_1 e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t + c_2 e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t
\end{aligned}
$$



$$\lim_{t\to\infty} y(t) = 0$$

(decay oscillation, $\lambda < 0$)

(unbounded, oscillating betw[een] $-\infty$ and ∞

ex4: Solve the DE:

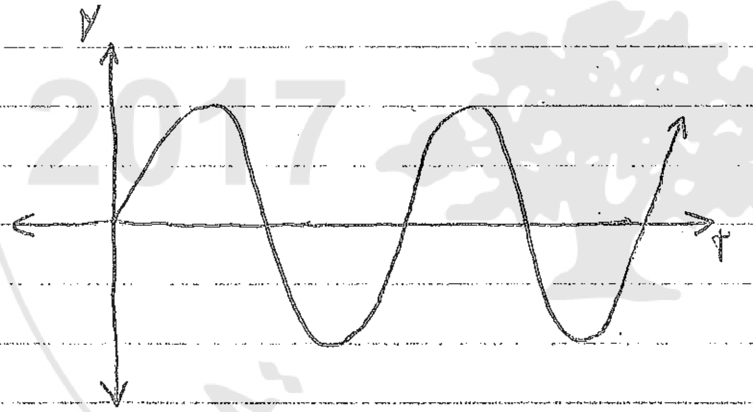
$$y'' + 4y = 0$$

$$r^2 + 4 = 0$$
$$r_{1,2} = \mp 2i$$

$$\lambda = 0, \mu = 2$$

the general solution is:

$$y(t) = c_1 \cos 2t + c_2 \sin 2t$$



(oscillating, $\lambda = 0$)

ex5: Solve the DE:

$$16y'' - 8y' + 145y = 0 \quad , \; y(0) = -2 \; , \; y'(0) = 1$$

$$16r^2 - 8r + 145 = 0$$

$$r_{1,2} = \frac{8 \mp \sqrt{64 - (4)(16)(145)}}{32}$$

$$r_{1,2} = \frac{1}{4} \mp 3i$$

$$\lambda = \frac{1}{4} \; , \; \mu = 3$$

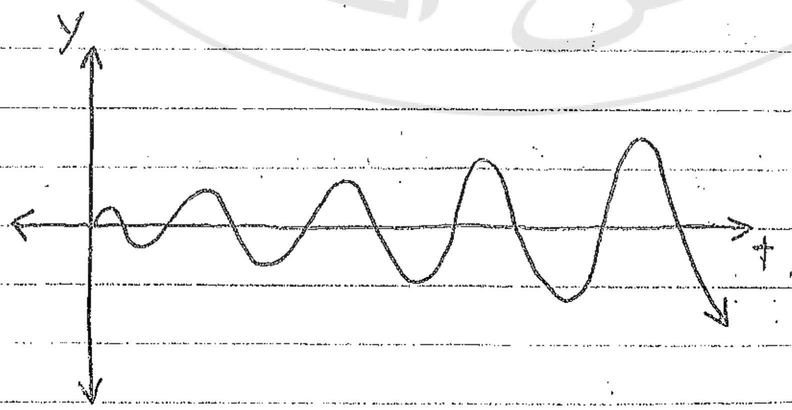
the general solution is:

$$y(t) = c_1 e^{\frac{1}{4}t} \cos 3t + c_2 e^{\frac{1}{4}t} \sin 3t$$

$$y(0) = -2 \implies c_1 = -2$$

$$y'(0) = 1 \implies c_2 = \frac{1}{2}$$

$$\text{So}, \; y(t) = -2 e^{\frac{1}{4}t} \cos 3t + \frac{1}{2} e^{\frac{1}{4}t} \sin 3t$$



$\lim_{t \to \infty} y(t)$ is unbounded

(growth oscillation, $\lambda > 0$)

(3.4) Repeated Roots ; Reduction of Order

ex1: Solve the DE:

$$y'' + 2y' + 1 = 0 \quad , y(0) = 1 , y'(0) = 1$$

If $y = e^{rt}$, then the characteristic equation is:

$$r^2 + 2r + 1 = 0$$
$$(r+1)^2 = 0$$
$$(r+1)(r+1) = 0$$
$$r = r_1 = r_2 = -1$$

the general solution is:

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}$$

$$y(0) = 1 , y'(0) = 1 \implies c_1 = 1 , c_2 = 2$$

So, $y(t) = e^{-t} + 2te^{-t}$

$$\lim_{t \to \infty} y(t) = 0$$

ex2: Find the solution of:

$$y'' - y' + \frac{1}{4} y = 0 \quad , y(0) = 2 , y'(0) = \frac{1}{2}$$

$$r^2 - r + \frac{1}{4} = 0$$
$$(r - \tfrac{1}{2})^2 = 0$$
$$(r - \tfrac{1}{2})(r - \tfrac{1}{2}) = 0$$
$$r = r_1 = r_2 = \frac{1}{2}$$

the general solution is:

$$y(t) = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$$

$y(0) = 2,\ y'(0) = \frac{1}{2} \Longrightarrow c_1 = 2,\ c_2 = \frac{-1}{2}$

So, $y(t) = 2 e^{\frac{1}{2}t} - \frac{1}{2} t e^{\frac{1}{2}t}$

$$\lim_{t \to \infty} y(t) = -\infty$$

* Reduction of Order:

Recall that a second order linear homogeneos DE with constant coefficients has the form:

$$ay'' + by' + cy = 0$$

the roots are the solution of the characteristic equation:

$$ar^2 + br + c = 0$$

$$\Longrightarrow r_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

1) If $b^2 - 4ac > 0$, then:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

note: the value $(b^2 - 4ac)$ is called the "discriminant".

2) If $b^2 - 4ac < 0$, then:

$$y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$$

3) If $b^2 - 4ac = 0$, then:

$$r_{1,2} = \frac{-b}{2a} = r \quad \text{(repeated)}$$

So, one solution is $y_1(t) = e^{rt} = e^{\left(\frac{-b}{2a}\right)t}$

How to find the second solution $y_2(t)$?

We use the method of reduction order to obtain the second independent solution.

Assume that $y_2(t) = V(t)\, y_1(t)$ is the second solution of the DE:

$$y'' + P(t) y' + q(t) y = 0 \quad --- (1)$$

So, we need to find $V(t)$, since $y_1(t)$ is a solution.

$$y_2(t) = V(t)\, y_1(t)$$

$$y_2'(t) = V'(t)\, y_1(t) + y_1'(t)\, V(t)$$

$$y_2''(t) = V''(t)\, y_1 + 2V'(t)\, y_1'(t) + V(t)\, y_1''(t)$$

Substitute y_2, y_2', y_2'' in the DE (1) and rearrange the terms to obtain the following equation:

$$y_1 V'' + (2y_1' + P(t)y_1)V' + (y_1'' + P(t)y_1' + q(t)y_1) = 0$$

But, $y_1'' + P(t)y_1' + q(t)y_1 = 0$, since y_1 is a solution for the DE (1), so the equation becomes:

$$y_1 V'' + (2y_1' + P(t)y_1)V' = 0$$

Now, if we let $W = V'$ --- (2)

then the last equation becomes:

$$y_1 W' + (2y_1' + P(t)y_1) W = 0 \quad --- (3)$$

So, to find the second solution, what we need to do is:

First: Solve equation (3) for W.
Second: Solve equation (2) for V'.
Third: Solve for V (integrate V').
Fourth: Find $y_2(t)$ $(y_2(t) = V(t)y_1(t))$.

ex: Given $y_1(t) = \frac{1}{t}$ is a solution for the DE:

$$t^2 y'' + 3ty' + y = 0 , \quad t > 0$$

Find the second independent solution.

$$y_1 = \frac{1}{t} \Rightarrow y_1' = \frac{-1}{t^2}$$

$$P(t) = \frac{3t}{t^2} = \frac{3}{t}$$

$$y_1 w' + \left(2y_1' + P(t)\, y_1\right) w = 0$$

$$\frac{1}{t} w' + \left(\frac{-2}{t^2} + \frac{3}{t^2}\right) w = 0$$

$$\frac{1}{t} w' + \frac{1}{t^2} w = 0$$

$$w' + \frac{1}{t} w = 0$$

$$\frac{dw}{dt} = \frac{-w}{t}$$

$$\int \frac{dw}{w} = -\int \frac{dt}{t}$$

$$\ln|w| = -\ln|t| + c_1$$
$$\ln w = -\ln t + c_1$$
$$\ln w = \ln \frac{1}{t} + c_1$$

$$w = \frac{1}{t} e^{c_1}$$

$$\boxed{w = \frac{c_1}{t}}$$

Now, $w = v'$

$$v' = \frac{c_1}{t}$$

$$v = \int \frac{c_1}{t}\, dt$$

$$V = c_1 \ln|t| + c_2$$
$$V = c_1 \ln t + c_2$$

$$y_2(t) = V(t)\, y_1(t)$$
$$y_2(t) = (c_1 \ln t + c_2) \cdot \frac{1}{t}$$

$$y_2(t) = c_1 \frac{\ln t}{t} + \frac{c_2}{t}$$

Note that the second term on the right side of the last equation is a multiple of $y_1(t)$ and can be dropped, but the first term provides a new solution $y_2 = \frac{\ln t}{t}$, which is the second solution.

(3.5) Nonhomogeneous DE's ; The Method of Undertermined Coefficients

A second order linear DE has the general form:

$$y'' + P(t)\,y' + q(t)\,y = g(t) \quad \text{---} (1)$$

where P, q and g are continuous functions on an open interval I.

The corresponding ~~DE is~~ homogeneous DE is:

$$y'' + P(t)\,y' + q(t)\,y = 0 \quad \text{---} (2)$$

Th 3.5.1:

1) If Y_1 & Y_2 are two solutions of DE(1), then $Y_1 - Y_2$ is a solution of DE(2).

2) If y_1 & y_2 are a fundamental set of solutions of DE(2), then:

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$

proof: 1) $L[Y_1] = g(t)$, $L[Y_2] = g(t)$, since Y_1 & Y_2 are two solutions of DE (1).

$$\begin{aligned} L[Y_1 - Y_2] &= L[Y_1] - L[Y_2] \\ &= g(t) - g(t) \\ &= 0 \end{aligned}$$

So, $Y_1 - Y_2$ is a solution of DE(2).

2) Since $Y_1 - Y_2$ is a solution of DE (2) and y_1 & y_2 are fundamental set of solutions, we can write $Y_1 - Y_2$ as a linear combination of y_1 & y_2, thus:

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2$$

* The Method of Undetermined Coefficients:

Consider the non-homogeneous second order linear DE:

$$y'' + ay' + by = g(t), \quad a, b \in \mathbb{R} \quad --- (3)$$

There are two methods to solve the DE (3):

1) The method of variation of parameters (next section). It works for general functions $g(t)$.

2) The method of undetermined coefficients (this section). It works only if $g(t)$ is one of the following functions:

1) Exponential functions.
2) Polynomial functions.
3) Trigonometric functions (sin or cos only).
4) Multiple or sum or subtraction of (1), (2) & (3).

Note that in the method of undetermined coefficients, we must have a & b constants.

The general solution of the DE(3) is given by:

$$y(t) = y_h(t) + y_p(t)$$

where $y_h(t)$ is the solution of the corresponding homogeneous equation obtained by solving:

$$y'' + ay' + by = 0$$

note: $y_h(t)$ may be also denoted by $y_c(t)$ (the complementary solution).

and $y_p(t)$ (or $Y(t)$) is the particular solution that depends totally on the form of $g(t)$ given in DE(3).

So, we have three cases:

1) If $g(t) = Ce^{\lambda t}$, $C, \lambda \in \mathbb{R}$, then we let $y_p = Ae^{\lambda t}$ and to find A, we substitute y_p, y_p', y_p'' in DE(3).

2) If $g(t) = a_n t^n + a_{n-1} t^{n-1} + \dots + a_1 t + a_0$, where

$a_i \in \mathbb{R}$, for all i, then we let:

$$y_p = A_n t^n + A_{n-1} t^{n-1} + \dots + A_1 t + A_0$$

and to find A_i, for all i, we substitute y_p, y_p', y_p'' in DE(3).

3) If $g(t) = C \sin(\mu t)$ or $g(t) = C \cos(\mu t)$, then we let:

$$y_p = A \cos(\mu t) + B \sin(\mu t)$$

and to find A & B, we substitute y_p, y_p', y_p'' in DE(3)

Remark: y_p must be independent from y_h, if not, then we multiply y_p by t (or may be t^2 or t^3 ...) whenever y_p & y_h become dependent.

ex1: Find the general solution of the DE:

$$y'' - 3y' - 4y = 3e^{2t}, \quad y(0) = \frac{9}{2}, \quad y'(0) = -1$$

The general solution is:

$$y(t) = y_h(t) + y_p(t)$$

to find $y_h(t)$, we have to solve:

$$y'' - 3y' - 4y = 0$$

$$r^2 - 3r - 4 = 0$$
$$(r-4)(r+1) = 0$$
$$r_1 = 4, \; r_2 = -1$$

So, $y_h(t) = c_1 e^{4t} + c_2 e^{-t}$

to find $y_p(t)$, we let $y_p(t) = Ae^{2t}$ (y_p is independent)

to find A:

$$y_p = Ae^{2t}, \quad y_p' = 2Ae^{2t}, \quad y_p'' = 4Ae^{2t}$$

Now, we substitute y_p, y_p', y_p'' in the DE:

$$y_p'' - 3y_p' - 4y_p = 3e^{2t}$$
$$\cancel{4Ae^{2t}} - 6Ae^{2t} - \cancel{4Ae^{2t}} = 3e^{2t}$$
$$-6A\cancel{e^{2t}} = 3\cancel{e^{2t}}$$
$$\Longrightarrow \boxed{A = \frac{-1}{2}}$$

So, $y_p(t) = \frac{-1}{2} e^{2t}$.

the general solution is:

$$y(t) = y_h(t) + y_p(t)$$
$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t}$$

to find c_1 & c_2, we use the initial conditions:

$$y(0) = \frac{9}{2} \Longrightarrow c_1 + c_2 = 5$$

$$y'(0) = -1 \Longrightarrow c_2 = 4c_1$$

$$\Longrightarrow c_1 = 1, \quad c_2 = 4$$

So, $y(t) = e^{4t} + 4e^{-t} - \frac{1}{2} e^{2t}$.

ex2: Find the particular solution of the DE:

$$y'' - 3y' - 4y = 2\sin t$$

First, we have to find $y_h(t)$.

$$y_h(t) = c_1 e^{4t} + c_2 e^{-t} \quad (\text{from ex1})$$

to find y_p, we let:

$$y_p = A\sin t + B\cos t \quad (y_p \text{ is independent})$$

to find A & B:

$$y_p = A\sin t + B\cos t$$
$$y_p' = A\cos t - B\sin t$$
$$y_p'' = -A\sin t - B\cos t$$

Now, we substitute y_p, y_p', y_p'' in the DE:

$$y_p'' - 3y_p' - 4y_p = 2\sin t$$

$$-A\sin t - B\cos t - 3A\cos t + 3B\sin t - 4A\sin t - 4B\cos t = 2\sin t$$

$$(-5A + 3B)\sin t + (-5B - 3A)\cos t = 2\sin t$$

So, $-5A + 3B = 2$

and $-5B - 3A = 0$

$$\Longrightarrow A = \frac{-5}{17}, \quad B = \frac{3}{17}$$

So, $y_p(t) = \frac{-5}{17} \sin t + \frac{3}{17} \cos t$

ex3: Find a particular solution of the DE:

$$y'' - 3y' - 4y = 4t^2 - 1$$

First, we need to find $y_h(t)$.

$$y_h(t) = c_1 e^{4t} + c_2 e^{-t} \quad (\text{from ex1})$$

to find $y_p(t)$, we let:

$$y_p(t) = At^2 + Bt + C \qquad (y_p \text{ is independent})$$

to find A, B & C:

$$y_p(t) = At^2 + Bt + C \,, \quad y_p'(t) = 2At + B \,, \quad y_p''(t) = 2A$$

Now, we substitute y_p, y_p', y_p'' in the DE:

$$y_p'' - 3y_p' - 4y_p = 4t^2 - 1$$

$$\cancel{At^2 + Bt + C}$$

$$2A - 3(2At + B) - 4(At^2 + Bt + C) = 4t^2 - 1$$

$$2A - 6At - 3B - 4At^2 - 4Bt - 4C = 4t^2 - 1$$

$$(-4A)t^2 + (-6A - 4B)t + (2A - 3B - 4C) = 4t^2 - 1$$

So,
$$-4A = 4 \quad --- (1)$$
$$-6A - 4B = 0 \quad --- (2)$$
$$2A - 3B - 4C = -1 \quad --- (3)$$

$\Longrightarrow \; A = -1, \; B = \frac{3}{2}, \; C = \frac{-11}{8}$

So, $y_p = -t^2 + \frac{3}{2}t - \frac{11}{8}$

ex4: Find y_p for the DE:

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

First, we have to find y_h.

$y_h = c_1 e^{4t} + c_2 e^{-t}$ (from ex1)

to find y_p, we let:

$y_p(t) = e^t (A\cos 2t + B\sin 2t)$ (y_p is independ

to find A & B, we find y_p' & y_p'' and then we substitute y_p, y_p', y_p'' in the DE to get:

$$A = \frac{10}{13}, \quad B = \frac{2}{13}$$

So, $y_p(t) = e^t\left(\frac{10}{13}\cos 2t + \frac{2}{13}\sin 2t\right)$

ex5: Find $y_p(t)$ for the following DE:

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$$

First, we have to find y_h.

$$y_h = c_1 e^{4t} + c_2 e^{-t}$$

to find y_p, we divide the DE above into 3 sub DE's:

1) $y'' - 3y' - 4y = 3e^{2t}$

with $y_{p_1} = \frac{-1}{2} e^{2t}$ (from ex1).

2) $y'' - 3y' - 4y = 2\sin t$

with $y_{p_2} = \frac{-5}{17}\sin t + \frac{3}{17}\cos t$ (from ex2)

3) $y'' - 3y' - 4y = -8e^{t}\cos 2t$

with $y_{p_3} = e^{t}\left(\frac{10}{13}\cos 2t + \frac{2}{13}\sin 2t\right)$ (from ex4)

Hence, the particular solution of the DE is:

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t) + y_{p_3}(t)$$

$$y_p(t) = \frac{-1}{2}e^{2t} + \frac{-5}{17}\sin t + \frac{3}{17}\cos t + e^{t}\left(\frac{10}{13}\cos 2t + \frac{2}{13}\sin 2t\right)$$

ex6: Find y_p for the following DE's without finding the values of A, B & C.

1) $y'' - 5y' + 6y = 3e^{4x}$

to find $y_h(x)$:

$r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$r_1 = 2$, $r_2 = 3$

So, $y_h = c_1 e^{2x} + c_2 e^{3x}$

$y_p = A e^{4x}$ (y_p is independent from y_h)

2) $y'' - 5y' + 6y = 10e^{3x}$

$y_h = c_1 e^{2x} + c_2 e^{3x}$ (from ex6 (1))

$y_p = A e^{3x} \cdot x$ (since it is dependent)

3) $y'' - 6y' + 9y = 7e^{3x}$

to find $y_h(x)$:

$r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$(r-3)(r-3) = 0$

$r = r_1 = r_2 = 3$

So, $y_h(x) = c_1 e^{3x} + c_2 x e^{3x}$

$y_p(x) = A e^{3x} \cdot x^2$ (since it is dependent)

4) $y'' + y' = 10t^2$

to find $y_h(t)$:

$r^2 + r = 0$
$r(r+1) = 0$
$r_1 = 0, r_2 = -1$

So, $y_h = c_1 + c_2 e^{-t}$

$y_p = (At^2 + Bt + C) \cdot t$ (since it is dependent)

Note that y_p is dependent, because there is a constant in both y_p & y_h.

5) $y''' + y'' = 10x^2$

to find $y_h(x)$:

$r^3 + r^2 = 0$
$r^2(r+1) = 0$
$r_{1,2} = 0, r_3 = -1$

So, $y_h = c_1 + c_2 x + c_3 e^{-x}$ (we multiplied c_2 by x, since it is dependent)

$y_p = (Ax^2 + Bx + C) \cdot x^2$ (since it is dependent)

6) $y'' + y = 20 \cos 3x$

to find $y_h(x)$:

$r^2 + 1 = 0$
$r_{1,2} = \mp i$

$\lambda = 0, \mu = 1$

So, $y_h = c_1 \cos x + c_2 \sin x$

$y_p = A \cos 3x + B \sin 3x$ (y_p is independent from y_h)

7) $y'' + y = 5x \sin x$

$y_h = c_1 \cos x + c_2 \sin x$ (from ex6 (6))

$y_p = (Ax + B)(C \sin x + D \cos x) \cdot x$ (since it is dep.)

Note that if,

$y_p = (Ax + B)(C \sin x + D \cos x)$
$y_p = Ax(C \sin x + D \cos x) + B(C \sin x + D \cos x)$

$\Longrightarrow$ the second term is dependent, so we need to multiply by x to get independent y_p.

(3.6) Variation of Parameters

Recall that a linear second order DE has the form:

$$y'' + P(t)y' + q(t)y = g(t) \quad --- (1)$$

In case where:

1) P & q are constants.
2) $g(t)$ is one of the functions exp, sin, cos or polynomial.

we have studied the method of undetermined coefficients to solve such DE's.

How can we solve equation (1) if $g(t)$ is a more general function than exp, sin, cos or polynomial? or if P & q aren't constants?

In this case, we use the method of variation of parameters.

This method is characterized in the following theorem:

Th 3.6.1:

Consider the DE (1), if y_1 & y_2 are the fundamental set of solutions for the corresponding homogeneous DE:

$$y'' + P(t)y' + q(t)y = 0$$

then, the general solution of DE (1) is:

$$y(t) = y_h(t) + y_p(t)$$

where:

$$y_h(t) = c_1 y_1 + c_2 y_2$$

$$y_p(t) = V_1 y_1 + V_2 y_2$$

$$V_1(t) = -\int \frac{y_2(t)\, g(t)}{W(y_1, y_2)(t)}\, dt$$

$$V_2(t) = \int \frac{y_1(t)\, g(t)}{W(y_1, y_2)(t)}\, dt$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

ex: Find the general solution of the DE:

$$x^2 y'' + x y' - y = x \ln x \quad , \quad x > 0$$

if $y_1 = x$ & $y_2 = \frac{1}{x}$ are two independent solutions for the homogeneous DE:

$$x^2 y'' + x y' - y = 0 \quad , \quad x > 0$$

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = \frac{\ln x}{x}$$

So, $g(x) = \frac{\ln x}{x}$

$$y_h(x) = c_1 y_1 + c_2 y_2$$

$$y_h(x) = c_1 x + \frac{c_2}{x}$$

$$y_p(x) = V_1(x)\, y_1(x) + V_2(x)\, y_2(x)$$

$$W(y_1, y_2) = \begin{vmatrix} x & \frac{1}{x} \\ 1 & \frac{-1}{x^2} \end{vmatrix}$$

$$= \frac{-1}{x} - \frac{1}{x}$$

$$= \frac{-2}{x}$$

$$V_1(x) = -\int \frac{y_2(x)\, g(x)}{W(y_1, y_2)(x)}\, dx$$

$$= -\int \frac{\frac{1}{x} \cdot \frac{\ln x}{x}}{\frac{-2}{x}}\, dx$$

$$= \frac{1}{2} \int \frac{\ln x}{x}\, dx$$

$$= \frac{1}{4} (\ln x)^2 + c_1$$

$$V_2(x) = \int \frac{y_1(x)\, g(x)}{W(y_1, y_2)(x)}\, dx$$

$$= \int \frac{\cancel{x} \cdot \frac{\ln x}{\cancel{x}}}{\frac{-2}{x}}\, dx$$

$$= \frac{-1}{2} \int x \ln x\, dx$$

$$= \frac{-x^2}{8} (2 \ln x - 1) + c_2$$

Now, $y_p(x) = V_1(x)\, y_1(x) + V_2(x)\, y_2(x)$

So, $y_p(x) = x\left[\frac{1}{4}(\ln x)^2 + c_1\right] + \frac{1}{x}\left[\frac{-x^2}{8}(2\ln x - 1) + c_2\right]$

$$y(x) = y_h(x) + y_p(x)$$

زهراء حليلي

د. عبد الرحيم موسى

The general solution is:

$$y(x) = y_h(x) + y_p(x)$$

Chapter 4.

4.1 Higher order linear equations:-

The nth order linear DE has the general form:-

$$\boxed{y^{(n)} + P_{n-1}(t)\, y^{(n-1)} + \dots\ P_1(t)\, y' + P_0(t)\, y = g(t)} \quad \dots (A)$$

with corresponding homogeneous DE:-

$$\boxed{y^{(n)} + P_{n-1}(t)\, y^{(n-1)} + \dots + P_1(t)\, y' + P_0(t)\, y = 0} \quad \dots (B)$$

The DE (A) requires n initial conditions:-

$$y(t_0) = y_0 \,,\ y'(t_0) = y_0' \,, \dots\ y^{(n-1)}(t_0) = y_0^{(n-1)} \quad \dots (C)$$

Remark:- All mathmatical theory that we have learnt for the 2^{nd} order linear DE's apply perfectly well for the n^{th} order linear DE in (A).

So, it is enough to remind y ~~ass~~ with the corresponding general Theorems without proof.

Th. 4.1.1 If $P_0(t), P_1(t), \dots P_{n-1}(t), g(t)$ are ~~conditions~~ continuous functions on an open interval I. There exists a unique solution $y=\phi(t)$ that satisfies the DE (A) together with the initial conditions given in C.

The general solution for the homogeneous DE (B) is given by:

$$y_h(t) = c_1y_1(t) + c_2y_2(t) + \dots + c_ny_n(t).$$

To find $c_1, c_2, \dots, c_n$ we use the initial conditions given in (C)

$$y(t_0) = y_0 \Rightarrow c_1y_1(t_0) + c_2y_2(t_0) + \dots + c_ny_n(t_0) = y_0$$

$$y'(t_0) = y_0' \Rightarrow c_1y_1'(t_0) + c_2y_2'(t_0) + \dots + c_ny_n'(t_0) = y_0'$$

$$\vdots$$

$$y^{(n-1)}(t_0) = y^{(n-1)}(t_0) \Rightarrow c_1y_1^{(n-1)}(t_0) + c_2y_2^{(n-1)}(t_0) + \dots c_ny_n^{(n-1)}(t_0) = y_0^{(n-1)}$$

* For the constants $c_1, c_2, \dots c_n$ to make sence we must have $W(y_1, y_2, \dots y_n)(t_0) \neq 0$

Recall that * can be written as

$$\underbrace{\begin{bmatrix} y_1(t_0) & y_2(t_0) & \dots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \dots & y_n'(t_0) \\ \vdots & & & \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \dots & y_n^{(n-1)}(t_0) \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(n-1)} \end{bmatrix}.$$

This System has a unique solution iff $|A| \neq 0$

W

but $|A| = W(y_1, \dots, y_n)(t_0) \neq 0$

Remember that ① if $W(y_1, y_2 \dots y_n)(t_0) \neq 0$ for some $t_0 \in I \Rightarrow W(t) \neq 0 \ \forall t \in I$ (Abel's Thm)

② if $W(t_0) = 0$ for some $t_0 \in I \Rightarrow W(t) = 0 \ \forall t \in I$

Ex Determine an interval in which the solution of the DE:

$$t^2 y^{(4)} + t y^{(3)} + 5y = \sin t$$ is sure to exist

$$\Rightarrow y^{(4)} + \frac{1}{t} y^{(3)} + \frac{5}{t^2} y = \frac{\sin t}{t^2}$$

0

$I = (-\infty, 0) \cup (0, \infty)$

Defn. A functions $f_1, f_2 \dots f_n$ are lin. indep

if $c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$

implies that $c_1 = c_2 = \dots = c_n = 0$

$\Rightarrow f_1, f_2 \dots f_n$ are lin. indep. iff

$W(f_1, f_2 \dots f_n)(t) \neq 0$

$$W(f_1, f_2, f_3 \dots f_n) = \begin{vmatrix} f_1 & f_2 & f_3 & \dots & f_n \\ f_1' & \vdots & \vdots & & \vdots \\ & & & & f_n^{(n-1)} \\ f_1^{(n-1)} & f_2^{(n-1)} & - & - & f_n \end{vmatrix} \neq 0$$

Defn: A functions $f_1, f_2 \dots f_n$ are lin. dep

if $\exists$ $c_1, c_2, c_3 \dots c_n$ not All zero s.t.

$c_1 f_1 + c_2 f_2 + \dots c_n f_n = 0$.

if $f_1, f_2 \dots f_n$ are lin. dep.

$W(f_1, f_2, \dots f_n) = 0$

Ex. Show that $1, t, t^3$ form a fundamental set of solutions for the DE: $t^2 y''' + t y'' = 0$

if $t \neq 0$.

$y_1 = 1$ is a solution since $-t^2(0) + t(0) = 0$

$y_2 = t$ " " " $-t^2(0) + t(0) = 0$

$y_3 = t^3$ " " " $-t^3(6) + t(6t) = 0$

$\therefore$ $1, t, t^3$ are solutions.

now to check the indep.

$$W(1, t, t^3) = \begin{vmatrix} 1 & t & t^3 \\ 0 & 1 & 3t^2 \\ 0 & 0 & 6t \end{vmatrix} = 6t \neq 0$$

since $t \neq 0$

$\therefore$ y_1, y_2, y_3 are lin. indep.

$\therefore$ They form a fundamental set of solutions

Ex. Show that $f_1 = 1$, $f_2 = t$, $f_3 = t^2$ are lin. indep.

$$W = \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0 \quad \text{lin. indep.}$$

4.2 Homogeneous Equations with constant coefficients:

Find the general solution of

① $y^{(4)} + 2y''' - 13y'' - 14y' + 24y = 0$

$y(0) = 1, \quad y'(0) = -1 \quad y''(0) = 0 \quad y'''(0) = -1$

Ch. eq. $r^4 + 2r^3 - 13r^2 - 14r + 24 = 0$

$(r-1)(r+2)(r-3)(r+4) = 0$

$r_1 = 1 \quad r_2 = -2 \quad r_3 = 3 \quad r_4 = -4.$

The general solution "Homogeneous" is:-

$y(t) = c_1 e^{t} + c_2 e^{-2t} + c_3 e^{3t} + c_4 e^{-4t}$

$y(0) = 1 \Leftrightarrow c_1 + c_2 + c_3 + c_4 = 1$

$y'(0) = -1 \Leftrightarrow c_1 - 2c_2 + 3c_3 - 4c_4 = -1$

$y''(0) = 0 \Leftrightarrow c_1 + 4c_2 + 9c_3 + 16c_4 = 0$

$y'''(0) = -1 \Leftrightarrow c_1 - 8c_2 + 27c_3 - 64c_4 = -1$

$c_1 = \frac{1}{2}, \quad c_2 = 4/5, \quad c_3 = -11/70, \quad c_4 = -1/7$

$y(t) = \frac{1}{2} e^{t} + \frac{4}{5} e^{-2t} - \frac{11}{70} e^{3t} - \frac{1}{7} e^{-4t}$

② $y^{(4)} - y = 0 \qquad y(0) = 7/2, \quad y'(0) = -4, \quad y''(0) = 5/2$

$y'''(0) = -2$

Ch. eq. $r^4 - 1 = 0$

$(r^2 - 1)(r^2 + 1) = 0$

$r^2 = 1 \qquad r^2 = -1$

$r_{1,2} = \mp 1 \qquad r_{3,4} = \mp i$

The general solution

$y(t) = c_1 e^{t} + c_2 e^{-t} + c_3 \cos t - c_4 \sin t$

Using initial condition we get:

$y(t) = 3e^{-t} + \frac{1}{2} \cos t - \sin t$

$c_4 = -1$

$c_3 = \frac{1}{2}$

$c_2 = 3$

$c_1 = 0$

③ $y^{(iv)} + 2y'' + y = 0$

Ch. eq: $r^4 + 2r^2 + 1 = 0$

$(r^2 + 1)^2 = 0$

$r_{1,2} = \mp i \qquad r_{3,4} = \mp i$

$y(t) = c_1 \cos t - c_2 \sin t + c_3 t \cos t - c_4 t \sin t$

4) $y^{(4)} + y^{(2)} = 0$

$r^4 + r^2 = 0$

$r^2(r^2+1) = 0$

$r_{1,2} = 0 \qquad r_{3,4} = \mp i$

$\therefore y(t) = c_1 + c_2 t + c_3 \cos t + c_4 \sin t$

5) $y^{(iv)} - y''' + y'' + y' = 0$

$r^4 - r^3 + r^2 + r = 0$

$r(r^3 - r^2 - r + 1) = 0$

$r(r^2(r-1) - (r-1)) = 0$

$r((r-1)(r^2-1)) = 0$

$r_1 = 0, \quad r_2 = 1 \quad r_{3,4} = \mp 1$

$y(t) = c_1 + c_2 e^t + c_3 t e^t + c_4 e^{-t}$

6) $y''' + y'' = 10x^2 \qquad y(0) = y'(0) = y''(0) = 1$

$y_h(x)$:- $\quad r^3 + r^2 = 0$

$r^2(r+1) = 0$

$r_1 = r_2 = 0 \qquad r_3 = -1$

$y_h(x) = c_1 + c_2 x + c_3 e^{-x}$

$y_p(x) = (Ax^2 + Bx + C)x^2 \qquad = Ax^4 + Bx^3 + Cx^2$

We substitute ~~y_p, y_p'~~, y_p', y_p'' in eq to find A, B, C

$\Rightarrow A = \frac{5}{6} \qquad B = -10/3 \quad , \quad C = 10$

$\Rightarrow y_p(x) = \frac{5}{6}x^4 - \frac{10}{3}x^3 + 10x^2$

The general solution is

$y(x) = y_h(x) + y_p(x)$

$= c_1 + c_2 x + c_3 e^{-x} + \frac{5}{6}x^4 - \frac{10}{3}x^3 + 10x^2$

to find c_1, c_2, c_3 we use the initial conditions to obtain

$c_1 = 20, \quad c_2 = -18, \quad c_3 = -19$

Ex. 2 express the following complex numbers in the form of $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ Euler formula.

1) $-1 + \sqrt{3}\,i \qquad (x + yi)$

length $= \sqrt{x^2+y^2}$

$= \sqrt{1+3} = 2$

$-1 + \sqrt{3}i = 2\left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) \qquad \theta = \frac{2\pi}{3} + 2\pi n$

$n = 0, \mp 1, \mp 2, -$

$= 2\left[\cos \frac{2\pi}{3} + 2\pi n + \sin\left(\frac{2\pi}{3} + 2\pi n\right) i\right]$

$2\, e^{i\left(\frac{2\pi}{3} + 2\pi n\right)}$

$2\left[\cos\left(2\frac{\pi}{3} + 2\pi n\right) + i\sin\left(\frac{2\pi}{3} + 2\pi n\right)\right]$

② $-3 = -3+0i$

length $= \sqrt{9+0} = 3$

$3[-1+0i]$ $\theta = \pi + 2\pi n$

$3e^{i(\pi+2\pi n)} = 3[\cos(2\pi n+\pi) + i\sin(\pi+2\pi n)]$

ex find the general solution of

$y^4 + y = 0$

$r^4 + 1 = 0$

$\cancel{(r^2-1)(r^2-1) = 0}$

$\cancel{r_1 = r_2 = 1 \quad r_2 = r_3 = (1+0i)}$

$r^4 = -1$

$\cancel{r_1 = r_2 = 1 \quad r_3 = r_4 = i}$ $r = (-1)^{\frac{1}{4}}$

length $= \sqrt{(1+0)} = 1$

$-1 = -1+0i$ $\theta = \pi + 2\pi n$

$e^{i(\pi + 2\pi n)}$

$(-1)^{\frac{1}{4}} = e^{i\left(\frac{\pi}{4} + \frac{8\pi n}{2}\right)} = \cos\left(\frac{\pi}{4} + \frac{\pi}{2}n\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{2}n\right)$

$n = 0, 1, 2, 3$

To find r_1, r_2, r_3, r_4 we substitute $n = 0, 1, 2, 3$.

When $n = 0$

$r_1 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

$n = 1$ $r_2 = \cos\left(\frac{\pi}{4} + \frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{2}\right)$

$= \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

$n = 2$ $r_3 = \cos\left(\frac{\pi}{4} + \pi\right) + i\sin\left(\frac{\pi}{4} + \pi\right)$

$r_3 = \frac{-1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}i$

$n = 3$ $r_4 = \cos\left(\frac{\pi}{4} + \frac{3\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{3\pi}{2}\right)$

$r_4 = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$r_{1,4} = \frac{1}{\sqrt{2}} \mp \frac{1}{\sqrt{2}}i$ $r_{2,3} = \frac{-1}{\sqrt{2}} \mp \frac{1}{\sqrt{2}}i$

The general solution

$y(t) = C_1 e^{\frac{1}{\sqrt{2}}t}\cos\frac{1}{\sqrt{2}}t + C_2 e^{\frac{1}{\sqrt{2}}t}\sin\frac{t}{\sqrt{2}} + C_3 e^{-\frac{t}{\sqrt{2}}}\cos\frac{t}{\sqrt{2}} + C_4 e^{-t/\sqrt{2}}\cos\frac{t}{\sqrt{2}}$

Ex: determine the two roots of

$$(1-i)^{\frac{1}{2}}$$

$$1-i = \sqrt{2}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right] \qquad \text{length} = \sqrt{1+1} = \sqrt{2}$$

$$\theta = -\frac{\pi}{4} + 2\pi n$$

$$(1-i) = \sqrt{2}\, e^{i(\frac{\pi}{4}+2\pi n)}$$

$$(1-i)^{\frac{1}{2}} = \left(\sqrt{2}\, e^{i(-\frac{\pi}{4}+2\pi n)}\right)^{\frac{1}{2}}$$

$$2^{\frac{1}{4}}\, e^{i(-\frac{\pi}{8}+\pi n)}$$

$$2^{\frac{1}{4}}\left(\cos(-\tfrac{\pi}{8}+\pi n) + i\sin(-\tfrac{\pi}{8}+\pi n)\right)$$

the two roots r_1, r_2 are when

$n=0$, $r_1 = 2^{\frac{1}{4}}\left(\cos -\frac{\pi}{8} + i\sin -\frac{\pi}{8}\right) = 2^{\frac{1}{4}}\left[0 + 0i\right]$

$n=1$, $r_2 = 2^{\frac{1}{4}}\left(\cos \frac{7\pi}{8} + i\sin \frac{7\pi}{8}\right) = 2^{\frac{1}{4}}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right]$

4.3 The method of ~~unde~~ undetermined coefficient

Recall that a linear nonhomogeneous DE of order n with constant coefficients has the general form

$$a_n y^{(n)} + a_{(n-1)} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t) \quad \text{---} \; (1)$$

Now we can still use the method of undetermined coefficients to find a particular solution of DE (1) iff $g(t)$ is $\sin t$, $\cos t$, polynomial...

Ex. find the general solution of

$$y''' - 3y'' + 3y' - y = 4e^{t}$$

$$y_h(t) = r^3 - 3r^2 + 3r - 1 = 0$$

$$(r-1)^3 = 0$$

$$r_1 = r_2 = r_3 = 1$$

$$y_h(t) = c_1 e^{t} + c_2 t e^{t} + c_3 t^2 e^{t}$$

To find $y_p(t)$

$$y_p(t) = A e^{t} \cdot t^3 \qquad \text{(to check independency).}$$

$$y_p' = At^3 e^{t} + 3At^2 e^{t}$$

$$y_p'' = At^3 e^{t} + 6At^2 e^{t} + 6Ate^{t}$$

$$y_p''' = At^3 e^{t} + 9At^2 e^{t} + 18Ate^{t} + 6Ae^{t}$$

Substitute y_p, y_p', y_p'', y_p''' in the DE above.

$6Ae^t = 4e^t \iff A = 2/3$

$y_p(t) = \frac{2}{3} t^3 e^t$

general solution $y(t) = y_p(t) + y_h(t)$

[2] find the particular solution of

$y^{(4)} + 8y'' + 16y = 2\sin t - 3\cos t$

$r^4 + 8r^2 + 16 = 0$

$(r^2+4)(r^2+4) = 0$

$r^2 + 4 = 0$

$r_{1,2} = \mp 2i \qquad r_{3,4} = \mp 2i \qquad \lambda = 0 \quad \mu = 2$

$y_h(t) = c_1 \cos 2t + c_2 \sin 2t + (c_3 \cos 2t + c_4 \sin 2t)t$

To find $y_p(t)$.

We let $y_p(t) = (A\cos t + B\sin t)$ ... ⇒ independent

$y_p' = -A\sin t + B\cos t$

$y_p'' = -A\cos t - B\sin t$

$y_p''' = A\sin t - B\cos t$

$y_p^{(4)} = A\cos t + B\sin t$

(substitute in DE above)

$9A\sin t + 9B\cos t = 2\sin t - 3\cos t$

$A = 2/9 \qquad B = \frac{-3}{9} = -\frac{1}{3}$

$y_p(t) = \frac{2}{9}\cos t - \frac{1}{3}\sin t$

[3] Find y_p for the DE:

$y^{(4)} + 8y'' + 16y = 2\sin 2t - 3\cos 2t$

we know $y_h(t)$

$y_p(t) = \cancel{A\cos} (A\sin 2t + B\cos 2t)t^2$

find $y_p', y_p'', y_p''', y_p^{(4)}$ & substitute above to obtain $\boxed{A = -1/16,\ B = 3/32}$

Hence $y_p(t) = \left(-\frac{1}{16}\sin 2t + \frac{3}{32}\cos 2t\right)t^2$

[4] $y''' - 4y' = t + 3\cos t + e^{-2t}$ --- (*)

$y_h(t)$: $r^3 - 4r = 0$

$r(r^2-4) = 0 \qquad r_1 = 0,\ r_2 = 2,\ r_3 = -2$

$y_h(t) = c_1 + c_2 e^{2t} + c_3 e^{-2t}$

To find y_p of t.

3 cases.

① $y''' - 4y' = t \Rightarrow y_{p_1}(t) = (A_0 t + A_1)t$

find y_{p_1}', y_{p_1}''' & find A_0, A_1

$A_0 = -1/8$ $\boxed{A_1 = 0}$

$y_{p_1} = -\frac{1}{8}t^2$

② $y''' - 4y' = 3\cos t$

$y_{p_2}(t) = B_1\cos t + B_2\sin t$

find y_{p_2}', y_{p_2}''' & obtain $B_1 = 0, B_2 = -\frac{3}{5}$

$y_{p_2} = \frac{-3}{5}\sin t.$

③ $y''' - 4y' = e^{-2t}$

$y_{p_3} = Dte^{-2t}$ find y_{p_3}', y_p''' to obtain

$D = \frac{1}{8}$

$y_{p_3} = \frac{1}{8}te^{-2t}$

Hence y_p for the DE (*) is

$y_p(t) = y_{p_1} + y_{p_2} + y_{p_3}$

$y_p(t) = \frac{-1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}te^{-2t}.$

Ch.5

5.1 Review of Power series

Recall that finding a solution for a linear DE yields to find the fundamental set of solutions in case where the DE are with constant coefficient.

So this Chapter we will try to find the fundamental set of solutions, y_1 & y_2, for a 2^{nd} linear DE with variable coefficients, and we write these solution in terms of power series.

Ex. $y'' + y = 0$

Ch. Eq $r^2 + 1 = 0$ $r_{1,2} = \mp i$

$$y(t) = c_1\cos t + c_2\sin t = c_1\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} + c_2\sum_{n=0}^{\infty}(-1)^n\frac{x^{(2n+1)}}{(2n+1)!}$$

The fundamental set of solutions are

$y_1(t) = \cos t$ $y_2(t) = \sin t$

Ex. $a_n = \frac{1}{n}$, $n = 1, 2, \ldots$

sequence

$a_n \to 0$ as $n \to \infty$

Ex. $\sum_{n=1}^{\infty} \frac{1}{n}$ Series

"harmonic series" Diverges

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right) + \cdots$$

$>\frac{1}{2}$ $>\frac{1}{2}$ $>\frac{1}{2}$

$\to \infty$

after 75 million terms it will have a sum of 20.

series $\to$ finite / infinite.

Power series about the point x_0 "center" has the form:

$$\boxed{\sum_{n=0}^{\infty} a_n (x-x_0)^n} \quad *$$

the power series $*$ converges if $\lim_{n\to\infty} \sum_{0} a_n (x-x_0)^n$ exists.

To check for the absolute convergence for the Power series $*$, we apply Ratio Test "RT".

Ratio Test $\quad L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, $a_n \neq 0$

if $L<1$ then the series converges absolutely

* if $L>1$ the series diverges.
* if $L=1$ the test is inconclusive.

Ex. for which values of x does the power series $\sum_{n=1}^{\infty} (-1)^{n+1} n (x-2)^n$ converges.

Ratio Test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(-1)^{n+2}(n+1)(x-2)^{n+1}}{(-1)^{n+1} n (x-2)^n}\right|$$

$$|x-2| \lim_{n\to\infty} \frac{n+1}{n} = |x-2| < 1$$

$|x-2| < 1 \to \rho$

$-1 < x-2 < 1$

$1 < x < 3$

$1-\rho$, $\rho = 1$, 1, x_0 (2), 3

the series converges abs for all $1<x<3$

* The series diverges for all $|x-2|>1$
* when $x=1 \Rightarrow \sum_{n=1}^{\infty} (-1)^{2n+1} n = -\sum_{n=1} n$ diverges by nth term Test
$= -\sum n$
* when $x=3 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n+1} n$ diverges by nth term test & Alternating series test

interval of convergence

diverges | converges absolutely $\rho = 1$ | diverges
1 ... 3

ex. determine the radius of convergence of the Power series :-

① $\sum \frac{(x+1)^n}{n 2^n}$.

IC

the series diverge ($x_0 - \rho$ ... x_0 ... $x_0 + \rho$) the series diverges

may converge & maybe divergence

Apply R.T.

$$\lim_{n \to \infty} \left| \frac{(x+1)^{n+1}}{(n+1) 2^{(n+1)}} \cdot \frac{n\, 2^n}{(x+1)^n} \right|$$

$$\frac{|x+1|}{2} \lim_{n \to \infty} \frac{n}{(n+1)} = \frac{|x+1|}{2}$$

$$\frac{|x+1|}{2} < 1 \qquad |x-1| < 2$$

$$2 - < x - 1 < 2$$

$$\boxed{-1 < x < 3}$$

The power series converges abs for all $x \in (-3, 1)$ and diverges for all $|x+1| > 2$

when $x = -3$, $\sum \frac{(x+1)^n}{n 2^n} = \sum_{n=1} \frac{(-2)^n}{n 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$\lim_{n \to \infty} a_n = 0$, converges by A.S.T.

A.S.T ($\lim_{n \to \infty}$ ...) Alternating

when $x = 1$ $\Rightarrow \sum_{n=1} \frac{1}{n}$ (Harmonic Series)

Divergent harmonic series

IC: $[-3, 1)$

② $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$. Apply R.T

$$\lim \frac{(-1)^n x^{n+1}}{(n+1)} \cdot \frac{n}{(-1)^{n-1} x^n}$$

$|x| < 1$

$\rho = 1$

③ $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ Apply RT

$$\lim \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n \to \infty} \frac{1}{(n+1)} = 0 < 1$$

$\boxed{\rho = \infty}$ IC $= (-\infty, \infty)$

④ $\sum_{n=0}^{\infty} n!\, x^n$

$$\lim_{n \to \infty} \left| \frac{(n+1)!\; x^{n+1}}{n!\; x^n} \right| = |x| \lim_{n \to \infty} n+1$$

if $x = 0$ converges

$\rho = 0$ div $\;\overset{}{\underset{0}{\circ}}\;$ div

Shifting Indeces:

It is not important which indeces we use for a power series.

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{k=0}^{\infty} a_k (x-x_0)^k$$

Ex. $\displaystyle\sum_{n=1}^{\infty} a_n (x+1)^{n-1} = \sum_{n=0}^{\infty} a_{n+1} (x+1)^n$

$a_1 + a_2(x+1)$ $a_1 + a_2(x+1)$

Ex. Write the series $\displaystyle\sum_{n=0}^{\infty} (n+1) a_n x^{n+3}$ as a sum in values x^n

$$\sum_{3}^{\infty} (n-2) x^n a_{n-3}$$

5.2 Series Solution near Ordinary Point I.

* Recall we have used the method of undetermined coefficient to find the general solution of 2nd order linear DE with constant coefficients.

Here, we will find a series solution for the DE

$$P(x) y'' + Q(x) y' + R(x) y = 0 \quad \text{--- (1)}$$

in the neighborhood of an ordinary point x_0 in the case where the DE (1) is with variable coefficients.

Note: the point x_0 is an ordinary point if $P(x_0) \neq 0$.

if $P(x_0) = 0 \Rightarrow x_0$ is called singular point.
(next sections)

in the DE (1), the functions $P(x), Q(x), R(x)$ are assumed to be continuous on an interval I around x_0

* In this sense, the DE (1) may be written as:

$$y'' + p(x) y' + q(x) y = 0, \quad \text{where} \quad p(x) = \frac{Q(x)}{P(x)}$$

$$\& \quad q(x) = \frac{R(x)}{P(x)}$$

Therefor :- by Th. 3.2.1, there exists a unique solution that satisfies the DE ① together with the initial conditions.

$y(x_0) = y_0$, $y'(x_0) = y'_0$

To find such a solution in terms of power series we assume such a solution has the form:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + \dots$$

defined on an interval of convergence $|x - x_0| < \rho$ where ρ is the radius of convergence which is assumed to be positive.

Ex. find a series solution for the DE

$y'' + y = 0$ $\quad -\infty < x < \infty$

* $P(x) = 1 \neq 0 \quad \forall x \in (-\infty, \infty)$

Hence, every point is an ordinary point.

So we choose x_0 to be the simplest choice $\boxed{x_0 = 0}$

$r^2 + 1 = 0$, $r = \mp i$

$y(x) = \underbrace{c_1 \cos x}_{y_1(x)} + \underbrace{c_2 \sin x}_{y_2(x)}$

series $\quad$ series

* the power series solution will now have the form:

$$y(x) = \sum_{0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

For $y(x)$ to a solution for the DE * :

$\Rightarrow$ $y(x)$ must satisfy it :-

$$y'(x) = \sum_{1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute y, y'' in the DE *

$$\sum_{2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{0}^{\infty} a_n x^n = 0$$

shifting indices

$$\sum_{0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{0}^{\infty} a_n x^n = 0$$

$$= \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n = 0$$

Equating the coefficients of each power of x in the LHS to its corresponding terms in the RHS :

$$(n+2)(n+1) a_{n+2} + a_n = 0 \qquad , n = 0, 1, 2, \dots$$

$$\boxed{a_{n+2} = \frac{-a_n}{(n+2)(n+1)} \qquad , n = 0, 1, 2, 3}$$

$\hookrightarrow$ recurrence relation.

we just need to find a_2, a_3, a_4.

when $n=0$: $a_2 = \frac{-a_0}{2(1)}$ when $n=1$: $a_3 = \frac{-a_1}{(3)(2)}$

$n=2 \Rightarrow a_4 = \frac{-a_2}{(4)(3)} = \frac{+a_0}{4!}$

$n=3 \Rightarrow a_5 = \frac{-a_3}{(5)(4)} = \frac{a_1}{5!}$

$a_{2k} = \frac{(-1)^k a_0}{(2k)!}$, $k=0,1,2.$

$a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!}$, $k=0,1,2 \ldots$

So the solution becomes

$$y(x) = \sum_0^\infty a_n x^n = \sum_0^\infty a_{2n} x^{2n} + \sum_0^\infty a_{2n+1} x^{2n+1}$$

$$= \sum_0^\infty \frac{(-1)^n a_0}{(2n)!} x^{2n} + \sum_0^\infty \frac{(-1)^n a_1}{(2n+1)!} x^{2n+1}$$

$$= a_0 \sum_0^\infty \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

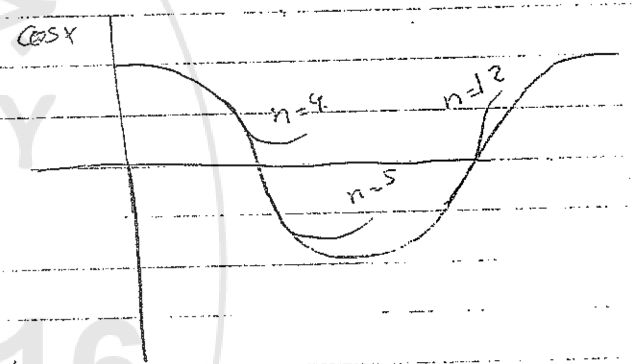
$$y(x) = a_0 \underbrace{\cos x}_{y_1(x)} + a_1 \underbrace{\sin x}_{y_2(x)}$$

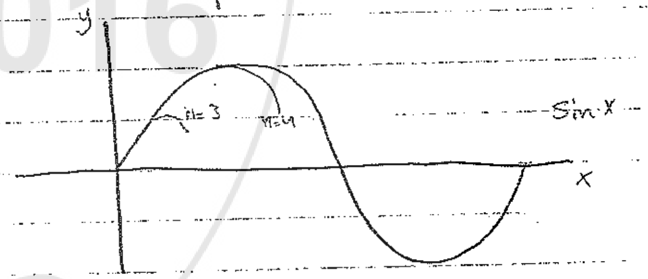
$$= a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right] + a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right]$$

$$W(y_1,y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow$ y_1, y_2 They are linearly independen

$\Rightarrow$ They are fundamental set of solutions





Ex. Find a power series solution of Power x. for the Airy's Eq:

$$y'' - xy = 0 \qquad -\infty < x < \infty$$

$P(x) = 1 \Rightarrow$ Any point is an ordinary point. the simplest choice is $x_0 = 0$.

The power series solution is

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\Rightarrow \sum_{2}^{\infty} n(n-1) a_n x^{n-1} - x \sum_{0}^{\infty} a_n x^n$$

اذا كان

$$\sum \quad x \sum a_n (x-1)^n$$
$$((x-1)+1) \sum$$

نوزع ونكمل

$$= \sum_{2}^{\infty} n(n-1) a_n x^{n-1} - \sum_{0}^{\infty} a_n x^{n+1}$$

$$\sum_{0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 \quad \sum_{1} [(n+2)(n+1) a_{n+2} - a_{n-1}] x^n = 0 \quad , n = 2, 3, \ldots$$

كل x

$$\boxed{a_2 = 0} \quad , \quad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \quad , n = 1, 2, 3, \ldots$$

$$a_5 = a_8 = a_{11} = \cdots = 0$$

reccurance relation

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 \left[1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{(2)(3)\ldots(3n-1)(3n)} \right] + a_1 \left[x + \sum \frac{x^{3n+1}}{3(4)\ldots(3n+1)} \right]$$

$$W(y_1, y_2)(0) \begin{vmatrix} y_1(0) & y_2(0) \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\Rightarrow \quad y_1, y_2$ is fundumental

$\boxed{\text{5.2 continued}}$

Ex. find a series solution of Power $x-1$ for the Airy's equation $y'' - xy = 0$

$P(x) = 1$ any point is ordinary point.

$\Rightarrow x_0 = 1$ is an ordinary point.

The power series solution is

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n = a_0 + a_1(x-1) + a_2(x-1)^2 + \cdots$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$$

Substitute y and y'' in the DE above to obtain:-

$$\sum n(n-1) a_n (x-1)^{n-2} = ((x-1)+1)\sum_{n=0}^{\infty} a_n (x-1)^n$$

$$\sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{0}^{\infty} a_n (x-1)^{n+1} + \sum_{0}^{\infty} a_n (x-1)^n$$

Shifting Indices

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n = \sum_{1}^{\infty} a_{n-1} (x-1)^n + \sum_{0}^{\infty} a_n (x-1)^n$$

$$2(1) a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n = \sum_{1}^{\infty} a_{n-1} (x-1)^n + a_0 + \sum_{n=1}^{\infty} a_n (x-1)^n$$

$$\boxed{a_2 = \frac{a_0}{2}}$$

$$a_{n+2} = \frac{a_{n-1}(x-1)^n + a_n (x-1)^n}{(n+2)(n+1)(x-1)^n} = \frac{\cancel{a_{n+1}}\, a_{n-1} + a_n}{(n+1)(n+2)}$$

$n = 1, 2, 3, \dots$

recurrence relation

When $n=1$

$$a_3 = \frac{a_0 + a_1}{3 \cdot 2}$$

$n=2$

$$a_4 = \frac{a_1 + a_2}{4(3)} = \frac{a_1}{4(3)} + \frac{a_2}{4(3)} = \frac{a_1}{4(3)} + \frac{a_0}{4(3)(2)}$$

$n=3$ $$a_5 = \frac{a_2 + a_3}{5(4)} = \frac{\frac{a_0}{2} + \frac{a_0 + a_1}{3(2)}}{5(4)} = \frac{4a_0 + a_1}{5(4)(3)(2)}$$

The solution becomes:-

$$y(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 + a_5(x-1)^5 +$$

$$= a_0 + a_1(x-1) + \frac{a_0}{2}(x-1)^2 + \frac{a_0}{3(2)}(x-1)^3 + \frac{a_1}{3(2)}(x-1)^3 + \frac{a_1}{4(3)}(x-1)^4 + \frac{a_0}{4(3)(2)}(x-1)^4$$

$$+ \frac{4a_0}{5!}(x-1)^5 + \frac{a_1}{5!}(x-1)^5 + \dots$$

$$a_0\left[1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \frac{(x-1)^4}{4!} + 4\frac{(x-1)^5}{5!} + \dots\right]$$

$$+ a_1\left[(x-1) + \frac{(x-1)^3}{3!} + \frac{2(x-1)^4}{4!} + \frac{(x-1)^5}{5!}\right]$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

$W(y_1,y_2)(1) = \begin{vmatrix} y_1(1) & y_2(1) \\ y_1'(1) & y_2'(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\Rightarrow$ thus y_1, y_2 form a fundamental set of solutions.

5.3 Series solution near an Ordinary point

Recall that we have seen how to find a power series solution for the DE: $P(x)y'' + R(x)y' + Q(x)y = 0$ (*) in case where P, Q, R are polynomials, in the neighbourhood of an ordinary point x_0, $P(x_0) \neq 0$

Hence, * can be written as:

$$y'' + p(x)y' + q(x)y = 0$$

where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are analytic function.

Defn. The function $p(x)$ (resp. $q(x)$) is analytic function about x_0 if $p(x)$ (resp. $q(x)$) has a taylor expantion that converges to $p(x)$ (resp. $q(x)$) in the interval of convergence $|x-x_0| < \rho$

$$p(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n = p_0 + p_1(x-x_0) + p_2(x-x_0)^2 + \dots$$

$$q(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n = q_0 + q_1(x-x_0) + q_2(x-x_0)^2 + \dots$$

ex. [1] $\sin x, \cos x, e^x$ are analytic everywhere. (Interval of convergence $x \in (-\infty, \infty)$, & $\rho = \infty$)

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

[2] $\frac{1}{x}$ is analytic everywhere except at $x_0 = 0$.

[3] $\tan x$ " " " " odd multiple of $\frac{\pi}{2}$

Recall the power series solution

$\phi(x) = y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ (A) to the DE

$y'' + p(x) y' + q(x) y = 0$

One can show that:

$\phi^{(m)}(x_0) = m!\, a_m$

Proof: Differentiate $\phi(x)$ given in (A) m times:

$\phi'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$

$\phi''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$

$\phi'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2) a_n (x-x_0)^{n-3}$

$\phi^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)(n-2)\cdots(n-m+1) a_n (x-x_0)^{n-m}$

$\phi^{(m)}(x) = m(m-1)(m-2)\cdots(m-m+1)\, a_m (x-x_0)^{m-m} + \sum_{m+1}^{\infty} n(n-1)(n-2)\cdots(n-m+1) a_n (x-x_0)^{n-m}$

$\phi^{(m)}(x) = m!\, a_m + \Sigma \ldots$

$\phi^{(m)}(x_0) = m!\, a_m + 0$

$$\boxed{\phi^{(m)}(x_0) = m!\, a_m}$$

Note that since $p(x)$ & $q(x)$ are analytic at x_0, it follows that we can differentiate them as much as we like at x_0.

How to find $\phi^{(m)}(x)$ if $\phi(x)$ is a solution to the DE: $y'' + p(x) y' + q(x) y = 0$

Since $\phi(x)$ is a solution to the DE above

$\Rightarrow \phi'' + p(x)\phi'(x) + q(x)\phi(x) = 0$

$\phi'' = -p(x)\phi'(x) - q(x)\phi(x)$ — (B)

from B $\Rightarrow$

$\phi''(x_0) = -p(x_0)\phi'(x_0) - q(x_0)\phi(x_0)$

$= -p(x_0) a_1 - q(x_0) a_0$

To find $\phi'''(x_0)$

we differentiate B to find $y'''(x)$ then we substitute x_0.

5.3 continued.

Ex. 1 Determine $\phi''(x_0)$, $\phi'''(x_0)$, $\phi^{(4)}(x_0)$ for $x_0 = 0$, if $y(x) = \phi(x)$ is a solution for the IVP :-

$$y'' + xy' + y = 0 \qquad y(0) = 1 \quad , \; y'(0) = 0$$

$*$ $y(0) = 1 = y_0 = \phi(0) = a_0$
$y'(0) = 0 = y_0' = \phi'(0) = a_1$

$*$ Since $y = \phi(x)$ is a solution $\Rightarrow$
$$\phi''(x) + x\phi'(x) + \phi(x) = 0$$

$$\phi''(x) = -x\phi'(x) - \phi(x) \quad \text{--- } 1$$
$$\phi''(0) = -(0)\phi'(0) - \phi(0)$$
$$\phi''(0) = -1$$

to find $\phi'''(0)$ we first differentiate 1

$$\phi''' = -x\phi''(x) - \phi'(x) - \phi'(x) \quad \text{--- } 2$$
$$= -x\phi''(x) - 2\phi'(x)$$

$$\phi'''(0) = -0(\phi''(0)) - 2\phi'(0) = 0$$

To find $\phi^{(4)}(0)$ we differentiate eq. 2

$$\phi^{(4)}(x) = -x\phi'''(x) - 3\phi''(x)$$

$$\phi^{(4)}(0) = -0 - 3\phi''(0) = 3$$

2 find the first ~~non~~ two non zero terms of the series solution (about x_0)

The series solution about $x_0 = 0$ has the form :-
$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$
$$1 + 0x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$
$$-1/2 \qquad 0$$

$$\phi^{(m)}(x_0) = m!\, a_m$$
$$\phi''(0) = 2! * a_2$$
$$-1 = 2a_2$$
$$\boxed{-1/2 = a_2}$$

$$\phi'''(0) = 3!\, a_3$$
$$0 = 3!\, a_3$$
$$\boxed{0 = a_3}$$

$$\phi^{(4)}(0) = 4!\, a_4$$
$$\frac{3}{4!} = a_4$$
$$\boxed{\frac{1}{8} = a_4}$$

$$= \underbrace{1 - \frac{x^2}{2}}_{\text{first two non zero}} + \frac{x^4}{8}$$

first 3 non zero

Ex. what is the radius of convergence of the Taylor series for $(1+x^2)^{-1}$ at $x_0=0$

Solution 1

$(1+x^2)^{-1} = \frac{1}{1+x^2} = \frac{1}{1+(x)^2} = 1+x^2+x^4+x^6+\dots$

geometric

$\frac{1}{1-x}$ ratio

$(-x^2)$

$|x^2|<1$

$-1<x<1$

$\boxed{\rho=1}$

Or $\frac{1}{1+x^2} = \sum_{0}^{\infty}(-1)^n x^{2n}$

Apply R.T

$\lim_{n\to\infty} \left|\frac{x^{2n+2}}{x^{2n}}\right| = \lim_{n\to\infty} x^2 = |x^2|<1 \Rightarrow \rho$

$-1<x^2<1$

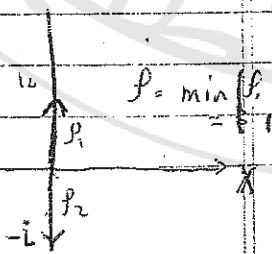
$\boxed{-1<x<1}$

Solution 2

Polynomial:- $1+x^2=0$ $\Leftrightarrow$ root

$x=\mp i$

$\rho=\min(\rho_1,\rho_2) = 1$



Note:- If P and Q are polynomials, then the radius of convergence for $p=\frac{Q(x)}{P(x)}$ and $q=\frac{R(x)}{P(x)}$ about x_0 is the difference between x_0 and the nearest zero of P.

Ex. what is the radius of convergence of the Taylor series of $(x^2-2x+2)^{-1}$ about:-

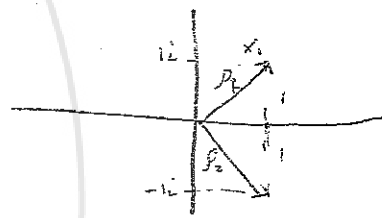
① $x_0=0$

$x^2-2x+2=0$ $\quad r=\frac{2\mp\sqrt{4-8}}{2}$

$x=1\mp i$

$\rho_1=\sqrt{2}$

$\rho_2=\sqrt{2}$



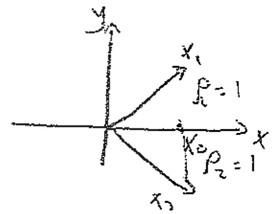
$\rho=\min(\rho_1,\rho_2)=\sqrt{2}$

② $x_0=1$

$\rho=\min(1,1)$

$\boxed{\rho=1}$

converge Abs.



Hence, the radius of convergence for the series Solution $y(x)=\sum_{a}^{n} a_n(x-1)^n$ is 1

Thm 5.3.1

If x_0 is an Ordinary point for the DE

$$\boxed{P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0} \quad (A)$$

$\Rightarrow p(x) = \dfrac{Q(x)}{P(x)}$, $q(x) = \dfrac{R(x)}{P(x)}$

$\Rightarrow$ p, q are analytic at x_0.

then, the general solution of the DE (A) is

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$
$$= a_0 y_1(x) + a_1 y_2(x)$$

where:- * a_0 and a_1 are arbitrary (عشوائي)

* $y_1(x)$ & $y_2(x)$ are analytic at x_0. (Power series solutions that are)

and $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions.

Furthermore, the radius of convergence for each of the series solutions (y_1 and y_2) is at least as the min of the radius of convergence of p and q.

radius & (radii)

* Note that:-

Thm 5.3.1 is a general theorem.

(P, Q, R) can be polynomials or not!!

Ex. determine a lower pound for the radius of convergence of the series solution:-

i) for $y'' - xy = 0$ about $x_0 = 1$

$P(x) = 1$, $Q(x) = 0$, $R(x) = -x$

(All are polynomials)

$\Rightarrow x_0 = 1$, $P(1) = 1$

every point is an ordinary point

$p = \dfrac{0}{1} = 0$, $\rho = \infty$

$q = \dfrac{-x}{1} = -x$, $\rho = \infty$

are analytic everywhere.

Hence, the radius of convergence for p, q is infinite.

by Thm 5.3.1, the radius of convergence for the series solution $\sum_0^{\infty} a_n (x-1)^n$ is infinite.

2) for $(x^2+3x)y'' + y' + y = 0$ about $x_0 = -1$

$P(x) = x^2 + 3x$, $Q(x) = 1$, $R(x) = 1$

$x^2 + 3x = 0$ (all are polynomials)

$\boxed{x=0}$ $\boxed{x=-3}$

Singular points. $x_0 = -1$ is an ordinary point.

$P(x) = \frac{1}{x(x+3)}$, $q(x) = \frac{1}{x(x+3)}$

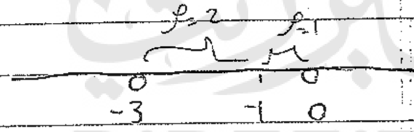
have singular points at $x = 0, 3$

but q, p at $x = -1$ are analytic.

(They have taylor series)

$\rho = \min(1, 2)$

$\boxed{\rho = 1}$



Hence, the radius of convergence for the taylor series expantion for $P(x)$, $q(x)$ is one.

therefore, by Th. 5.3.1 the radius of convergence for the series solution $\sum_{0}^{\infty} a_n(x+1)^n$ is at least 1.

Ex. Find a lower bound for the radius of convergence for the series solution of the DE

1) $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

about $x_0 = 0$.

$P(x) = 1 - x^2$ $\quad Q(x) = 2x$ $\quad R(x) = \alpha(\alpha+1)$

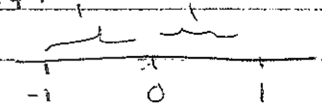
(all polynomials

$P(x)$ has a singular points only when $x = \mp 1$

Hence $x_0 = 0$ is Ordinary point

$p(x) = \frac{-2x}{1-x^2}$ and $q(x) = \frac{\alpha(\alpha+1)}{1-x^2}$

are analytic at $x = 0$ but they have singular points at $x = \mp 1$



* Hence, the radius of convergence for the taylor series expansion of $p(x)$ and $q(x)$ is 1.

* Therefore, by Th. 5.3.1 the radius of convergence for the series solution $\sum_{n=0}^{\infty} a_n x^n$ is at least one.

② $(1+x^2)y'' + 2xy' + 4x^2y = 0$ about $x_0 = -\frac{1}{2}$

$P(x) = 1 + x^2$ $\quad Q(x) = 2x$ $\quad R(x) = 4x^2$

"All are polynomials"

$P(x)$ has singular points at $1 + x^2 = 0 \Rightarrow x = \mp i$

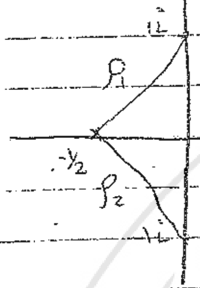
Hence, $x_0 = -\frac{1}{2}$ is an ordinary point.

$p(x) = \frac{2x}{1+x^2}$ $\quad q(x) = \frac{4x^2}{1+x^2}$

are analytic at $x_0 = -\frac{1}{2}$ but they have singular points at $x = \mp i$

$\lambda = 0$ x-axis

$u = 1$ (y-axis)



$\rho_1 = \rho_2 = \sqrt{\frac{1}{4}+1}$

$\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$

Hence the radius of convergence for the taylor seriese expansion of $p(x)$ and $q(x)$ is $\frac{\sqrt{5}}{2}$.

Therefore, by Th. 5.3.1, the radius of convergence for the series solution $\sum a_n (x+\frac{1}{2})^n$ is at least $\frac{\sqrt{5}}{2}$.

③ $y'' + \sin x \, y' + (1+x^2) y = 0$

about $x_0 = 0$

$P(x) = 1 \quad Q(x) = \sin x \quad R(x) = 1+x^2$

« Not All polynomials »

Any point is an ordinary point since P is never zero.

$p(x) = \frac{\sin x}{1} = \sin x$. which is analytic everywhere

since the taylor series expansion of $\sin x = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Hence, its radius of convergence is infinite.

$q(x) = \frac{1+x^2}{1} = 1+x^2$

$\sum \frac{f^{(n)}(0)\, x^n}{n!}$

$= 1 + 0x + \frac{2x^2}{2} = 1+x^2$ taylor Series.

Hence, $q(x)$ has taylor series expansion (itself) which is analytic everywhere.

Hence, its radius of convergence is ∞

~~Hence~~,

Thus, by Thm. 5.3.1 the radius of convergence for the series solution $\sum_0^\infty a_n x^n$ is infinite.

5.4 Euler equation; singular regular points

In this section, we will find the solution in the neighborhood of a singular point for a given DE.

* We will consider a simple DE that has a singular point at $x=0$ which is Euler equation.

* Euler equation has the form $\Rightarrow$

$$x^2 y'' + \alpha x y' + \beta y = 0 \quad \text{——} (1)$$

$P(x) = x^2$ $\boxed{= 0 \text{ iff } x = 0}$ singular point.

To find the solution for the DE (1) in the neighborhood of the singular point $x=0$, we assume a solution of the form $\boxed{y = x^r}$ — (2)

for (2) to be solution it must Satisfy (1) :-

$y = \ln r$

$(e^y)^r = (x)^r$

$\boxed{e^{ry} = x^r}$,

$y = x^r$

$y' = r x^{r-1}$

$y'' = r(r-1) x^{r-2}$

Substitute y, y', y'' in the DE (1) to obtain

$$x^2 r(r-1) x^{r-2} + \alpha x r x^{r-1} + \beta x^r = 0$$

$$\underbrace{x^r}_{\text{never zero}} [r^2 - r + \alpha r + \beta] = 0$$

$$r^2 + (\alpha - 1) r + \beta = 0 \quad \text{——} (3)$$

* if r is a root for (3), then x^r is a solution for (1).

So we have (3) cases for (3) :-

[1] if $r_1 \neq r_2 \in \mathbb{R}$, then the general solution of the DE (1) is :-

$$y(x) = c_1 |x^{r_1}| + c_2 |x^{r_2}|$$

[2] if $r_1 = r_2 = r$ then, the general solution of the DE (1) is

$$y(x) = c_1 |x|^r + c_2 |x|^r \ln|x|.$$

[3] if $r_{1,2} = \lambda \mp \mu i$, then the general solution of the DE (1) is :-

$$y(x) = c_1 |x^{\lambda}| \cos(\mu \ln|x|) + c_2 |x^{\lambda}| \sin(\mu \ln|x|).$$

ex. Solve the DE :-

(1) $2x^2 y'' + 3x y' - y = 0$, $x > 0$

(Euler) :-

$$x^2 + \frac{3}{2} x y' - \frac{1}{2} y = 0$$

eq (3) becomes

$$r^2 + \frac{1}{2} r - \frac{1}{2} = 0$$

$$(r+1)(r - \tfrac{1}{2}) = 0$$

$$r_1 = -1 \qquad r_2 = \tfrac{1}{2}$$

The general solution is

$$y(x) = c_1 x^{-1} + c_2 x^{\frac{1}{2}}$$

$$= \frac{c_1}{x} + c_2 \sqrt{x}.$$

(2) $x^2y'' + 5xy' + 4y = 0$, $x>0$
(Euler)

$r^2 + 4r + 4 = 0$
$(r+2)^2 = 0$ $\boxed{r_1 = r_2 = -2}$

The general solution is:
$y(x) = \frac{c_1}{x^2} + \frac{c_2}{x^2}\ln x$, $x>0$

(3) $x^2y'' + xy' + y = 0$, $x>0$
"Euler"

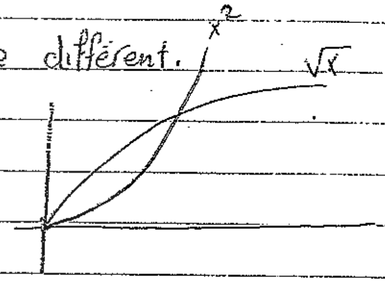
$r^2 + 0 + 1 = 0$
$r_1 = r_2 = \mp i$
$\lambda = 0$, $\mu = 1$
The solution:
$y(x) = c_1 x^0 \cos(1\ln x) + c_2 x^0 \sin(\ln x)$
$c_1 \cos(\ln x) + c_2 \sin(\ln x)$

To study the qualitative behavior for the solution of the Euler equation in the neighbourhood of the singular point $x=0$.
We should note that such a behavior depends totally on the roots r_1 and r_2 in the DE (3)

* without loss of generality, we will consider the case where $x>0$

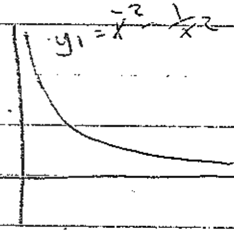
So we have the following cases:-

$\boxed{A}$: r_1 and r_2 are positive different.



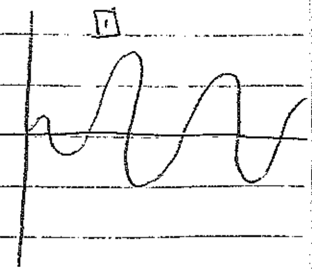
a) as $x \to 0$, $y \to 0$.
the solution

B) r_1 and r_2 are negative

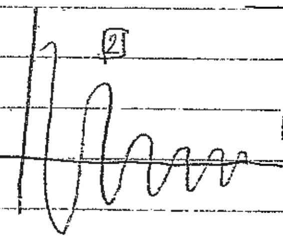


the solution
as $x \to 0$, $y \to \infty$
(~~becomes unbounded~~)

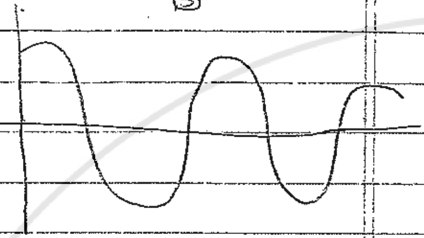
c) $r_{1,2} = \lambda \mp \mu i$
in this case we have 3 subcases:-
$\boxed{1}$ $\lambda > 0$
as $x \to 0$, the solution $y \to 0$



$\boxed{2}$ $\lambda < 0$,
as $x \to 0$
the solution y
becomes unbounded



[3] $\lambda = 0$



as $x \to 0$, the solution y oscillate with constant amplitude.

d) $r_1 = r_2 = r$.

one solution $y_1 = x^r$

second solution $y_2 = x^r \ln x$

we have two subcases:-

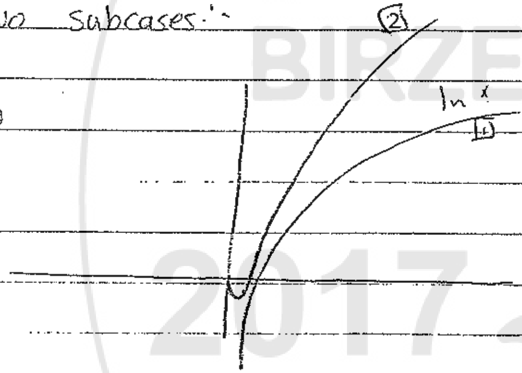
[1] if $r < 0$

$y_2 = \ln x$

as $x \to 0$

the solution

$y \to -\infty$



[2] if $r > 0$

eg $y_2 = x \ln x$

as $x \to 0$, solution $y \to 0$

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0} x = 0$$

5.4 continued

Euler eq. $x^2 y'' + \alpha x y' + \beta y = 0$.

* let $x > 0$. In case where:-

[A] $r_1 \neq r_2 \in \mathbb{R}$, then the general solution of equation is

$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$.

$$W(y_1, y_2)(x) = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1 - 1} & r_2 x^{r_2 - 1} \end{vmatrix} = r_2 x^{r_1 + r_2 - 1} - r_1 x^{r_1 + r_2 - 1}$$

$$= (r_2 - r_1) x^{r_1 + r_2 - 1} \neq 0$$

$\Rightarrow y_1 = x^{r_1}$, $y_2 = x^{r_2}$ are independent.

[B] $r_1 = r_2 = r \Rightarrow$ One solution is $y_1 = x^r$, $y_2 = x^r \ln x$ We will prove tha

to find the second solution y_2 we can use the reduction of order method, or we can use the following method:-

Recall that the Euler equation is

$$L[y] = x^2 y'' + \alpha x y' + \beta y = 0$$

$$L[x^r] = x^r \underbrace{[r^2 + (\alpha - 1) r + \beta]}_{F(r)} = 0$$

$$L[x^r] = x^r F(r)$$

$F(r) = x^2 - 2x + 1 = 0$

$(x-1)(x-1) = 0$

$(x-1)^2$

repeated root

$= x^r (r-r_1)^2$

since $F(r)$ has repeated roots.

$L[x^r] = x^r (r-r_1)^2$

$(x^2)' = 2x$

$(2^x)' = 2^x \ln x$

$x^r = e^{r\ln x}$

$\frac{d}{dr} x^r = \frac{d}{dr} e^{r\ln x}$

$= \ln x \, e^{r\ln x} = \ln x \, e^{\ln x^r} = \ln x \, x^r$

$L\left[\frac{d}{dr} x^r\right] = \frac{d}{dr} x^r (r-r_1)^2$

$L[\ln x \, x^r] = x^r \cdot 2(r-r_1) + (r-r_1)^2 x^r \ln x$

$L[\ln x \, x^{r_1}] = x^{r_1} 2(r_1-r_1) + (r_1-r_1)^2 x^{r_1} \ln x$

$= 0$

So the second solution $y_2 = \ln x \, x^{r_1}$

$W[y_1,y_2](x) = \begin{vmatrix} x^{r_1} & x^{r_1}\ln x \\ r_1 x^{r_1-1} & x^{r_1-1}(r_1\ln x + 1) \end{vmatrix} = x^{2r_1-1} \neq 0$

Hence, y_1, y_2 are independent solutions.

Solution behavior and Singular Point.

* Recall ~~that~~ the DE :-

$P(x)y'' + Q(x)y' + R(x)y = 0$

If x_0 is a singular point "$P(x_0)=0$", then if we try to find a series solution in the ~~neighbou~~ neighbourhood of the singular point x_0, we will find out that it is impossible, and the reason for that underlines in the fact that $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are not analytic at x_0.

* In this case we need more information about the functions $p(x)$ and $q(x)$.

* If we assume that the singularity of the functi[ons] $p(x)$ and $q(x)$ is a "week singularity", (this means that the singularity at x_0 is not too severe.

* In such situation we will have to classify the singular points into :- ⇒

Regular Singular point
or Irregular singular point.

* Consider the DE:-

$P(x)y'' + Q(x)y' + R(x)y = 0$
where x_0 is a singular point.

1. If P, Q, R, are all polynomials then x_0 is a regular singular point.

if $\lim_{x\to x_0} (x-x_0)\frac{Q(x)}{P(x)} < \infty$ finite and (finite)

$\lim_{x\to x_0} (x-x_0)^2 \frac{R(x)}{P(x)} < \infty$ finite. (finite)

2. If P, Q, R are more general functions than Polynomials then x_0 is a regular singular point if:-

$(x-x_0)\frac{Q(x)}{P(x)}$ and $(x-x_0)^2 \frac{R(x)}{P(x)}$

are analytic at x_0

Remark:- If x_0 is not regular singular point then it is ~~no~~ irregular singular point

Ex. Find the singular points for the following DE's and classify them to regular or irregular singular points.

1. $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$
[Bessel Eq].

$\Rightarrow$ Singular point is $x=0$
$\Rightarrow P(x) = x^2, \quad Q(x) = x \quad R(x) = x^2 - \alpha^2$ (All Are Polynomials

① $\lim_{x\to 0} (x-0)\frac{x}{x^2} = 1 < \infty$

② $\lim_{x\to 0} (x-x_0)^2 \frac{x^2-\alpha^2}{x^2} = \lim_{x\to 0} x^2 - \alpha^2 = -\alpha^2 < \infty$

$x_0 = 0$ is regular singular point

[2] $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$
(Legendre Eq.)

$\Rightarrow$ Singular points $x = \mp 1$.
$\Rightarrow$ $P(x) = 1-x^2$, $Q(x) = -2x$, $R(x) = \alpha(\alpha+1)$
All polynomials.

① $x_1 = 1$, $\lim_{x\to 1} (x-1) \frac{-2x}{1-x^2}$ $- \cancel{\lim \frac{2x^2+2x}{}}$

$\lim_{x\to 1} \frac{\overset{-1}{\cancel{(x-1)}}\,-2x}{\cancel{(1-x)}(1+x)} = \lim_{x\to 1} \frac{2x}{1+x} = 1$

☆ $\lim_{x\to 1} (x-1)^2 \frac{\alpha(\alpha+1)}{1-x^2} = \lim_{x\to 1} \frac{-\alpha(x-1)(\alpha+1)}{x+1} = 0$

$\Rightarrow$ $x_1 = 1$ is aregular singular point

$x_2 = -1$ Similarly
$x_2 = -1$ is aregular point.

[3] $2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$
Singular points. $x = 0, 2$
$P(x) = 2x(x-2)^2$; $Q(x) = 3x$
$R(x) = x-2$ All polynomials

$x_1 = 0$.
$\lim_{x\to 0} x \cdot \frac{3x}{2x(x-2)^2} = 0$

$\lim_{x\to 0} x^2 \frac{x-2}{2x(x-2)^2} = 0$

$\Rightarrow$ $x_1 = 0$ is aregular singular point.

$x_2 = 2$.
$\lim_{x\to 2} \cancel{(x-2)} \frac{3x}{2x(x-2)^{\cancel{2}}} = \lim_{x\to 2} \frac{3}{2(x-2)} = DNE$

$\lim_{x\to 2^+}$ تطلع ∞ شو نقطا
$\lim_{x\to 2^-}$ تطلع $-\infty$

$x = 2$ is irregular singular point.

[4] $(x - \frac{\pi}{2})y'' + \cos x\, y' + \sin x\, y = 0$

Singular point :- $x = \frac{\pi}{2}$

$P(x) = (x - \frac{\pi}{2})^2$, $Q(x) = \cos x$ $R(x) = \sin x$
Not All polynomials.

Not all polynomials:-

(2) $\left(x-\frac{\pi}{2}\right)\dfrac{\cos x}{\left(x-\frac{\pi}{2}\right)^2} = \dfrac{\cos x}{x-\frac{\pi}{2}}$

$\left(x-\frac{\pi}{2}\right)^2 \dfrac{\sin x}{\left(x-\frac{\pi}{2}\right)^2} = \sin x$

we need to show that these functions are analytic at $x=\frac{\pi}{2}$.

i.e we need to show that these functions have taylor series about $x_0=\frac{\pi}{2}$

* Taylor series expansion for $\cos x$ about $x=\frac{\pi}{2}$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}$$

لأنها حول $\frac{\pi}{2}$, مش صفر

~~shifting indices~~

$$\frac{\cos x}{x-\frac{\pi}{2}} = -1 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\left(x-\frac{\pi}{2}\right)^{2n}}{(2n+1)!}$$

Which converges $\forall$ all x

$\Rightarrow$ it is analytic at $x=\frac{\pi}{2}$

$\sin x$ has taylor series expansion for every

$$\sin x = \sum_{0}^{\infty} \frac{(-1)^{2n+1} x^{2n+1}}{(2n+1)!}$$

which converges for all x.

Hence it is analytic at $x=\frac{\pi}{2}$

$\Rightarrow$ Hence $x=\frac{\pi}{2}$ is regular singular point

5.5 Series Solution Near Regular Singular Point.

Consider the DE:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (1)$$

Assume $x_0=0$ is a regular singular point for the DE (1).

Now, we will find a series solution for the DE (1) near the regular singular point $x_0=0$.

* Since $x=0$ is a regular singular point,

$\Rightarrow \lim_{x\to x_0}(x-x_0)\dfrac{Q(x)}{P(x)}$ and $\lim_{x\to x_0}(x-x_0)^2\dfrac{R(x)}{P(x)}$ are fini

II

$\Rightarrow \lim_{x \to 0} x\frac{Q(x)}{P(x)}$, $\lim_{x \to x_0} x^2\frac{R(x)}{P(x)}$

Are finite

And

(2) $xp(x)$ and $x^2q(x)$ are analytic

$\Rightarrow$ they have taylor series expansion about $x_0 = 0$

~~$xp(x) = \sum_{n=0}^{\infty} a_n p_n$~~

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n = p_0 + p_1 x + p_2 x^2 + \cdots$$

$$x^2 q(x) = \sum_{n=0}^{\infty} q_n x^n = q_0 + q_1 x + q_2 x^2 + \cdots \quad (A)$$

where radius of convergence ρ and interval of convergence $|x - x_0| < \rho$

$|x| < \rho$ $\qquad -\rho < x < \rho$

converge absolutely

$-\rho \quad 0 \quad \rho$

Now devide the DE (1) by $P(x)$ and multiply by x^2

$$x^2 y'' + x^2 p(x) y' + x^2 q(x) y = 0$$

$$x^2 y'' + x(xp(x)) y' + x^2 q(x) y = 0$$

Substitute (A) in the last equation to obtain:-

$$x^2 y'' + x\left(\sum_{n=0}^{\infty} p_n x^n\right) y' + \left(\sum_{n=0}^{\infty} q_n x^n\right) y = 0 \quad \text{--- (B)}$$

$$x^2 y'' + x[p_0 + p_1 x + p_2 x^2 + \ldots] y' + [q_0 + q_1 x + q_2 x^2 \ldots] y = 0.$$

To arrive Euler equation and simplify computation we set

$$p_1 = p_2 = \cdots = q_1 = q_2 = \cdots = 0$$

Hence:-

$$\boxed{x^2 y'' + p_0 x y' + q_0 y = 0} \quad (C) \quad \text{--- Euler equation}$$

Note that:-

(1) $p_0 = \lim_{x \to 0} x \frac{Q(x)}{P(x)}$ and

$$q_0 = \lim_{x \to 0} x^2 \frac{R(x)}{P(x)}$$

[2] the essential character of the series solution of the DE (C) is identical to the series solution for the DE (B)

~~Summary~~

[3] the general solution is then given by : about a regular singular point $x_0=0$

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$$

i.e:- $$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

* In this case we just need to find:

a) r:- is the exponent of the first term which is the solution of indicial equation.

b) a_n :- the recurance relation.

c) the radius of convergence for the series solution

Ex. consider the DE:-

$$2x^2y'' - xy' + (1+x)y = 0$$

find a series solution at $x_0=0$.

Note that $x_0=0$ is a singular point.

* we need to show that $x_0=0$ is a regular singular point.

$\Rightarrow P(x) = 2x^2, \quad Q(x) = -x, \quad R(x) = 1+x$

$$\lim_{x\to 0} x \cdot \frac{-x}{2x^2} = \frac{-1}{2} \quad \text{(finite) and}$$

$$\lim_{x\to 0} x^2 \cdot \frac{(1+x)}{2x^2} = \frac{1}{2} \quad \text{(finite)}$$

$\therefore x_0 = 0$ is a regular singular point.

* The series solution about the regular singular point $x_0=0$ $\quad y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$

* to find the indicial equation:

Two methods:-

[1] Euler:- $x^2y'' + P_0 xy' + q_0 y = 0$

$$P_0 = \lim_{x\to 0} x \frac{Q(x)}{P(x)} = \frac{-1}{2}$$

$$q_0 = \lim_{x\to 0} \frac{x^2 R(x)}{P(x)} = \frac{1}{2}$$

$\Rightarrow$ Euler becomes:-

$$x^2y'' - \frac{1}{2}xy' + \frac{1}{2}y = 0$$

$$\boxed{r^2 - \frac{3}{2}r + \frac{1}{2} = 0}$$ the Indicial equation

$$2r^2 - 3r + 1 = 0$$
$$(2r-1)(r-1) = 0$$

$\boxed{r_1 = 1}$ largest root $\quad \boxed{r_2 = \frac{1}{2}}$ smallest root $\rightarrow$ exponents of singularity

(2) the second method appears when we try to find the series solution and it will be coefficient of a_0

* we will find the recurance relation:-

~~$y(x) = \sum_{n=0}^{\infty}(n+1)a_n x^{n+r-1}$~~

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r} = a_0x^r + a_1x^{r+1} + a_2x^{r+2} + \cdots$$

$$y'(x) = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y''(x) = \sum_0^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

Now Substitute y, y', y'' in the DE above

$$2x^2y'' - xy' + (1+x)y = 0$$

$$\Rightarrow 2\sum_0^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_0^{\infty}(n+r)a_n x^{n+r} + \sum_0^{\infty}a_n x^{n+r} + \sum_0^{\infty}a_n x^{n+r+1} = 0$$

$$+\sum a_{n-1}x^{n+r} = 0$$

Indicial equation

$$a_0[2r(r-1) - r + 1]x^r + \sum_{n=1}^{\infty}\left[2a_n(r+n)(r+n-1) - a_n(r+n) + a_n + a_{n-1}\right]x^{n+r} = 0$$

a_0 never zero.

$a_0 \neq 0$ since

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} = x^r\sum_{n=0}^{\infty}a_n x^n$$

$$= x^r[a_0 + a_1x + a_2x^2 + \cdots]$$

Hence :- $a_0 \neq 0$.

first:- the coefficient of x^r

$$2r(r-1) - r + 1 = 0 \quad \text{since } a_0 \neq 0$$

$2r^2 - 3r + 1 = 0$ Indicial equation.

$r_1 = 1$, $r_2 = \frac{1}{2}$

Second

the coefficient of x^{n+r}

$$2a_n(n+r)(r+n-1) - a_n(n+r) + a_n + a_{n-1} = 0$$

$, n = 1, 2, 3, \dots$

$$a_n = \frac{-a_{n-1}}{2(n+r)(n+r-1) - (n+r) + 1} , n = 1, 2, 3, \dots$$

$$\boxed{= \frac{-a_{n-1}}{(2(n+r)-1)((n+r)-1)} \quad n = 1, 2, 3 \dots}$$

Recurance relation *

* Find the series solution $(x > 0)$ corresponding to the largest root:-

Substitute $r_1 = 1$ in the recurance relation $* \Rightarrow$

$$a_n = \frac{-a_{n-1}}{(2n+1)n} , n = 1, 2, 3, \dots$$

$n = 1$, $\Rightarrow a_1 = \frac{-a_0}{3(1)}$

$n = 2$, $a_2 = \frac{-a_1}{5(2)} = \frac{a_0}{5(3)(2)(1)}$

$n = 3$ $a_3 = \frac{-a_2}{(7)(3)} = \frac{-a_0}{3(5)(7)(3)(2)(1)}$

$$a_n = \frac{(-1)^n a_0}{(3)(5)(7)\cdots(2n+1)n!}$$

for $x > 0$, one ~~first~~ solution to our DE is

$$y_1(x) = \sum_{0}^{\infty} a_n x^{n+1} = a_0 x + \sum_{1} \frac{(-1)^n a_0 x^{n+1}}{(3)(5)(7)\cdots(2n+1)n!}$$

$$\underbrace{y_1(x) = a_0 x \left[1 + \sum \frac{(-1)^n x^n}{3(5)(7)\cdots(2n+1)n!} \right]}_{y_1(x)}$$

Now to determine the radius of convergence, we just apply R.T

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x|}{(2n+3)(n+1)} = 0 < 1$$

So the radius of convergence is infinite.

$\Rightarrow y_1(x)$ converges absolutely for all x.

* find the series solution corresponding to the smallest root.

$r_2 = \frac{1}{2}$ Substitute in recurance relation

$$a_n = \frac{-a_{n-1}}{n(2n-1)} \quad , n = 1, 2, 3 \dots$$

When $n=1 \Rightarrow a_1 = \frac{-a_0}{(1)(1)}$

$n=2 \Rightarrow a_2 = \frac{-a_1}{2(3)}$

$n=3 \Rightarrow a_3 = \frac{-a_0}{(1)(2)(3)\,(1)(3)(5)}$

$\vdots$

$$a_n = \frac{(-1)^n a_0}{(1)(3)(5)\dots(2n-1)\,n!} \quad , n = 1, 2, 3 \dots$$

$$y_2(x) = \sum_0^\infty a_n x^{n+\frac{1}{2}}$$

$$= a_0 x^{\frac{1}{2}} \boxed{1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(1)(3)(5)\dots(2n-1)\,n!}}$$

the general solution is:-

$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad , x > 0$

Note that x and $x^{\frac{1}{2}}$ are indepenedent. it follows that y_1 and y_2 are independent solutions.

$\Rightarrow$ They are fundamental.

Ch 6

6.1 The Definition of Laplace Transform

⇒ Improper Integral

$$\int_0^\infty f(t)dt = \lim_{b\to\infty} \int_0^b f(t)dt$$

The Improper integral converges if:-

1) $\int_0^b f(t)dt$ exists and

2) $\lim_{b\to\infty} \int_0^b f(t)dt$ exist

If we assume $f(t)$ is a picewise continuous function on $[a,b]$ $\forall$ $b>0$, then

Now Condition 2 can be acheived using Some theorems like direct comparison Tests:-

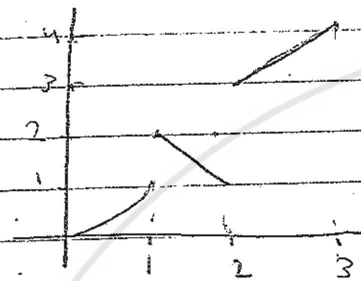
$0 \le f(t) \le g(t)$ $\forall$ $t \in [0,\infty]$

* If $\int_0^\infty g(t)dt$ converges, then $\int_0^\infty f(t)dt$ converges too.

* If $\int_0^\infty f(t)dt$ diverges, then $\int_0^\infty g(t)dt$ diverges too

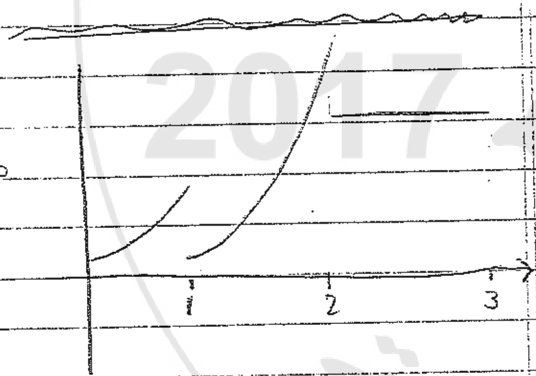
Ex. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 3-t & 1 \le t \le 2 \\ t+1 & 2 \le t \le 3. \end{cases}$ Is f peicewise continuous is Piecewise Continuous

(lim does not go to ∞)



Ex. $f(t) = \begin{cases} t^2+1 , & 0 \le t \le 1 \\ \frac{1}{2-t} , & 1 < t \le 2 \\ 4 & 2 < t \le 3 \end{cases}$

is not piecewise cont. since $\lim\limits_{t \to 2^-} f(t) = \infty$



Defn The Laplace transform of ~~a piece wise~~ a function $f(t)$ is $F(S)$ defined by

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Where e^{-st} is the kernal of the Laplace transform since it is assumed previously to be a solution for a 2^{nd} order Linear D.E with constant coefficients.

$f(t) \longrightarrow \boxed{L[f(t)]} \longrightarrow F(s)$

inputs — Machine — Output

Ex. find Laplace transform of ?

1) $f(t) = C$, C is constant.

$$L\{f(t)\} = L\{C\} = F(s) = \int_0^\infty e^{-st} C\, dt$$

$$C \lim_{b\to\infty} \int_0^b e^{-st} dt = C \lim_{b\to\infty} \left. \frac{-e^{-st}}{s} \right]_0^b$$

$$C \lim_{b\to\infty} \left[\frac{-e^{-sb}}{s} + \frac{1}{s} \right] = \frac{c}{s} \quad , s > 0$$

Hence, $L\{C\} = \frac{c}{s}$, $s > 0$

Ex. $L\{\sqrt{2}\} = \frac{\sqrt{2}}{s}$

$L\{\pi\} = \frac{\pi}{s}$

$L\{-3\} = \frac{-3}{s}$

$L^{-1}\left[\frac{c}{s}\right] = c = f(t)$

$L^{-1}\left[\frac{5}{s}\right] = 5$, $L^{-1}\left[\frac{1}{s}\right] = 1$

(2) $f(t) = t$

$L\{f(t)\} = L\{t\} = F(s) = \int_0^\infty e^{-st} \cdot t \, dt$

$= \lim_{b\to\infty} \int_0^b t e^{-st} dt$

t	e^{-st}
1	$\frac{-e^{-st}}{s}$
0	$\frac{e^{-st}}{s^2}$

$\lim_{b\to\infty} \left. \frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^b$

$\lim_{b\to\infty} \frac{-be^{-bs}}{s} - \frac{e^{-bs}}{s^2} + \frac{1}{s^2}$

$= \frac{1}{s^2}$

Hence, $L\{t\} = \frac{1}{s^2}$, $s > 0$

find $L\{t^2\} = \frac{2!}{s^3}$

$L\{t^3\} = \frac{3!}{s^4}$

$L\{t^n\} = \frac{n!}{s^{n+1}}$

Ex. find $L\{2t\}$

$= 2L\{t\} = 2\left(\frac{1}{s^2}\right) = \frac{2}{s^2}$

$L^{-1}\left(\frac{6}{s^4}\right) = L^{-1}\left(\frac{3!}{s^4}\right) = t^3$

$L^{-1}\left(\frac{6}{s^3}\right) = L^{-1}\left(\frac{3 \cdot 2!}{s^3}\right) = 3L^{-1}\left(\frac{2!}{s^3}\right) = 3t^2$

$L^{-1}\left(\frac{1}{s^5}\right) = L^{-1}\left[\frac{4!}{4!\, s^5}\right] = \frac{1}{4!} L^{-1}\left[\frac{4!}{s^5}\right]$

$\frac{1}{4!} t^4 = \frac{t^4}{4!}$

$* \quad L\left\{\frac{t^4}{20}\right\} = \frac{1}{20} L\{t^4\}$

$\frac{1}{20} \cdot \frac{4!}{s^5} = \frac{6}{5s^5}$

(3) $f(t) = e^{at} = L\{f(t)\}$

$= L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt$

$\lim_{b\to\infty} \int_0^b e^{-(s-a)t} dt = \frac{1}{s-a}, \quad s > a$

Hence, $L\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

Ex. $L\{e^{2t}\} = \frac{1}{s-2}$

$L\{e^{\pi t}\} = \frac{1}{s-\pi}$

$L\{e^{-\sqrt{3}t}\} = \frac{1}{s+\sqrt{3}}$

$L^{-1}\left(\frac{5}{s-1}\right) = 5e^{t}$

$L^{-1}\left(\frac{-2}{s+3}\right) = -2e^{-3t}$

4) $f(t) = \sin at, \quad t \geq 0$

$L\{\sin at\} = F(s) = \int_0^\infty e^{-st} \sin at \, dt$

$\lim_{b\to\infty} \int_0^b e^{-st} \sin at \, dt$

$e^{-st} \quad \sin at$

$-se^{-st} \quad \frac{-1}{a}\cos at$

$s^2 e^{-st} \rightarrow \frac{-1}{a^2}\sin at$

نتنقف لما يصير زي الأصل

$\lim_{b\to\infty} \left(\frac{-1}{a} e^{-st}\cos at - \frac{s}{a^2} e^{-st}\sin at + \int_0^b \frac{-s^2}{a^2} e^{-st}\sin at \, dt\right)$

$F(s) = \lim_{b\to\infty} \left[\frac{-1}{a} e^{-sb}\cos ab - \frac{s}{a^2} e^{-sb}\sin ab - \left(\frac{-1}{a} - 0\right)\right] - \frac{s^2}{a^2} F(s)$

$\left[1 + \frac{s^2}{a^2}\right] F(s) = \frac{a}{\cancel{s^2+a^2}} \quad \frac{1}{a}$

$F(s) = \frac{a}{s^2+a^2}$

Similarly $L\{\cos at\} = \frac{s}{s^2+a^2}$

Ex. $L\{\sin 2t\} = \frac{2}{s^2+4}$

$L\{\sin 3t\} = \frac{3}{s^2+9}$

$L^{-1}\left(\frac{5}{s^2+1}\right) = 5 L^{-1}\left(\frac{1}{s^2+1}\right) = 5\sin t$

$L^{-1}\left(\frac{1}{s^2+3}\right) = \frac{1}{\sqrt{3}} L\left(\frac{\sqrt{3}}{s^2+3}\right) = \frac{1}{\sqrt{3}}\sin\sqrt{3}t$

$L\{\cos 2t\} = \frac{s}{s^2+4}$

$L^{-1}\left(\frac{2s}{s^2+1}\right) = 2L^{-1}\left(\frac{s}{s^2+1}\right) = 2\cos t$

$L^{-1}\left(\frac{2s}{s^2+4}\right) = 2\cos 2t$

Result. $L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 F_1(s) + c_2 F_2(s)$

Proof: $L\{c_1 f_1(t) + c_2 f_2(t)\} = \cancel{c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}}$

$= \int e^{-st}(c_1 f_1(t) + c_2 f_2(t))\,dt$

$= c_1 \int e^{-st} f_1(t)\,dt + c_2 \int e^{-st} f_2(t)\,dt$

$c_1 L[f_1(t)\} + c_2 L\{f_2(t)\}$

$c_1 F_1(s) + c_2 F_2(s)$

find laplace transform for

$f(t) = 5e^{-2t} - 3\sin 4t + 5$

$L\{f(t)\} = 5L[e^{-2t}] - 3L[\sin 4t] + L[5]$

$5\left(\frac{1}{s+2}\right) - 3\left(\frac{4}{s^2+16}\right) + \frac{5}{s}$

Ex. $L^{-1}\left(\frac{5}{s+2}\right) = 5L^{-1}\left(\frac{1}{s+2}\right) = 5e^{-2t}$

Ex. $L^{-1}\left(\frac{12}{s^2+16}\right) = 3L^{-1}\left[\frac{4}{s^2+16}\right] = 3\sin 4t$

Ex. (A) find $L\{\sinh at\}$

$L\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \frac{1}{2} L\{e^{at} - e^{-at}\}$

$\frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$

$\frac{1}{2}\left\{\frac{s+a-s+a}{s^2-a^2}\right\} = \frac{1}{2}\cdot\frac{2}{s^2-a^2}$

$= \frac{a}{s^2-a^2}$

$L\{\sinh 3t\} = \frac{3}{s^2-9}$

$L^{-1}\left(\frac{5}{s^2-7}\right) = \frac{5}{\sqrt{7}}\, L\left(\frac{\sqrt{7}}{s^2-7}\right)$

$= \frac{5}{\sqrt{7}}\sinh\sqrt{7}\,t$

(B) $L\{\cosh at\} = \frac{s}{s^2-a^2}$

Proof $L[\cosh at] = L\left[\frac{e^{at}+e^{-at}}{2}\right] = \frac{s}{s^2-a^2}$

Ex. $L\{\cosh\sqrt{2}\,t\} = \frac{s}{s^2-2}$

(2) $L^{-1}\left(\frac{2s-1}{s^2-11}\right) = L^{-1}\left(\frac{2s}{s^2-11}\right) - L^{-1}\left(\frac{1}{s^2-11}\right)$

$2\cosh\sqrt{11}\,t - \frac{1}{\sqrt{11}}\sinh\sqrt{11}\,t$

Ex. let $h(t) = 2\sin 3t - 10t^2 + 5e^{-3t} + 2 + \cosh t$

Find $H(s)$

$H(s) = 2L[\sin 3t] - 10\,L[t^2] + 5\,L\{e^{-3t}\} + L\{2\} + L\{\cosh t\}$

$2\cdot\frac{3}{s^2+9} - 10\cdot\frac{2!}{s^3} + \frac{5}{s+3} + \frac{2}{s} + \frac{s}{s^2-1}$

See the table P. 317

$f(t)$ $F(s)$

6.2

Ex. find the following

① $L^{-1}\left(\frac{10}{s-5}\right) = 10\, L^{-1}\left(\frac{1}{s-5}\right) = 10e^{5t}$

$L^{-1}(F(s)) = f(t)$

② $L^{-1}\left(\frac{3}{s}\right) = 3$

③ $L^{-1}\left(\frac{3}{s^2}\right) = 3\, L^{-1}\left(\frac{1}{s^2}\right) = 3t$

④ $L^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{3}\, L^{-1}\left(\frac{3}{s^2+9}\right) = \frac{1}{3}\sin 3t$

⑤ $L^{-1}\left(\frac{3}{s^3}\right) = \frac{3}{2}\, L^{-1}\left(\frac{2}{s^3}\right) = \frac{3}{2}t^2$

⑥ $L^{-1}\left(\frac{6}{s^4}\right) = t^3$

⑦ $L^{-1}\left(\frac{8}{s^3}\right) = \frac{8}{2}\, L^{-1}\left(\frac{2!}{s^3}\right) = 4t^2$

8) $L^{-1}\left(\frac{1}{s^2-5s+6}\right) = L^{-1}\left(\frac{1}{(s-3)(s-2)}\right)$

$\frac{1}{(s-3)(s-2)} = \frac{A}{(s-3)} + \frac{B}{(s-2)}$

$\boxed{A = 1}$

$\boxed{B = -1}$

$L^{-1}\left(\frac{1}{s-3} - \frac{1}{s-2}\right) = L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right)$

$e^{3t} - e^{2t}$

⑨ $L^{-1}\left(\frac{4s+1}{s^2+9}\right) = L^{-1}\left(\frac{4s}{s^2+9}\right) + L^{-1}\left(\frac{1}{s^2+9}\right)$

$4\cos 3t + \frac{1}{3}\sin 3t$

10) $L^{-1}\left(\frac{4s+1}{s^2-9}\right) = 4\cosh 3t + \frac{1}{3}\sinh 3t$

Ex. consider the piecewise continuous function

$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ k, & t = 1 \\ 0, & t > 1 \end{cases}$ find $F(s)$

$$F(s) = L\{f(t)\}$$

$$= \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}\,dt + \int_1 e^{-st} k\,dt + \int^\infty 0\, e^{-st}\,dt$$

$$-\frac{1}{s} e^{-st}\Big]_0^1 = \frac{1-e^{-s}}{s}$$

Thm 6.2.1. If f is continuous and f' is apiecewise continuous on $[a,b]$ - -

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt = \lim_{b\to\infty} \int_0^b e^{-st} f'(t)\,dt$$

Since f' is apiece wise continuous on $[a,b]$ then f' has $t_1, t_2 \dots t_n$ points of discontinuity on $[a,b]$

Hence $L[f'(t)] \lim_{b\to\infty} \int_0^{t_1} e^{-st} f'(t)\,dt + \int_{t_1}^{2} e^{-st} f'(t)\,dt + \dots \int_{t_n}^{b} e^{-st} f'(t)\,dt$

Using Integral by Parts

$u = e^{-st}$ $\quad dv = f'(t)\,dt$

$du = -se^{-st}$ $\quad v = f(t)$

$$L\{f'(t)\} = \lim_{b\to\infty}\Big[e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \dots + e^{-st} f(t)\Big|_{t_n}^{b}$$

$$+ s\Big(\int_0^{t_1} e^{-st} f(t)\,dt + \int_{t_1}^{t_2} e^{-st} f(t)\,dt + \dots \int_{t_n}^{b} e^{-st} f(t)\,dt.$$

$$\lim_{b\to\infty}\Big[-f(0) + \underbrace{e^{-sb} f(b)}_{\text{go to } 0} + s\int_0^b e^{-st} f(t)\,dt\Big]$$

$$= L[f'(t)] = sL\{f(t)\} - f(0).$$

Similarly

$$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$$

Proof:- $L\{f''(t)\} = sL\{f'(t)\} - f'(0)$

$$s\Big[sL\{f(t)\} - f(0)\Big] - f'(0)$$

$$s^2 L\{f(t) - sf(0) - f'(0)\}$$

In general:

$$* \quad L\{f^{(n)}(t)\} = S^n L\{f(t)\} - S^{n-1}f(0) - S^{n-2}f'(0) - \dots - S f^{(n-2)}(0) - f^{(n-1)}(0).$$

$$L(y''') = S^3 L\{y\} - S^2 y(0) - S y'(0) - y''(0).$$

Note:-

We will now use laplace ~~transformation~~ to find the solution for linear IVP's.

* We start with the homogeneous ones.

Ex. Use laplace transform to solve the IVP

$$y'' - y' - 2y = 0$$

$y(0) = 1$, $y'(0) = 0$

Take laplace for both sides

$$L\{y''\} - L\{y'\} - L\{2y\} = L\{0\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) - [s L[y] - y(0)] - 2L[y] = 0$$

$$L[y]\left[S^2 - S - 2\right] - S + 1 = 0$$

$$L[y] = \frac{S-1}{S^2 - S - 2}$$

Charact eq

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r_1 = 2 \quad r_2 = -1$

$y(t) = c_1 e^{2t} + c_2 e^{-t}$

Using IC

$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$

$$y(t) = L^{-1}\left(\frac{S-1}{S^2 - S - 2}\right)$$

$$= L^{-1}\left(\frac{S-1}{(S-2)(S+1)}\right)$$

$$\frac{S-1}{(S-2)(S+1)} = \frac{A}{S-2} + \frac{B}{S+1}$$

$$A = \frac{2-1}{2+1} = \frac{1}{3}, \qquad B = \frac{-1-1}{-1-2} = \frac{2}{3}$$

$$L^{-1}\left(\frac{\frac{1}{3}}{S-2} + \frac{\frac{2}{3}}{S+1}\right) = \frac{1}{3}L^{-1}\left\{\frac{1}{S-2}\right\} + \frac{2}{3}L^{-1}\left(\frac{1}{S+1}\right)$$

$$\frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Ex. Use laplace transform to solve the IVP

$y'' + y = \sin 2t \qquad y(0) = 2, \quad y'(0) = 1$

Take laplace

$$L[y'' + y] = L\{\sin 2t\}$$

$$s^2 L\{y\} - s\,y(0) - y'(0) + L\{y\} = \frac{2}{s^2 + 4}$$

$$L\{y\}(s^2+1) - 2s - 1 = \frac{2}{s^2+4}$$

$$L\{y\} = \frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}$$

$$y = L^{-1}\left[\frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)}\right]$$

$$\frac{2s^3+s^2+8s+6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$2s^3+s^2+8s+6 = (a+c)s^3+(b+d)s^2+(4a+c)s+(4b+d)$$

$a+c=2$ $\quad b+d=1$

$4a+c=8$ $\quad 4b+d=6$

$a=2$, $b=\frac{5}{3}$, $c=0$, $d=-2/3$

$$\Rightarrow L^{-1}\left[\frac{2s+5/3}{s^2+1}\right] + L^{-1}\left[\frac{-2/3}{s^2+4}\right]$$

$$\underbrace{2\cos t + \frac{5}{3}\sin t}_{y_h} - \underbrace{\frac{1}{3}\sin 2t}_{y_p}$$

Ex. use laplace transform to solve the IVP

$y^{(4)} - y = 0$, $\quad y(0)=0$, $\quad y'(0)=1$ $\quad y''(0)=0$
$\quad y'''(0)=0$

$L[y^{(4)} - y] = 0$

$s^4 L[y] - s^3y(0) - s^2y'(0) - s\,y''(0) - y'''(0) - L[y] = 0$

$L[y]\,[s^4-1] - s^2 = 0$

$$L[y] = \frac{s^2}{s^4-1}$$

$$y(t) = L^{-1}\left[\frac{s^2}{s^4-1}\right] = L^{-1}\left[\frac{s^2}{(s^2-1)(s^2+1)}\right]$$

$$\frac{s^2}{(s^2-1)(s^2+1)} = \frac{As+B}{s^2-1} + \frac{Cs+d}{s^2+1}$$

$$s^2 = (a+c)s^3 + (b+d)s^2 + (a-c)s + b - d$$

$a=0$, $b=\frac{1}{2}$ $\quad c=0$, $d=\frac{1}{2}$

$$L^{-1}\left(\frac{\frac{1}{2}}{s^2-1}\right) + L^{-1}\left(\frac{\frac{1}{2}}{s^2+1}\right) = \frac{1}{2}\sinh t + \frac{1}{2}\sin t.$$

1^{st} shifting

$$\left.\begin{array}{l} L\{e^{at}f(t)\} = F(s-a) \\ e^{at}f(t) = L^{-1}[F(s-a)] \end{array}\right\} \ast$$

Ex. $L\{e^{3t}\cos 2t\}$

$F(s) = L\{\cos 2t\}$

$\frac{s}{s^2+4}$

$\Rightarrow F(s-3) = \frac{(s-3)}{(s-3)^2+4}$

Hence

$L^{-1}\left(\frac{s-2}{(s-2)^2+9}\right) = e^{2t}\cos 3t$

② Proof ($\ast$)

$L\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt$

$= \int_0^\infty e^{-(s-a)}f(t)\,dt$

$= F(s-a)$

2) $L\{e^{t}t^2\}$

$L\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$

$F(s-1) = \frac{2}{(s-1)^3}$

3) $L^{-1}\left(\frac{1}{(s-1)^2+1}\right) = e^{t}\sin t$

4) $L^{-1}\left[\frac{(s+3)}{(s+3)^2+5}\right]$ $\cos\sqrt{5}t\; e^{-3t}$

5) $L^{-1}\left[\frac{s+2-2}{(s-2)^2+9}\right] = L^{-1}\left[\frac{(s-2)}{(s-2)^2+9} + \frac{2}{(s-2)^2+9}\right]$

$e^{2t}\cos 3t + \frac{2}{3}\sin 3t\, e^{2t}$

5) $L^{-1}\left(\frac{2s-1}{s^2+2s+5}\right) = L^{-1}\left(\frac{2s-1}{(s+1)^2+4}\right)$

$= 2L^{-1}\left(\frac{s}{(s+1)^2+4}\right) - L^{-1}\left(\frac{1}{(s+1)^2+4}\right)$

$$= 2\,L^{-1}\left(\frac{(s+1)-1}{(s+1)^2+4}\right) - \frac{1}{2}\,L^{-1}\left(\frac{2}{(s+1)^2+4}\right)$$

$$2e^{-t}\cos 2t - e^{-t}\sin 2t - \frac{1}{2}e^{-t}\sin 2t$$

$$2e^{-t}\cos 2t - \frac{3}{2}\sin 2t\, e^{-t}.$$

Recall first shifting

$$L\{e^{at}f(t)\} = F(s-a)$$

$$e^{at}f(t) = L^{-1}(F(s-a))$$

ex. Find $L^{-1}\left(\frac{-10}{(s+1)^3}\right) = \frac{-10}{2!}t^2e^{-t} = -5t^2e^{-t}.$

Ex. Find the inverse transform of

$$G(s) = \frac{s+1}{s^2+2s+5}$$

$$L^{-1}(G(s)) = L^{-1}\left(\frac{s+1}{s^2+2s+5+1-1}\right) = L^{-1}\left(\frac{(s+1)}{(s+1)^2+4}\right) = \cos 2t\, e^{-t}$$

② Find $L\{e^{2t}\sin\sqrt{5}t\}$

$$F(s) = L\{\sin\sqrt{5}t\} = \frac{\sqrt{5}}{s^2+5}$$

$F(s-2)$

$$= \frac{\sqrt{5}}{(s-2)^2+5}$$

Ex. Find the inverse transform of

$$H(s) = \frac{4s-10}{s^2-6s+10}$$

$$h(s) = L^{-1}(H(s)) = L^{-1}\left(\frac{4s-10}{s^2-6s+10-9+9}\right) = L^{-1}\left(\frac{4s-10}{(s-3)^2+1}\right)$$

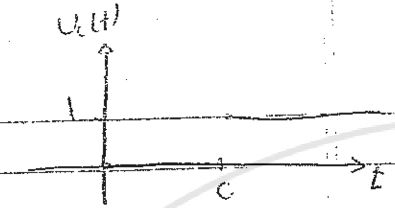
$$= L^{-1}\left(\frac{4(s-3)+2}{(s-3)^2+1}\right) = 4\,L^{-1}\frac{(s-3)}{(s-3)^2+1} + 2\,L^{-1}\left(\frac{1}{(s-3)^2+1}\right)$$

$$4e^{3t}\cos t + 2e^{3t}\sin t$$

6.3 Step functions:

Defn: The unit step function (the Heaviside function) is defined by:

$$U_c(t) = \begin{cases} 0 & 0 \le t < c \\ 1 & c \le t \end{cases}$$

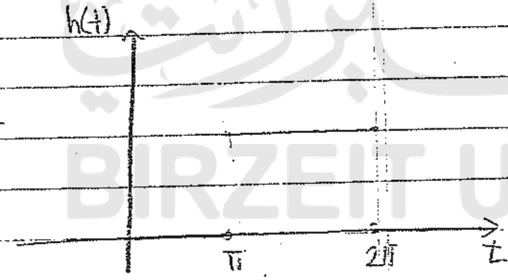


Sketch the graph of

$h(t) = u_{\pi}(t) - u_{2\pi}(t) \quad , t \geq 0$

$u_{\pi} = \begin{cases} 0 & , 0 \leq t < \pi \\ 1 & , t \geq \pi \end{cases} \quad ; \quad u_{2\pi} = \begin{cases} 0 & , 0 \leq t \leq 2\pi \\ 1 & , t \geq 2\pi \end{cases}$

$h(t) = \begin{cases} 0 & , 0 \leq t < \pi \\ 1 & , \pi \leq t \leq 2\pi \\ 0 & , 2\pi \leq t \end{cases}$



Express the following function in terms of $u_c(t)$

$f(t) = \begin{cases} 2 & , 0 \leq t < 1 \\ -1 & , 1 \leq t < 2 \\ 2 & , 2 \leq t \leq 3 \\ -1 & , 3 \leq t < 4 \\ 0 & , 4 \leq t \end{cases}$

$f(t) = 2 - 3\,u_1(t) + 3\,u_2(t) - 3\,u_3(t) + u_4(t)$

Find Laplace transform for $u_c(t)$

$L\{u_c(t)\} = \int_0^{\infty} e^{-st} u_c(t)\,dt$

$= \int_0^{c} e^{-st}(0)\,dt + \int_c^{\infty} e^{-st}\,dt = \int_c^{\infty} e^{-st}\,dt$

$= \lim_{b\to\infty} \int_c^{b} e^{-st}\,dt = \lim_{b\to\infty} \frac{-1}{s} e^{-st}\Big|_c^b = \lim_{b\to\infty} \frac{-e^{-sb}}{s} + \frac{e^{-bs}}{s}$

$= \frac{e^{-cs}}{s}$

Result $L\{u_c(t)\} = \frac{e^{-cs}}{s}$

$L\{u_2(t)\} = \frac{e^{-2s}}{s}$

$L\{u_{\pi}(t)\} = \frac{e^{-\pi s}}{s}$

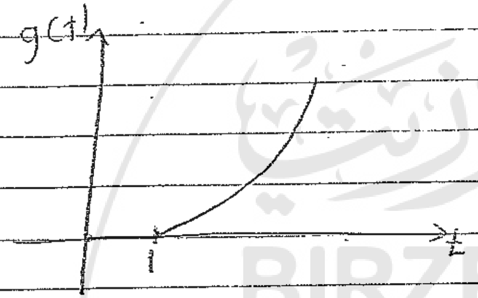
$L^{-1}\left(\frac{e^{-3s}}{s}\right) = u_3(t)$

ex. let $f(t) = t^2$, $t \geq 0$. Sketch the graph of

$g(t) = u_1(t)\, f(t-1)$

$g(t) = u_1(t)\,(t-1)^2$

$= \begin{cases} 0, & 0 \le t < 1 \\ (t-1)^2, & 1 \le t \end{cases}$



2nd shifting

$L\{u_c(t)\, f(t-c)\} = e^{-cs}\, L\{f(t)\}$
$= e^{-cs}\, F(s)$

$L^{-1}(e^{-cs}F(s)) = u_c(t)\, f(t-c)$

Proof:- $L\{u_c(t)\, f(t-c)\} = \int_0^\infty e^{-st} u_c(t)\, f(t-c)\, dt$

$= \int_c^\infty e^{-st} f(t-c)\, dt$

let $u = t-c$ $\quad t = u+c$

$du = dt$

$L\{u_c(t)\, f(t-c)\} = \int_0^\infty e^{-s(u+c)} f(u)\, du$

$e^{-cs}\int_0^\infty e^{-su} f(u)\, du \Rightarrow e^{-cs}F(s)$

Ex. Find:-

① $L\{u_2(t)\underbrace{(t-2)}_{f(t-2)}\} = e^{-2s}\, L\{t\} = \frac{e^{-2s}}{s^2}$

$f(t) = t$

② $L\{u_2(t)\,(t-1)\} = L\{u_2(t)\,((t-2)+1)\}$

$L\{u_2(t)(t-2)\} + L\{u_2(t)\}$

$e^{-2s}\, L\{t\} + \frac{e^{-2s}}{s} = e^{-2s}\cdot\frac{1}{s^2} + \frac{e^{-2s}}{s}$

③ $L\{u_2(t)\, t^2\} = L\{u_2(t)\,((t-2)+2)^2\}$

$L\{u_2(t)\,((t-2)^2 + 4(t-2) + 4)\}$

$L\{u_2(t)(t-2)^2\} + 4L\{u_2(t)(t-2)\} + 4L\{u_2(t)\}$

$e^{-2s}\,\frac{2}{s^3} + \frac{4e^{-2s}}{s^2} + \frac{4e^{-2s}}{s}$

Ex. find $L\{f(t)\}$ where

$$f(t) = \begin{cases} 0 , & 0 \le t < 1 \\ t-1 , & 1 \le t < 2 \\ 1 , & 2 \le t \end{cases}$$

$f(t) = (t-1)\, U_1 + (2-t)\, U_2$

$= U_1(t)(t-1) + U_2(t)(t-2)$

$L\{f(t)\} - L\{U_1(t)(t-1)\} - L\{U_2(t)(t-2)\}$

$$e^{-s}\frac{1}{s^2} - \frac{e^{-2s}}{s^2}$$

Ex. find

$$L^{-1}\left(\frac{3+e^{-7s}}{s^4}\right) = L^{-1}\left(\frac{3}{s^4}\right) + L^{-1}\left(\frac{e^{-7s}}{s^4}\right)$$

$$= \frac{1}{2}t^3 + U_7(t)\,\frac{(t-7)^3}{6}$$

$$\frac{1}{2}t^3 + \frac{1}{6}U_7(t)\,(t-7)^3$$

② $L^{-1}\left(\frac{2(s-1)e^{-2s}}{(s^2-2s+2)}\right)$

$$2L^{-1}\left(\frac{(s-1)\,e^{-2s}}{(s-1)^2+1}\right) = \cancel{2U_2(t)\cos(t)\, e^{t}}$$

$2U_2(t)e^{t-2}\cos(t-2)$.

Ex. find laplace transform of :-

① $S(t) = \begin{cases} 0 , & 0 \le t < 1 \\ (t-1)^2 & 1 \le t \end{cases}$

$f(t) = (t-1)^2\, U_1(t)$

$$L\{f(t)\} = L\{U_1(t)\,(t-1)^2\} = e^{-s}L(t^2) = e^{-s}\cdot\frac{2}{s^3}$$

ex. $f(t) = \begin{cases} \sin t , & 0 \le t < \frac{\pi}{4} \\ \sin t + \cos(t-\frac{\pi}{4}) , & \frac{\pi}{4} \le t \end{cases}$

$f(t) = \sin t + \cos(t-\frac{\pi}{4})\, U_{\frac{\pi}{4}}$

$$F(s) = L\{\sin t\} + L\{U_{\frac{\pi}{4}}(t)\,\cos(t-\tfrac{\pi}{4})\}$$

$$\frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\, L[\cos t]$$

$$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s} \cdot \frac{s}{s^2+1}$$

28) $$F(s) = \int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\}$$

$$= -\int_0^\infty t e^{-st} f(t)\,dt = (-1)^1 L\{t f(t)\}$$

$$F''(s) = \int_0^\infty t^2 e^{-st} f(t)\,dt = (-1)^2 L\{t^2 f(t)\}$$

$$F^{(n)}(s) = \int_0^\infty (-1)^n t^n e^{-st} f(t)\,dt = (-1)^n L\{t^n f(t)\}$$

Ex. 29

$$L\{t^2 e^{at}\} = F(s-a)$$

$$F(s) = L(t^2) = \frac{2}{s^3}$$

$$F(s-a) = \frac{2}{(s-a)^3}$$ // or

$$L\{t^2 e^{at}\} = (-1)^2 F''(s) = \frac{2}{(s-a)^3}$$

$$F(s) = L\{e^{at}\}$$

$$F(s) = \frac{1}{s-a}$$

$$F'(s) = \frac{-1}{(s-a)^2}$$

$$F''(s) = \frac{2}{(s-a)^3}$$

30) $$L\{t^2 \sin bt\} = F''(s)$$

$$F(s) = L\{\sin bt\} = \frac{b}{s^2+b^2}$$

$$F'(s) = \frac{-2bs}{s^2+b^2}$$

$$F''(s) = \frac{2b(3s^2-b^2)}{(s^2+b^2)^3}$$

33) $$L\{t e^{at} \sin bt\} = -(-1)F'(s) = \frac{2b(s-a)}{((s-a)^2+b^2)^2}$$

$$F(s) = L\{e^{at} \sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$F'(s) = \frac{-2b(s-a)}{((s-a)^2+b^2)^2}$$

$F' =$

6.4 DE's with Discontinuous Forcing functions:

* In This section we will use the second shifting to solve IVP's with discontinuous forcing functions $g(t)$.

Ex. Solve the IVP-

$$2y'' + y' + 2y = g(t) \qquad y(0)=0,\ y'(0)=0$$

where $g(t) = U_5(t) - U_{20}(t) = \begin{cases} 0, & 0 \le t < 5 \\ 1 & 5 \le t < 20 \\ 0 & 20 \le t \end{cases}$

Clearly g is discontinuous at $t = 5, 20$

$$U_5(t) = \begin{cases} 0, & 0 \le t < 5 \\ 1 & 5 \le t \end{cases}$$

Take laplace for both sides:

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{U_5(t)\} - L\{U_{20}(t)\}$$

$$2\left[s^2 L\{y\} - s y(0) - y'(0)\right] + sL\{y\} - y(0) + 2L\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$

$$L\{y\}\,(2s^2 + s + 2) = \frac{1}{s}\left(e^{-5s} - e^{-20s}\right)$$

$$L\{y\} = \frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)} = \frac{e^{-5s} - e^{-20s}}{2s(s^2+\frac{s}{2}+1)}$$

$$L\{y\} = \frac{e^{-5s} - e^{-20s}}{2s\left((s+\frac{1}{4})^2 + \frac{15}{16}\right)}$$

$$\frac{1}{2}\left[e^{-5s}H(s) - e^{-20s}H(s)\right] \quad \text{where} \quad H(s) = \frac{1}{s\left((s+\frac{1}{4})^2+\frac{15}{16}\right)}$$

$$y(t) = L^{-1}\left[\frac{1}{2}\left(e^{-5s}H(s) - e^{-20s}H(s)\right)\right]$$

$$\frac{1}{2}U_5(t)\,h(t-5) - \frac{1}{2}U_{20}(t)\,h(t-20)$$

we need to find $h(t) = L^{-1}(H(s))$

now $h(t) = L^{-1}(H(s)) = L^{-1}\left(\dfrac{1}{s(2s^2+s+2)}\right)$

$$\frac{1}{s(2s^2+s+2)} = \frac{A}{s} + \frac{Bs+C}{2s^2+s+2}$$

$$= \frac{\frac{1}{2}}{s} - \frac{s+\frac{1}{2}}{2s^2+s+2}.$$

$$h(t) = L^{-1}\left(\frac{\frac{1}{2}}{s}\right) - L^{-1}\left(\frac{s+\frac{1}{2}}{2s^2+s+2}\right)$$

$$\frac{1}{2} L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{\frac{1}{2}\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right)$$

$$\frac{1}{2}(1) - \frac{1}{2} L^{-1}\left(\frac{\left(s+\frac{1}{4}\right)+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right)$$

$$h(t) = \frac{1}{2} - \frac{1}{2} L^{-1}\left(\frac{s+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right) - \frac{1}{\sqrt{15}} L^{-1}\left(\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}\right)$$

$$h(t) = \frac{1}{2} - \frac{1}{2} e^{-\frac{1}{4}t}\cos\frac{\sqrt{15}}{4}t - \frac{1}{2\sqrt{15}} e^{-\frac{1}{4}t}\sin\frac{\sqrt{15}}{4}t$$

So the solution is

$$\phi(t) = \frac{1}{2} u_5(t)\, h(t-5) - \frac{1}{2} u_{20}(t)\, h(t-20)$$

where $h(t)$ is given by ↑

You can use computer to see that $\phi(t)$, $\phi'(t)$ are continuous at $t = 5, 20$

but $\phi''(5)$, $\phi''(20)$ are discontinuous.

Ex. Solve the IVP

$$y'' + 4y = g(t), \quad y(0) = 0, \quad y'(0) = 0$$

$$g(t) = \frac{t-5}{5} u_5(t) - \frac{t-10}{5} u_{10}(t)$$

$$= \begin{cases} 0, & 0 \le t < 5 \\ \frac{t-5}{5} & 5 \le t < 10 \\ 1, & 10 \le t \end{cases}$$

take laplace.

$$L\{y''\} + 4L\{y\} = L\{g(t)\}$$

$$L\{y\}(s^2+4) = \frac{1}{5} L\{u_5(t)(t-5)\} - \frac{1}{5} L\{u_{10}(t)(t-10)\}$$

$$= \frac{1}{5} e^{-5s} L\{t\} - \frac{1}{5} e^{-10s} L\{t\}$$

$$= \frac{1}{5} e^{-5s} \frac{1}{s^2} - \frac{1}{5} e^{-10s} \frac{1}{s^2}$$

$$L\{y\} = \frac{1}{5}\left(e^{-5s} - e^{-10s}\right) H(s), \quad H(s) = \frac{1}{s^2(s^2+4)}$$

$$y(t) = \frac{1}{5} L^{-1}\left(e^{-5s} H(s)\right) - \frac{1}{5} L^{-1}\left(e^{-10s} H(s)\right)$$

$$= \frac{1}{5} u_5(t)\, h(t-5) - \frac{1}{5} u_{10}(t)\, h(t-10)$$

We need only to find $h(t)$

$$h(t) = L^{-1}(H(s)) = L^{-1}\left(\frac{1}{s^2(s^2+4)}\right)$$

$$\frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$1 = (A+C)s^3 + (B+D)s^2 + 4As + 4B.$$

$A=0 \quad B=\frac{1}{4} \quad C=0 \quad D=-\frac{1}{4}$

$$h(t) = L^{-1}\left(\frac{\frac{1}{4}}{s^2}\right) + L^{-1}\left(\frac{-\frac{1}{4}}{s^2+4}\right)$$

$$h(t) = \frac{1}{4} t - \frac{1}{8} \sin 2t$$

we can see that $\phi(t)=y(t)$ ~~is continuous~~ $\phi'(t), \phi''(t)$ are continuous at $t=5, 10$

5.5) Impulse Function

* In some application, sometimes it is necessary to deal with impulse nature For example, the voltage that is acting for short time intervals.

* Such applications lead to a DE's of type
$ay'' + by' + c = d_\tau(t-t_0)$

where the forcing function $d_\tau(t-t_0)$

is given by $d_\tau(t-t_0) = \begin{cases} \frac{1}{2\tau}, & |t-t_0| < \tau \\ 0, & \text{o.w.} \end{cases}$

$|t-t_0| < \tau$

$t_0 - \tau < t < t_0 + \tau$

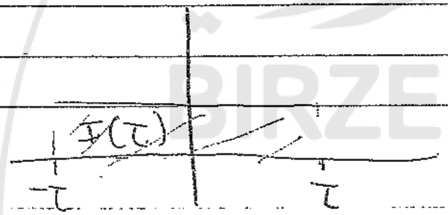
* In case where $t_0 = 0$. the forcing function becom

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau \\ 0, & \text{o.W} \end{cases}$$

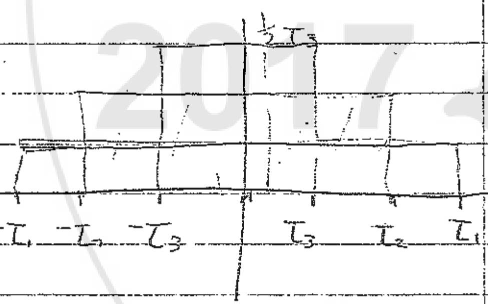
* To measure the strength of the forcing function, we use the integral

$$I(\tau) = \int_{-\infty}^{\infty} d_{\tau}(t)\, dt$$

$$= \int_{-\tau}^{\tau} \frac{1}{2\tau}\, dt = 1$$



$$\lim_{t \to 0} d_{\tau}(t) = 0$$



$$\lim_{\tau \to 0} I(\tau) = 1$$

* The unit Impulse function δ at arbitrary point t_0 is defined by

$$\delta(t) = \begin{cases} 1, & \text{if } t = t_0 \\ 0 & \text{if } t \neq t_0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$$

If $t_0 = 0$, then the unit impuls function becomes:

$$\delta(t) = \begin{cases} 1, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0 \end{cases}$$

* The Unit impulse function is an example of the Dirac function

$$\delta(t-c) = \begin{cases} 1, & \text{If } t = c \\ 0, & \text{if } t \neq c. \end{cases}$$

Note that

① $\delta(t) = \lim_{\tau \to 0} d_{\tau}(t)$, $t \neq 0$

Hence

② $\delta(t - t_0) = \lim_{\tau \to 0} d_{\tau}(t - t_0)$, $t \neq 0$.

Remark: $L\{\delta(t - t_0)\} = e^{-st_0}$

Proof:- $L\{\delta(t - t_0)\} = \int_0^{\infty} e^{-st}\, \delta(t - t_0)\, dt$

$$\lim_{\tau\to 0}\int_{0}^{\infty} e^{-st}\, d_{\tau}(t-t_0)\, dt$$

assume $t_0-\tau>0$
$t_0>\tau$

$$\lim_{\tau\to 0}\int_{t_0+\tau}^{t_0-\tau} e^{-st}\,\frac{1}{2\tau}\, dt$$

$$\lim_{\tau\to 0}\frac{1}{2\tau}\,\frac{-e^{-st}}{s}\Bigg|_{t_0+\tau}^{t_0+\tau}$$

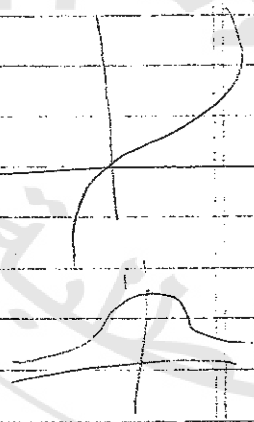
$$=\lim_{\tau\to 0}\frac{1}{2\tau}\left[\frac{e^{-s(t_0+\tau)}}{s}-\frac{e^{-(t_0-\tau)s}}{s}\right]$$

$$= e^{st_0}\lim_{\tau\to 0}\frac{1}{s\tau}\left(\frac{e^{+s\tau}-e^{-s\tau}}{2}\right)$$

$$= e^{-st_0}\lim_{\tau\to 0}\frac{\sinh(s\tau)}{s\tau}$$

$$e^{-st_0}\lim_{\tau\to 0}\frac{\cosh(s\tau).s}{s\tau}$$

$$= e^{-st_0}\cdot 1 = e^{-st_0}$$



ex. $L\{\delta(t-5)\} = e^{-5s}$

Ex. Find $L^{-1}(1)$

$$L^{-1}(1) = \delta(t)$$

$L\{1\} = \frac{1}{s}$

$L^{-1}(1) = \delta(t)$

$L\{\delta(t)\} = 1$

* Remark:- We can use this ($L\{\delta(t-t_0)\} = e^{-st_0}$) to find the integral of product unit impulse function with a continuous function $f(t)$:-

Result: $\int_{-\infty}^{\infty}\delta(t-t_0)\, f(t)\, dt = f(t_0)$.

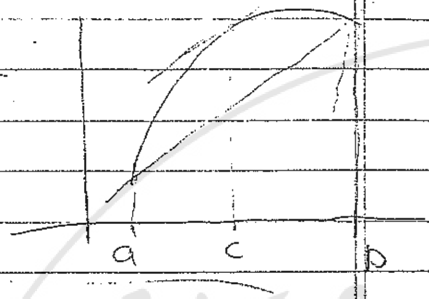
Proof:- $\int_{-\infty}^{\infty}\delta(t-t_0)\, f(t)\, dt = \lim_{\tau\to 0}\int_{-\infty}^{\infty} d_{\tau}(t-t_0)\, f(t)\, dt$

$$\lim_{\tau\to 0}\int_{t_0-\tau}^{t_0+\tau}\frac{1}{2\tau}\, f(t)\, dt = \lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)\, dt$$

(M.V.T)

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$F(c) = \frac{1}{b}\int_a^b f(t)\,dt$$



$*)\ \lim_{\tau\to 0} f(t^*)$ $\qquad t^* \in [t_0-\tau,\ t_0+\tau]$

$= f(t_0).$

Ex. $\cancel{L\{\}}\ \int_{-\infty}^{\infty} \delta(t-\frac{\pi}{2})\cos t\,dt$

$= \cos\frac{\pi}{2} = 0$

Ex. find the solution of

$2y'' + y' + 2y = \delta(t-5)$. $\qquad y(0)=0,\ y'(0)=0$

$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{\delta(t-5)\}$

$= e^{-5s}$

$$L\{y\} = \frac{e^{-5s}}{2s^2+s+2}.$$

$$y(t) = L^{-1}\left(\frac{\frac{1}{2}e^{-5s}}{(s+\frac{1}{4})^2+\frac{15}{16}}\right)$$

$$= \frac{1}{2}u_5(t)\, e^{-\frac{1}{4}(t-5)}\ \frac{\sqrt{15}}{4}(t-5)$$

6.6 The convolution Integrals.

Ex. If $f(t)=1$ and $g(t)=\sin t$.

then $L\{f(t)\} = F(s) = L\{1\} = \frac{1}{s}$.

$L\{g(t)\} = G(s) = L\{\sin t\} = \frac{1}{s^2+1}$

$L\{f(t)\,g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$

Does $L\{f(t)\,g(t)\} = L\{f(t)\}\,L\{g(t)\}$

No, $\frac{1}{s^2+1} \neq \frac{1}{s}\,\frac{1}{s^2+1}$

Defn. The convolution function of f and g is defined by $h(t) = (f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$

and $\int_0^t f(t-\tau)\, g(\tau)\, d\tau$ is called the convolution Integral.

Then, if $h(t) = \int_0^t f(t-\tau) \cdot g(\tau)\, d\tau$, then

$$L\{h(t)\} = H(s) = L\{f(t)\}\, L\{g(t)\}$$
$$= F(s)\; G(s)$$

Remark:-

$$f*g = g*f$$

$$\int_0^t f(t-\tau)\, g(\tau)\, d\tau = \int_0^t g(t-\tau)\, f(\tau)\, d\tau$$

Let $u = t-\tau$ when $\tau = 0$, $u = t$

$du = -d\tau$ $\quad \tau = t$, $u = 0$

$$-\int_t^0 f(u)\, g(t-u)\, du$$

$$= \int_0^t f(u)\, g(t-u)\, du \quad \#$$

Ex. Find laplace transform of

① $t * \sin 2t = h(t)$

$$L\{h(t)\} = L\{t * \sin 2t\} = L\left\{\int_0^t (t-\tau) \sin 2\tau\, d\tau\right\}$$

$$= L\{t\}\, L\{\sin 2t\}$$
$$\frac{1}{s^2} \cdot \frac{2}{s^2+4}$$

Hence:- $L^{-1}\left(\frac{1}{s^2} \cdot \frac{2}{s^2+4}\right) = \cancel{L^{-1}\left(\frac{1}{s^2}\right) L\{}$

$$L^{-1}\left(\frac{1}{s^2} \cdot \frac{2}{s^2+4}\right) = L^{-1}\left(L\{t\}\, L\{\sin 2t\}\right)$$

$$= t * \sin 2t = \int_0^t (t-\tau) \sin 2\tau\, d\tau$$

② $r(t) = \int_0^t (t-\tau)^2 \cos\tau\, d\tau$

$$R(s) = L\{r(t)\} = L\{t^2 * \cos t\} = L\{t^2\}\, L\{\cos t\}$$

$$\frac{2}{s^3} \cdot \frac{s}{s^2+1}$$

③ $h(t) = t * e^{t}$

$H(s) = L\{t\} L\{e^{t}\}$

$H(s) = \frac{1}{s^2} \frac{1}{s-1}$

Ex. Find the Inverse laplace transform of

$H(s) = \frac{2}{s^2(s-2)}$

$h(t) = L^{-1}(H(s)) = 2L^{-1}\left(\frac{1}{s^2} \frac{1}{s-2}\right)$

$= 2 L^{-1}\{ L\{t\} L\{e^{2t}\}\}$

$2 (t * e^{2t})$

$= 2\int_0^t (t-\tau) e^{2\tau} d\tau$

$2t \int_0^t e^{2\tau} dt - 2 \int_0^t \tau e^{2\tau} d\tau$

$2t \frac{1}{2} e^{2\tau}\Big|_0^t - 2\left(\frac{\tau}{2} e^{2\tau} - \frac{e^{2\tau}}{4}\right)\Big|_0^t$

$\tau \quad e^{2\tau}$

$1 \quad \frac{e^{2\tau}}{2}$

$0 \quad \frac{e^{2\tau}}{4}$

$-t + 2t e^{2t} - 2\left[\frac{t}{2} e^{2t} - \frac{e^{2t}}{4} + \frac{1}{4}\right]$

$= \frac{e^{2t} - 1}{2} \quad t$

Sol 2 $L^{-1}\left(\frac{2}{s^2(s-2)}\right)$ using partial fraction

$L^{-1}\left(\frac{\frac{1}{2}}{s-2} + \frac{-\frac{1}{2}s - 1}{s^2}\right)$

$\frac{2}{(s-2)s^2} = \frac{A}{s-2} + \frac{Bs+C}{s^2}$

$A = \frac{1}{2}, \ B = \frac{-1}{2}$

$C = -1$

$\frac{1}{2} e^{2t} - \frac{1}{2}(1) - t$

$= \frac{e^{2t} - 1}{2} - t$

Ex. Solve the IVP.

$y'' + 4y = g(t)$ $\quad y(0) = 3$, $y'(0) = -1$

take laplace :-

$L\{y''\} + 4L\{y\} = L\{g(t)\}$

$s^2 L\{y\} - s y(0) - y'(0) + 4L\{y\} = G(s)$

$L\{y\}(s^2 + 4) - 3s + 1 = G(s)$

$L\{y\}(s^2 + 4) = 3s - 1 + G(s)$

$L\{y\} = \frac{3s}{s^2+4} - \frac{1}{s^2+4} + \frac{1}{s^2+4} G(s)$

$$y(t) = 3L^{-1}\left(\frac{s}{s^2+4}\right) - \frac{1}{2}L^{-1}\left(\frac{2}{s^2+4}\right) + L^{-1}\left(\frac{1}{2}L\{\sin 2t\}L\{g(t)\}\right)$$

$$= 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\left(\sin 2t * g(t)\right)$$

$$y(t) = \underbrace{3\cos 2t - \frac{1}{2}\sin 2t}_{\text{Homogeneous}} + \underbrace{\frac{1}{2}\int_0^t \sin(2(t-\tau))\, g(\tau)\, d\tau}_{\text{Particular}}$$

Ex. 8) $F(s) = \frac{1}{s^4(s^2+1)}$

find $f(t)$.

$$f(t) = L^{-1}(F(s)) = L^{-1}\left(\frac{1}{s^4(s^2+1)}\right) = L^{-1}\left(\frac{1}{s^4} \quad \frac{1}{s^2+1}\right)$$

$$= L^{-1}\left\{\frac{1}{6} L\{t^3\} \; L\{\sin t\}\right\}$$

$$\frac{1}{6}\, t^3 * \sin t$$

$$= \frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau \; d\tau$$

9) $F(s) = \frac{s}{(s+1)(s^2+9)}$

$$f(t) = L^{-1}\left(\frac{s}{s^2+9} \quad \frac{1}{s+1}\right)$$

$$L^{-1}\left(L^{-1}\{e^{-t}\}\; L\{\cos 3t\}\right)$$

$$= e^{-t} * \cos 3t$$

$$\int_0^t e^{-(t-\tau)} \cos 3\tau \; d\tau.$$

Ex Solve the Integral eq. $\Rightarrow$ $\phi(t) + \int_0^t (t-\xi)\,\phi(\xi)\,d\xi = \sin 2t$

$$L\{\phi(t)\} + L\left\{\int_0^t (t-\xi)\,\phi(\xi)\,d\xi\right\} = L\{\sin 2t\}$$

$$L\{\phi(t)\} + L\{t\}L\{\phi(t)\} = \frac{2}{s^2+4}$$

$$L\{\phi(t)\}\left(1 + \frac{1}{s^2}\right) = \frac{2}{s^2+4}$$

$$L\{\phi(t)\} \cdot \frac{s^2+1}{s^2} = \frac{2}{s^2+4}$$

$$L\{\phi(t)\} = 2\left(\frac{s}{s^2+1}\right)\left(\frac{s}{s^2+4}\right)$$

$$\phi(t) = 2L^{-1}\left\{L(\cos t)\, L\{\cos 2t\}\right\}$$

$$= 2\cos t * \cos 2t$$

$$2\int_0^t \cos(t-\tau)\cos 2\tau \, d\tau$$

// OR you can use.

$$\phi(t) = L^{-1}\left(\frac{2s^2}{(s^2+1)(s^2+4)}\right)$$

$$L^{-1}\left(\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}\right)$$

$$L^{-1}\left(\frac{-2/3}{s^2+1} + \frac{8/3}{s^2+4}\right)$$

$$= \frac{-2}{3}\sin t + \frac{4}{3}\sin 2t$$

7.5 Homogeneous linear Systems with constant coefficients.

In this section we will find a solution for system of linear DE with constant coefficients.
~~That~~ That is, we solve $\boxed{x' = A_{n\times n} X}$ * where

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{\text{Coefficient matrix}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

As before, we will Assume exponential solution of the form

$X(t) = \xi e^{rt}$ to the System $*$

for this solution to Satisfy

$* \Rightarrow x'(t) = r\xi e^{rt}$

Hence, from $* \Rightarrow x' = Ax$

$r\xi e^{rt} = A\xi e^{rt}$

Rearrange $\Rightarrow$

$A\xi e^{rt} - r\xi e^{rt} = 0$

$(A - rI)\xi e^{rt} = 0$

$(A - rI)\xi = 0$ ——— 1

For having non-trivial solution "$\xi \neq 0$" for 1 the matrix $(A - rI)$ must be singular. Hence, $|A - rI| = 0$

r is called the eigenvalue of A and ξ is called the corresponding eigenvector.

~~ex.~~ In this section we only study real eigen values and different.

ex. find the solution of

$x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$

$x_1' = x_1 + x_2$

$x_2' = 4x_1 + x_2$

$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x' = Ax$

First, Assume exponential solution of the form

$X(t) = \xi e^{rt}$

Substitute in

$|A - rI| = 0$ $\quad A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $\quad rI \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$

$\begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0$

$(1-r)^2 - 4 = 0$

$(1-r)^2 = 4$

characteristic equation

$\boxed{r^2 - 2r - 3 = 0}$

$1 - r = 2 \quad , \quad 1 - r = -2$

$r_1 = -1 \quad \quad r_2 = 3$

To find the eigenvector $\xi^{(1)}$ corresponding to the eigenvalue $r_1 = 3$ we solve

$(A - r_1 I)\xi = 0$

$\begin{bmatrix} 1-3 & 1 \\ 4 & 1-3 \end{bmatrix} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2\xi_1 + \xi_2 = 0$$

$$\boxed{\xi_2 = 2\xi_1} \Leftrightarrow \xi_1 = \tfrac{1}{2}\xi_2$$

$$\xi^{(1)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\alpha \\ \alpha \end{pmatrix} = \frac{\alpha}{2}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

* To find eigenvector ξ^2 corresponding to the eigenvalue $r_2 = -1$ we solve

$$(A - r_2 I)\,\xi^{(2)} = 0$$

$$\begin{pmatrix} 1-1 & 1 \\ 4 & 1-1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2\xi_1 + \xi_2 = 0$$

$$\xi_1 = -\tfrac{1}{2}\xi_2$$

$$\xi^{(2)} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\alpha \\ \alpha \end{pmatrix} = \frac{-\alpha}{2}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\xi^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

first solution $x^{(1)}(t) = \xi^{(1)} e^{r_1 t}$
$= \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}$

Second solution $= \xi^{(2)} e^{r_2 t}$
$\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

the general solution is
$x(t) = c_1 x^{(1)}(t) = c_2 x^{(2)}(t)$

$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

If the initial condition $X(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$$X(0) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3(0)} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-(0)}$$

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\left.\begin{array}{l} c_1 + c_2 = 3 \\ +2c_1 - 2c_2 = -2 \end{array}\right\} \begin{array}{l} c_1 = 1 \\ c_2 = 2 \end{array}$$

Real Eigenvalues:-

Ex. solve $X' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix} X$

$-3x_1 + \sqrt{2}x_2$

$\sqrt{2}x_1 - 2x_2$

* Assume exp. solution $X(t) = \xi e^{rt}$
* To find the eigenvalues, we solve

$$|A - rI| = 0 \iff \begin{vmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{vmatrix} = 0$$

$$r^2 + 5r + 4 = 0$$

$$(r+1)(r+4) = 0$$

$$\left.\begin{array}{l} r_1 = -1 \\ r_2 = -4 \end{array}\right] \text{ real different}$$

* To find the eigen vector corresponding to the eigenvalues

$\lambda_1 = -1$ we solve $(A - r_1 I)x = 0$

$$\begin{bmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + \sqrt{2}x_2 = 0 \implies x_2 = \sqrt{2}x_1$$

$$\xi^{(1)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \sqrt{2}\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

First solution: $x^{(1)}(t) = \xi e^{r_1 t}$

$$x^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}$$

to find: $\lambda_2 = -4$ we solve $(A - r_2 I)x = 0$

$$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 + \sqrt{2}x_2 = 0 \qquad x_1 = -\sqrt{2}x_2$$

$$\xi^{(2)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{2}\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$$

$\Rightarrow$ Second solution

$$x^{(2)}(t) = \xi^{(2)} e^{r_2 t} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

the general solution is $X(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$

$$= c_1 \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}$$

as $t \to \infty$ the solution approach to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and it is Asymp. Stable b/c it has neg. eigenvalues

$$* W(x, x_2) = \begin{vmatrix} e^{-t} & \sqrt{2}e^{-4t} \\ \sqrt{2}e^{-t} & e^{-4t} \end{vmatrix} = e^{-5t} + 2e^{-5t} = 3e^{-5t} \neq 0$$

So $x^{(1)}(t), x^{(2)}(t)$ lin. indep.

$\Rightarrow$ Hence, they form fundamental set of solution.

7.6 Complex eigenvalues:

Solve $x' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} x$, find r_1, r_2:-

we solve $|A - rI| = 0$

$\begin{vmatrix} 1-r & -1 \\ 5 & -3-r \end{vmatrix} = 0 \iff r^2 + 2r + 2 = 0$

$r_1 = -1 - i$, $r_{1,2} = -1 \mp i$

$r_2 = -1 + i$

To find $\xi^{(1)}$ for $r_1 = -1 - i$ we solve $(A - r_1 I)x = 0$

$\iff \begin{pmatrix} 1-(-1-i) & -1 \\ 5 & -3-(-1-i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 2+i & -1 \\ 5 & -2+i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (2+i)x_1 - x_2 = 0$

$(2+i)x_1 = x_2$

$\xi^{(1)} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ (2+i)\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2+i \end{pmatrix}$

Similarly $\xi^{(2)} = \alpha \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

* The first solution: $x^{(1)}(t) = \xi^{(1)} e^{r_1 t} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(-1-i)t}$

$x^{(2)}(t) = \xi^{(2)} e^{r_2 t} = \begin{pmatrix} 1 \\ 2-i \end{pmatrix} e^{(-1+i)t}$

* but we need real values solutions, so we use Euler formula to help us , we take for example $x^{(1)}(t)$:-

$x^{(1)}(t) = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{(-1-i)t} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t} e^{-it}$

$\begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t} (\cos t \mp i \sin t) = \underbrace{e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix}}_{u(t)} + \underbrace{i e^{-t} \begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}}_{v(t)}$

$u(t)$ and $v(t)$ are real valued solutions:

* The general solution $x(t) = c_1 u(t) + c_2 v(t)$

$x(t) = c_1 e^{-t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -\sin t \\ \cos t - 2\sin t \end{pmatrix}$

$W(x^{(1)}, x^{(2)}) = \begin{vmatrix} e^{-t}\cos t & -e^{-t}\sin t \\ e^{-t}(2\cos t + \sin t) & e^{-t}(\cos t - 2\sin t) \end{vmatrix} = e^{-2t} \neq 0$

$\Rightarrow$ $u(t)$ and $v(t)$ are lin. indep. so they form fundamental set of solutions.

7.8 Repeated eigenvalues.

Consider the linear system homogeneous with co[nstant] coefficient $x' = Ax$ which has a repeated eigenvalu[e] $r_1 = r_2 = \rho$, with corresponding eigenvector ξ s.t $(A - \rho I)$ [$\xi = 0$]

Hence, one solution is $x^{(1)}(t) = \xi e^{\rho t}$

the second independent solution is

$x^{(2)}(t) = \xi t e^{\rho t} + \eta e^{\rho t}$, where η is called the generalized eigenvector associated with the repeated eigenvalue ρ s.t $(A - \rho I)\eta = \xi$

Ex. find the general solution of the IVP

$x_1' = x_1 - 4x_2$ $\quad x_1(0) = 3$ $\quad x_2(0) = 2$

$x_2' = 4x_1 - 7x_2$

$x' = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix} x$ To find the eigenvalues w[e] solve $|A - rI| = 0$

$\begin{vmatrix} 1-r & -4 \\ 4 & -7-r \end{vmatrix} = 0$ $\quad r^2 + 6r + 9 = 0$

$\iff (r+3)^2 = 0$ $\quad r_1 = r_2 = \boxed{\rho = -3}$

now, to find ξ we solve $(A-\rho I)\xi = 0$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 4x_1 - 4x_2 = 0$$

$$x_1 = x_2$$

$$\xi = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

* now to find η we solve $(A-\rho I)\eta = \xi$

$$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow 4\eta_1 + 4\eta_2 = 1$$

$\Rightarrow \cancel{\eta_2} \; \eta_1 = \eta_2 + \frac{1}{4}$ (η_2 → free)

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} k + \frac{1}{4} \\ k \end{pmatrix} = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence the second solution is $x^{(2)}(t) = \xi t e^{\rho t} + \eta e^{\rho t}$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} e^{-3t} + k\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}$$

$\boxed{x^{(1)} = \xi e^{\rho t} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t}}$ first sol.

no multiple of $x^{(1)}$

→ this term is a multiple of first solution

$$\Rightarrow x^{(2)}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} e^{-3t}$$

The general solution is

$$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$$

$$x(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} t e^{-3t} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} e^{-3t}\right)$$

To find c_1, c_2, we know $\cancel{x^{(1)}(0)}$ $x_0\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$x(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} \frac{1}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$c_1 + \frac{c_2}{4} = 3 \qquad \boxed{c_2 = 4}$$

$$\boxed{c_1 = 2}$$

The general solution becomes:

c_1, c_2 نعوض

$$* \; W(x_1, x_2) = \begin{vmatrix} e^{-3t} & te^{-3t} + \frac{1}{4}e^{-3t} \\ e^{-3t} & te^{-3t} \end{vmatrix} = \frac{-e^{-6t}}{4} \neq 0$$

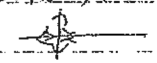
$x^{(1)}, x^{(2)}$ lin. ind. $\Rightarrow$ they form fundam. set of solutions.

Ch. 7 Review.

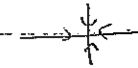
Solution behavior around origion:

(I) Real eigenvalues r_1 and r_2.

(a) if r_1 & r_2 are real positive, the the origion is unstable eq. point.



(b) if r_1 & r_2 are nagative then origion is asymptotic stable eq. point (node).



(c) If $r_1 r_2 < 0$, the the origion is Saddle point which is unstable eq. point