Differential Course "Math 331"

Chapter 1: Introduction:-

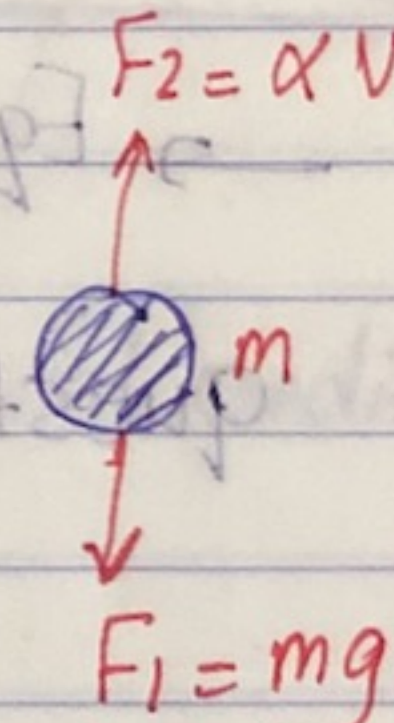
1.1: Some basic models and direction fields.

* Differential eqs are relations containing derivatives.

Ex:- Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

Sol:- let t = time (independent variable)
V = velocity (dependent variable)

2^{nd} Newton's law:



$$\boxed{F_{net} = ma} \quad \text{---} (1)$$

$$F_1 - F_2 = m\frac{dV}{dt}$$

$$\rightarrow m\frac{dV}{dt} = mg - \gamma V \rightarrow \boxed{\frac{dV}{dt} = g - \frac{\gamma}{m}V} \quad \text{---} (2)$$

where: g: the acceleration due to gravity
γ: drag coeffient
m: mass

RmK: ① Eq ②* is a ~~fis~~ first order d.e

② to solve Eq ②*, we need to find a function $V = V(t)$ that satisfies the eq "next section 1.2"

Our task :- investigate to the behaviour of the solution of Eq ② without solving it.

$\boxed{\lim_{t\to\infty} V(t) = ??}$, this is called direction field or slope field

Ex:- Take $m = 10$ Kg , $\gamma = 2$ Kg/s , $g = 9.8$ in eq ②

→ Eq ② becomes : $\boxed{\frac{dV}{dt} = 9.8 - \frac{1}{5} V}$ ---③

question : investigate the behaviour of Eq ③ "That is $\lim_{t\to\infty} V(t)$??"

sol:- Equilibrium solution:-

$\frac{dV}{dt} = 0 \rightarrow 9.8 - \frac{1}{5} V = 0$

$V(t) = 49$

If $V_0 < 49$, take $V_0 = 0$

$$\frac{dV}{dt} = 9.8 - \frac{1}{5}(0) = 9.8 > 0 \nearrow$$

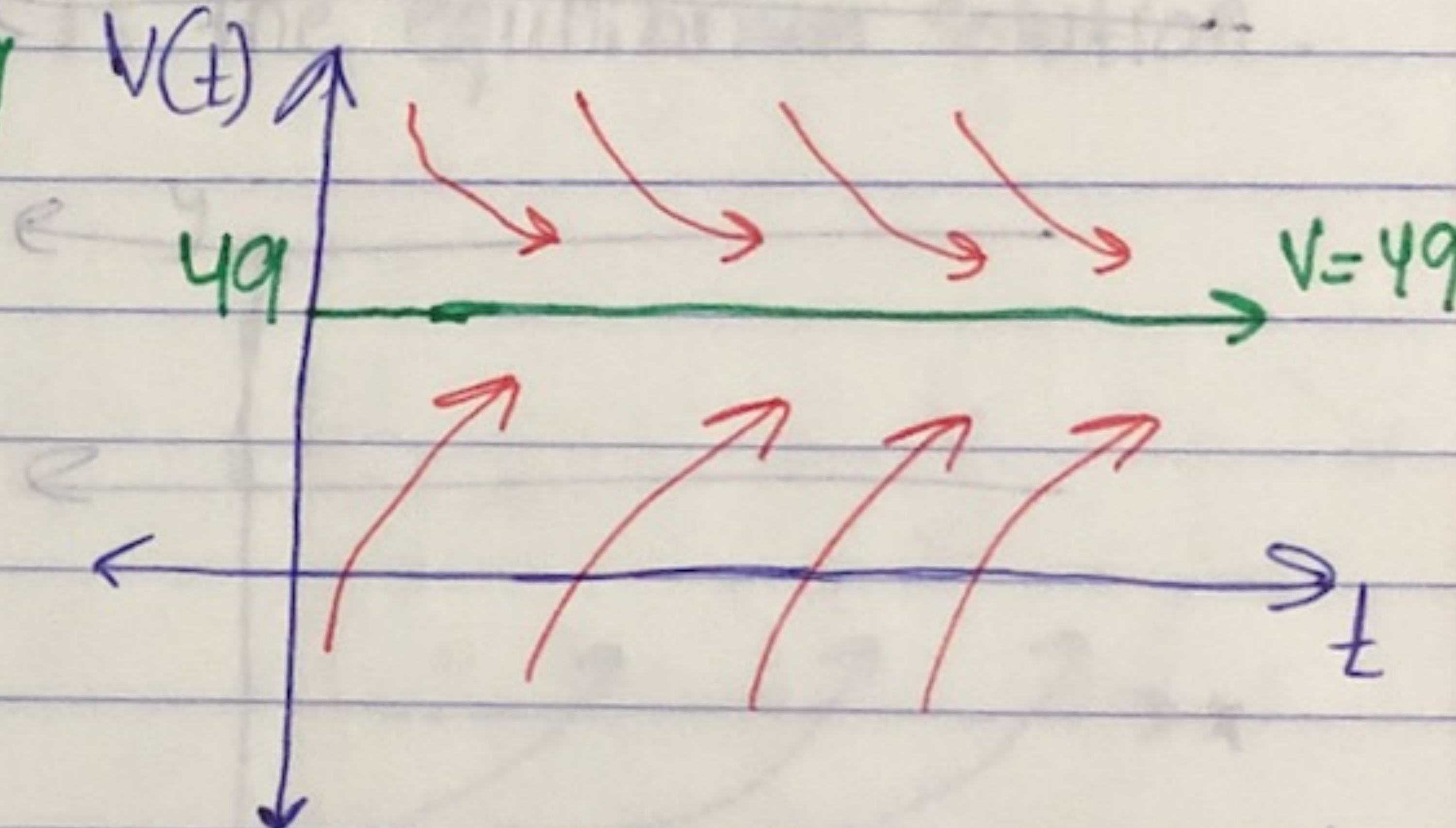
If $V_0 > 49$, take $V_0 = 50$

$$\frac{dV}{dt} = 9.8 - \frac{1}{5}(50) = -0.2 < 0 \searrow$$

as $t \to \infty$, $V(t) \to 49$

or $\lim_{t \to \infty} V(t) = 49$

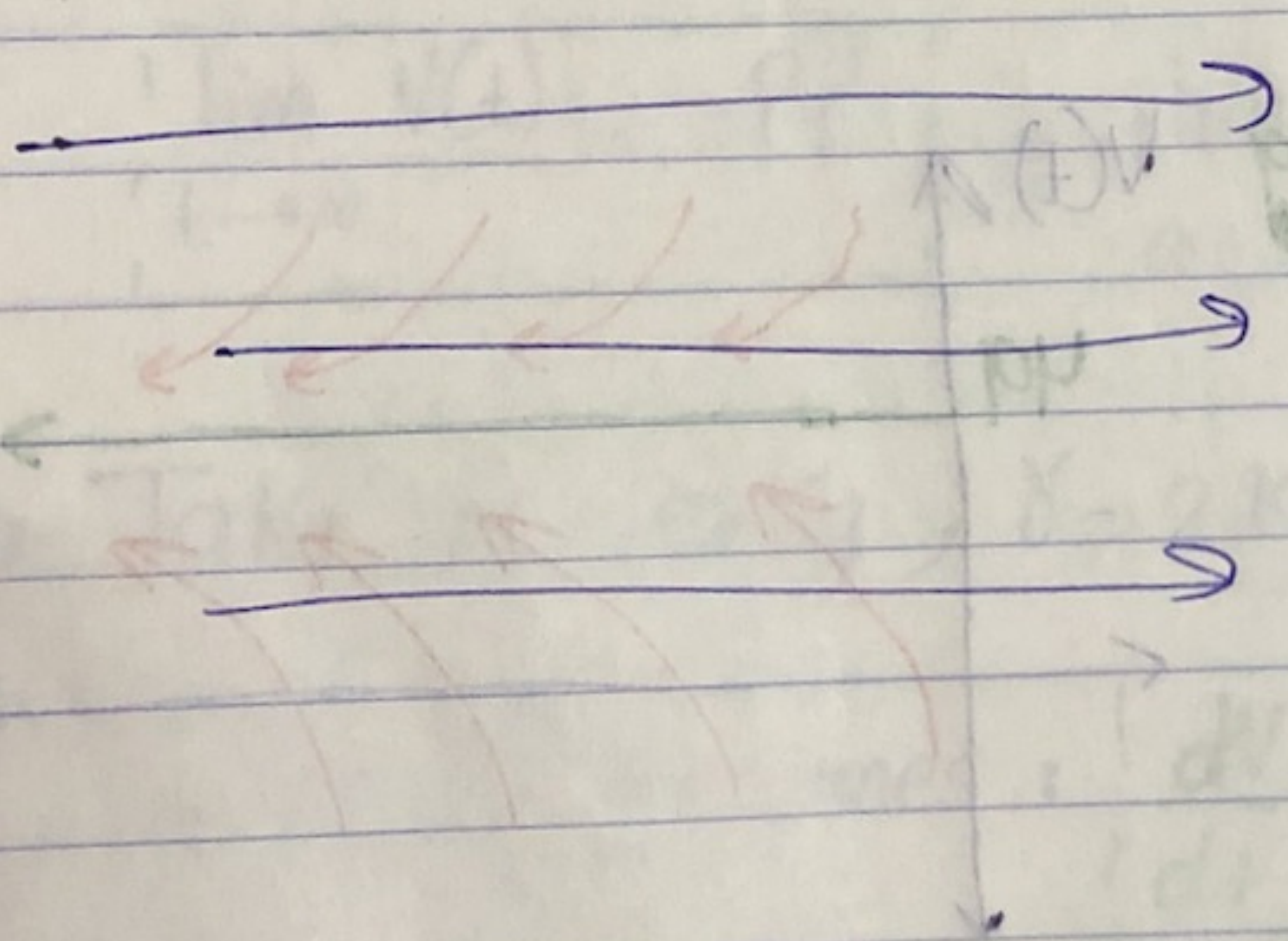
behaviour



RmK:- The solution doesn't depend on the intial condition V_0

Ex:- Draw a direction field for a given D.E.
Determine the behaviour of y as $t \to \infty$

① $\dfrac{dy}{dt} = 3 + 2y$



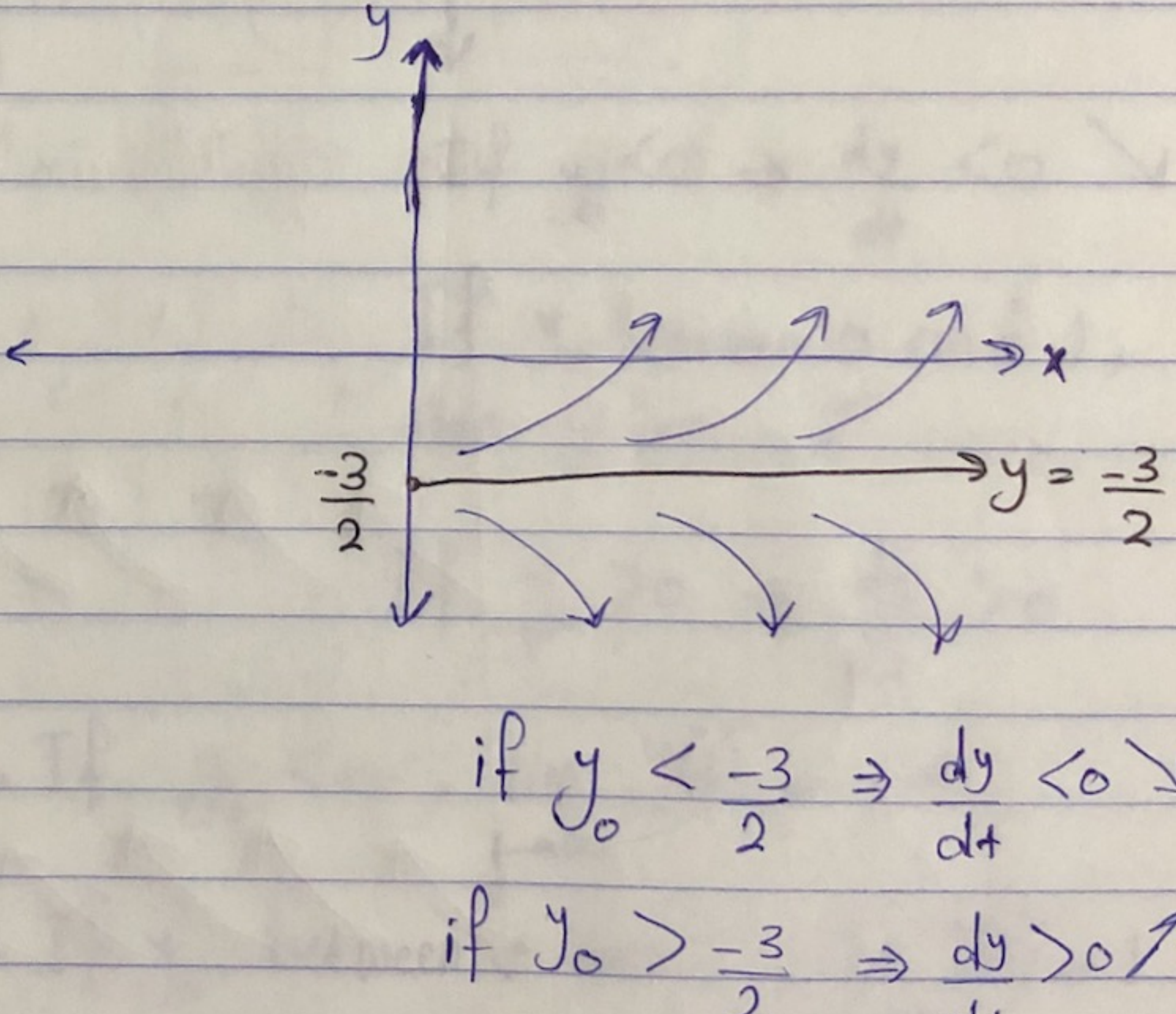
1.1 Continue

Behaviour $\lim_{t \to \infty} y(t) = ?$

Ex: Discuss the behaviour of y as $t \to \infty$

① $\frac{dy}{dt} = 3 + 2y$

$\frac{dy}{dt} = 0 \rightarrow y = \frac{-3}{2}$ is the equilibrium solution.



if $y_0 < \frac{-3}{2} \Rightarrow \frac{dy}{dt} < 0 \searrow$

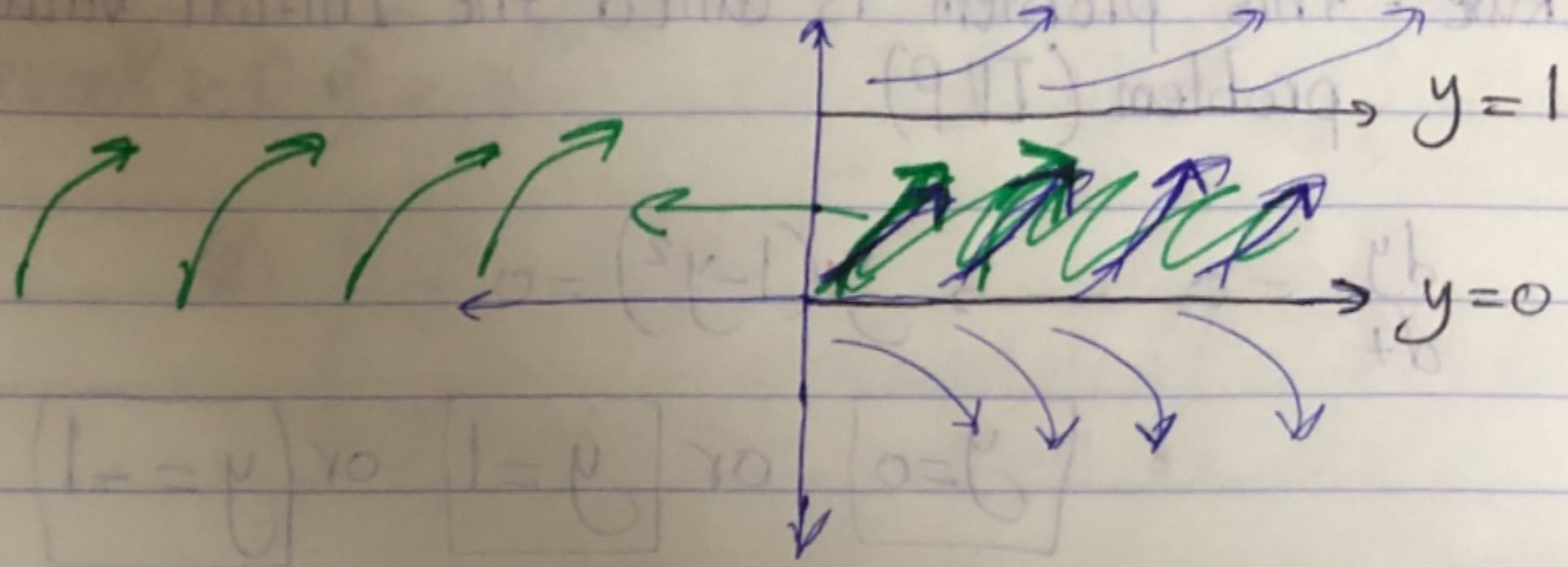
if $y_0 > \frac{-3}{2} \Rightarrow \frac{dy}{dt} > 0 \nearrow$

* If the intial value less than $\frac{-3}{2}$, then $y \to -\infty$ as $t \to \infty$

* If the intial value is greater than $\frac{-3}{2}$, then $y \to \infty$ as $t \to \infty$. Thus, the solution y diverges ~~[illegible]~~ from $\frac{-3}{2}$.

② $y' = y(y-1)^2$

$y' = 0 \rightarrow \boxed{y=0}$ or $\boxed{y=1}$ are the equilibruim solution.



If $y_0 < 0 \Rightarrow \frac{dy}{dt} < 0$ ↘

If y_0 between 0 and 1, then $y' > 0$ ↗

If $y_0 > 0 \Rightarrow \frac{dy}{dt} > 0$

Behaviour :- • If $y_0 < 0$, $\lim_{t\to\infty} y(t) = -\infty$

• If y_0 between 0 and 1, $\lim_{t\to\infty} y(t) = 1$

• If y_0 ~~betw~~ >1, $\lim_{t\to\infty} y(t) = \infty$

Notice that in the ex., the behaviour depends on the intial value.

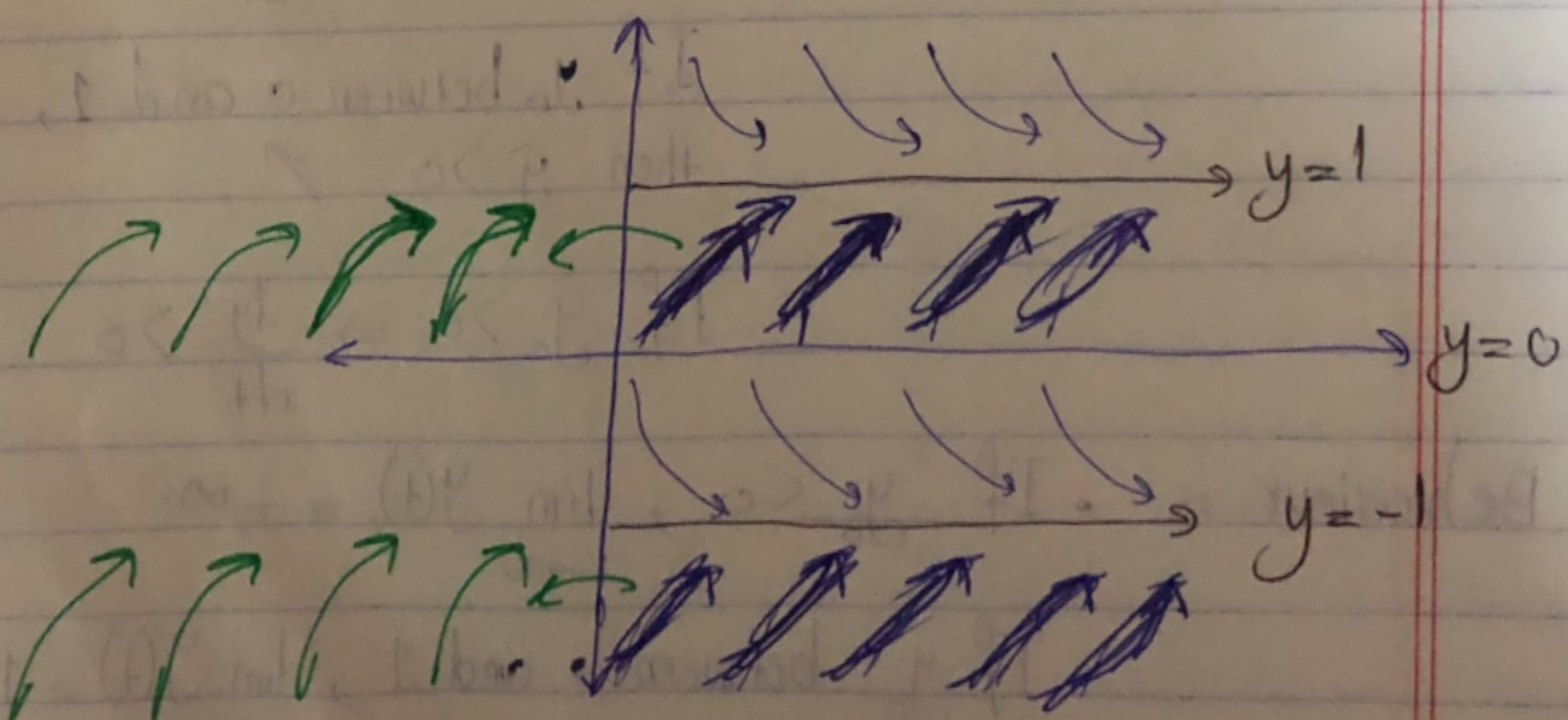
③ $\begin{cases} \dfrac{dy}{dt} = y(1-y^2) \\ y(0) = 2 \end{cases}$

Rule:- this problem is called the Initial value problem (IVP)

$$\frac{dy}{dt} = 0 \longrightarrow y(1-y^2) = 0$$

$$\boxed{y=0} \text{ or } \boxed{y=1} \text{ or } \boxed{y=-1}$$

are the equilibruim solution



Behaviour :-

- If the initial value is positive, then $y \longrightarrow 1$ as $t \to \infty$ "Stable"

Ex: (Field Mice and Owls)

Consider a population of field mice who inhabit a certain ~~rud~~ rural area. Assume that the mouse population increases at a rate proportional to the current popultion. Formulate a diff. eq. for this model

Sol:- P: population
t: time

$$\frac{dP}{dt} \propto P$$

$$\boxed{\frac{dP}{dt} = rP}$$ — ~~(*)~~, where r: rate constant (r is positive)
growth rate

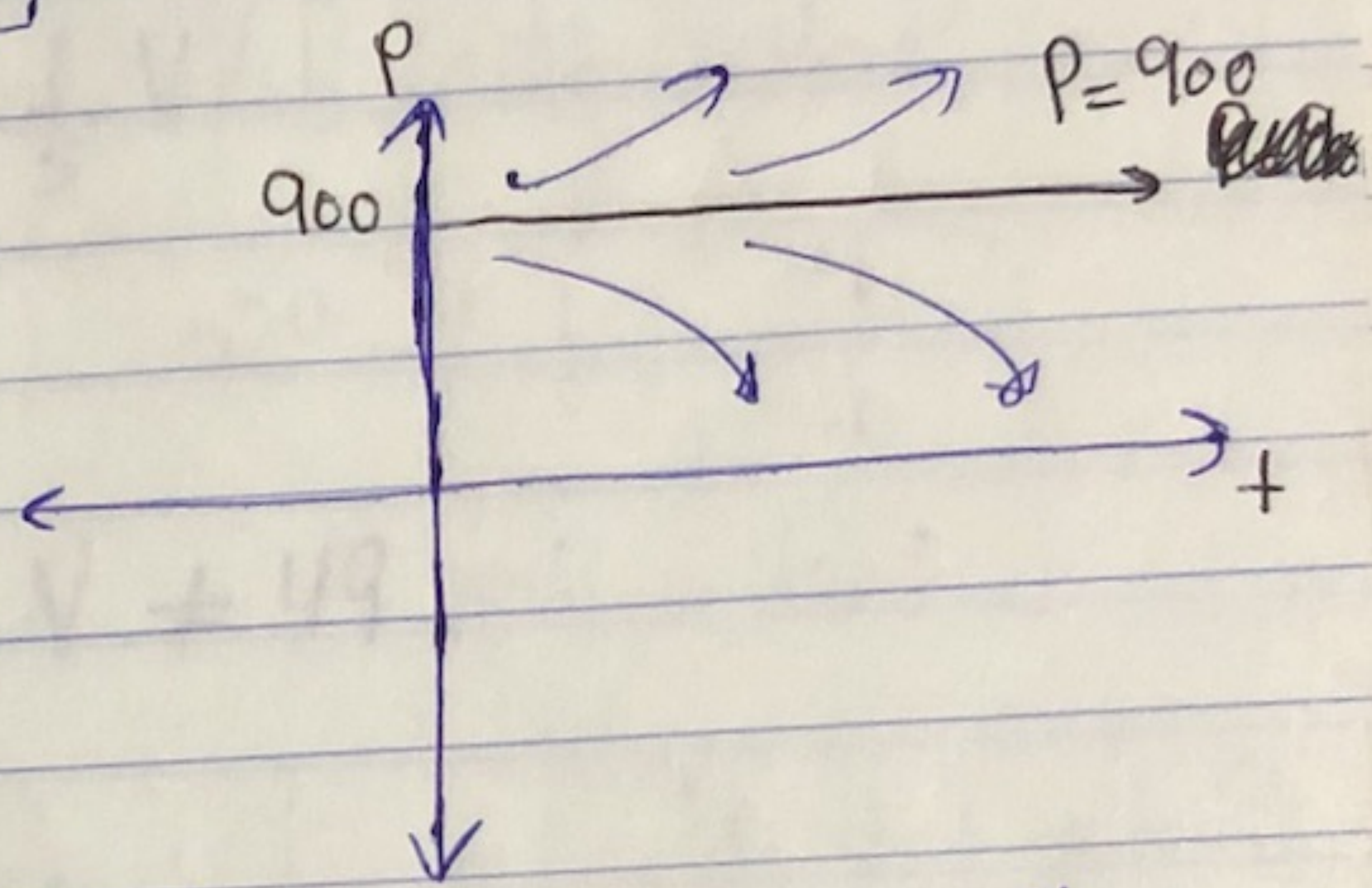
Ex: In the last ex., assume that $r = 0.5$/month and Owls are pro and they kill 15 field mice/day. So the D.E becomes :-

$$\boxed{\frac{dP}{dt} = \frac{1}{2}P - 450}$$ — (**)

Question:- Discuss the behaviour of $P(t)$ in (**)

$$\frac{dP}{dt} = 0 \longrightarrow \frac{1}{2}P - 450 = 0$$

$$\boxed{P = 900}$$



- If $P_0 > 900 \rightarrow \lim_{t \to \infty} P(t) = \infty$
- If $P_0 < 900 \rightarrow \lim_{t \to \infty} P(t) = 0$

(Not $-\infty$ since $P(t)$ is a Popultion)

(1.2) :- Solutions of some D.E's :-

consider $\boxed{\dfrac{dy}{dt} = ay - b}$, where a, b are constants

Ex: solve $\begin{cases} \dfrac{dV}{dt} = 9.8 - \dfrac{1}{5}V \\ V(0) = 850 \end{cases}$

Sol :- $\displaystyle\int \frac{dV}{9.8 - \frac{1}{5}V} = \int dt$, $V \neq 49$

$$-5 \int \frac{-\frac{1}{5}\, dV}{9.8 - \frac{1}{5}V} = t + C$$

$$-5 \ln\left|9.8 - \frac{1}{5}V\right| = t + C$$

$$\ln\left|9.8 - \frac{1}{5}V\right| = \frac{-t}{5} + C$$ generally constant

$$\left|9.8 - \frac{1}{5}V\right| = e^{\frac{-t}{5}} \cdot e^{C}$$

$$9.8 - \frac{1}{5}V = \pm e^{C}\, e^{\frac{-t}{5}}$$, $A = \pm e^{C}$

$$\frac{1}{5}V = 9.8 - A\, e^{\frac{-t}{5}}$$

$$V = 49 + \alpha\, e^{\frac{-t}{5}}$$, $\alpha = -5A$

$$V(0) = 49 + \alpha = 850$$

$$\alpha = 801$$

$$V(t) = 49 + 801\, e^{-\frac{t}{5}}$$

$$\lim_{t \to \infty} V(t) = 49$$

Ex:- Solve the IVP :-

$$\begin{cases} \frac{dP}{dt} = \frac{1}{2} P - 450 \\ P(0) = 800 \end{cases}$$

Sol :- $\int \frac{\frac{1}{2}\, dP}{\frac{1}{2} P - 450} = \int \frac{1}{2} dt$, $P \neq 900$

$$\ln \left| \frac{1}{2} P - 450 \right| = \frac{1}{2} t + C$$

$$\frac{1}{2} P - 450 = \pm e^{C} e^{\frac{1}{2}t}$$

$$\frac{1}{2} P = A e^{\frac{1}{2}t} + 450 \quad , A = \pm e^{C}$$

$$P = \alpha e^{\frac{1}{2}t} + 900 \quad , \alpha = 2A$$

$$P(t) = 900 + \alpha e^{\frac{1}{2}t}$$

$$P(0) = 800 \longrightarrow \boxed{\alpha = -100}$$

$$\longrightarrow P(t) = 900 - 100\, e^{\frac{1}{2}t} \longrightarrow$$

* notice that $\lim_{t \to \infty} P(t) = 0$ if P is population,

Otherwise $\lim_{t \to \infty} P(t) = -\infty$ if P is any fun. of t.

Ex:- Consider a population of field mice that is $\boxed{\frac{dP}{dt} = rP}$. Find the constant r if the population doubles in 20 days.

Sol: $\int \frac{dP}{P} = \int r \, dt$

$\ln|P| = rt + C \quad (P > 0)$

$P = e^{rt} e^{C}$

$\rightarrow P = A e^{rt}, \quad A = e^{C}$

$P(t) = A e^{rt}$

$P(20) = A e^{20r} = 2A$

$\rightarrow 20r = \ln 2$

$\rightarrow r = \frac{1}{20} \ln 2$

(Remark:- $P(0) = A$, $P(20) = 2A$)

1.3:- Classification of D.Es :-

① ODEs : Ordinary Differential Equations

The unknown function depends on one independent ~~value~~ variable and only ~~all~~ ordinary derivatives apper in the equation.

ex. $\frac{dP}{dt} = \frac{1}{2}P - 450$ is ode

ex. $\frac{dV}{dt} = 9.8 - \frac{1}{5}V$ is ode

ex. $\frac{d^3y}{dx^3} + x\frac{dy}{dx} = x^3$ is ode

② PDEs : Partial Differential ~~Eq~~ Equations

The unknown function depends on two or more independent variables and partial derivatives apper in the eq. .

ex. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ is p.d.e (Heat eq.)

($u = u(x, t)$ is the unknown function) ex. $\frac{a^2 \partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ is p.d.e (wave eq.)

($u = u(x,t)$ is the unknown function)

(3) System of D.E's :-

Two or more ~~unknow~~ unknown functions.

ex. (Lotka-Volterra) Eqs

$$\begin{cases} \frac{dx}{dt} = \alpha x - axy \\ \frac{dy}{dt} = -cy + \gamma xy \end{cases}$$

Rmk. ODEs (chapters "1-6")

PDEs (Math 332)

System (Chapter 7)

* The order of a D.E :-

is the order of the heighest derivative that appears in the eq.

* Linearity "Linear and nonlinear D.E's" :-

The Ode $\boxed{F(t, y, y', \ldots, y^{(n)})} = 0$ is said to be linear if F is a linear function of "$y, y', y'', \ldots, y^{(n)}$" → (*)

(*) An equation that is not of the form (*) is not linear and called nonlinear

Ex:- Determine the order of the d.e's - state whether the eq. is linear or nonlinear

① $y' - 2y = t^3$ " 1st order lin O.d.e

② $t^2 y'' + t y' - (\sin t) y = 0$

linear / order = 2

③ $\frac{dP}{dt} + t P^2 = 0$

nonlinear / order = 1

2nd order

(4) $\frac{d^2q}{dt^2} + \cos(q+t) = 0$ nonlinear / order = 2

(5) $y'' - e^{y} y' = 5x$ nonlinear / order = 2

لأنه يجب على كل من $(y''/y'/y \ldots)$ أن يعتمدوا فقط على constant أو independent variable ، فإذا اعتمدت على نفسها أو مشتقتها تصبح nonlinear.

(6) $\frac{d^3x}{dt^3} + \left(\frac{d^2x}{dt^2}\right)^5 + t^6 = x$

nonlinear / order = 3
third order

(7) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} = x^2 + y^2$

P.d.e / linear / order = 2

(8) $u \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = t$

P.d.e / nonlinear / 2nd order

(9) $(x + e^{y})\, dy = dx$

first case: $\frac{dy}{dx} = \frac{1}{(x + e^{y})}$ order = 1

"non linear in y"

second case: $\frac{dx}{dy} = x + e^{y}$

linear $\underline{\text{in } x}$ (order = 1)

* A solution of the O.d.e (*), on (α, β) is a function $y = \phi(t)$ such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist and satisfy (*) for every $t \in (\alpha, \beta)$

Ex. verify $\boxed{y = 3t + t^2}$ is a solution of the

d.e :- $\boxed{ty' - y = t^2}$

Sol:- $y' = 3 + 2t$

L.H.S :- $ty' - y = t(3+2t) - y$

$$= 3t + 2t^2 - 3t - t^2$$

$$= t^2 = \text{R.H.S}$$

Ex. verify that $\boxed{y = (\cos t)\ln(\cos t) + t\sin t}$

a solution of $\boxed{y'' + y = \sec t}$, $0 < t < \pi/2$

Sol:- $y' = (-\sin t)\ln(\cos t) + \cancel{\cos t} \cdot \frac{-\sin t}{\cancel{\cos t}} + \sin t + t\cos t$

$$= -(\sin t)\ln(\cos t) + t\cos t$$

$$y'' = -\cos t \cdot \ln\cos t - \sin t \cdot \frac{-\sin t}{\cos t} + \cos t - t\sin t$$

$\Rightarrow y'' = -\cos t \ln \cos t + \dfrac{\sin^2 t}{\cos t} + \cos t \cancel{2} - t \sin t$

L.H.S :- $y'' + y = \dfrac{\sin^2 t}{\cos t} + \cos t$

$= \dfrac{\sin^2 t + \cos^2 t}{\cos t}$

$= \dfrac{1}{\cos t}$

$= \sec t = R.H.S$

حسوبيك

Homework :- ① verify $y = \dfrac{\ln t}{t^2}$, $t > 0$ is a solution of $t^2 y'' + 5t y' + 4y = 0$

② Verify $y = \left(e^{-t^2} \left(\int_0^t e^{-r^2} dr\right) + e^{t^2}\right)$ is a solution of $y' - 2ty = 1$

* Some important questions dealing with D.E's :-

① Is there a solution of the d.e ? (Existence)

② If it exist, is it unique ? (uniquence)

③ How to find the solution if it exist ? Ans. ch.2 - Ch7