Ch. 2 :-

Chapter two :- 1st Order Differntial Equations

The general form of 1st order d.e is:

$$\boxed{\frac{dy}{dt} = f(t,y)} \text{ --- } (*)$$

* سيتم شرح 2.2 قبل 2.1 ....

(2.2) :- Seperable D.E's :-

If we can write $f(t,y)$ in $(*)$ as $f(t,y) = h(t)\, g(y)$

then $(*)$ becomes $\boxed{\frac{dy}{dt} = h(t)\, g(y)}$ --> $(**)$

$$\longrightarrow \int \frac{dy}{g(y)} = \int h(t)\, dt \quad , \; g(y) \neq 0$$

then $(**)$ is called seperable D.E's

Ex. Solve the IVP :-

$$\begin{cases} x\, e^{2x+\cos y} + (\sin y)\dfrac{dy}{dx} = 0 \\ y(0) = \pi/2 \end{cases}$$

Sol:- $x\, e^{2x} \cdot e^{\cos y}\, dx = -(\sin y)\, dy$

$$\int x e^{2x}\, dx = \int -\sin y\; e^{-\cos y}\, dy$$

$$-\int x e^{2x}\, dx = \int e^{-\cos y} \sin y\, dy$$

أجزاء

by parts

by substitutions

let $u = -\cos y$

$du = \sin y\, dy$

x	e^{2x}
1	$+\ \frac{e^{2x}}{2}$
0	$-\ \frac{e^{2x}}{4}$

$$\Rightarrow -x\frac{e^{2x}}{2} + \frac{e^{2x}}{4} = +\int e^{u}\, du$$

$$-\frac{x}{2} e^{2x} + \frac{e^{2x}}{4} = +e^{u} + C$$

$$-\frac{x}{2} e^{2x} + \frac{e^{2x}}{4} = e^{-\cos y} + C$$

$$y(0) = \frac{\pi}{2} \longrightarrow C = -\frac{3}{4}$$

* Recall:- the first order D.E has the form:-

$$\boxed{\frac{dy}{dt} = f(t,y)} \quad \text{---} (*)$$

If we can write $f(t,y) = g(t).h(y)$ then (*) becomes $\frac{dy}{dt} = g(t).h(y)$, and this eq. is seperable d.e.

Ex:- Solve the IVP:- $\begin{cases} \frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \\ y(0) = -1 \end{cases}$

then find the interval to which the solution is defined.

Sol:- $\frac{dy}{dx} = (3x^2+4x+2)\left(\frac{1}{2(y-1)}\right)$

so it's sep. D.E.

$$\int (2y-2)\,dy = \int (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$y(0) = -1 \Rightarrow 1+2 = C \rightarrow \boxed{C=3}$$

$$\longrightarrow \boxed{y^2 - 2y = x^3 + 2x^2 + 2x + 3}$$

↳ is the implicit ~~gen~~ solution

to find y explicitely, we complete the square in y.

$$\longrightarrow y^2 - 2y + \left(\frac{-2}{2}\right)^2 = x^3 + 2x^2 + 2x + 3 + \left(\frac{-2}{2}\right)^2$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$$

$$y(0) = 1 \pm \sqrt{-1 + 2 - 2 + 4}$$

$$y(0) = 1 \pm \sqrt{4} \quad \begin{matrix} \nearrow 3 & \times \\ \searrow -1 & \checkmark \end{matrix}$$

$$x^3 + 2x^2 + 2x + 4 \geq 0$$
$$x^2(x+2) + 2(x+2) \geq 0$$
$$(x^2+2)(x+2) \geq 0$$

$$\longrightarrow x + 2 \geq 0$$
$$x \geq -2$$

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}, \quad x \geq -2$$

Homework:- ① Solve the IVP:

$$\begin{cases} (x - xy^2) + (8y - x^2y)\,y' = 0 \\ y(2) = 2 \end{cases}$$

then find the interval in which the solution is defined.

Final Ans:- $y = \sqrt{\dfrac{20 - x^2}{8 - x^2}}$, $x \in (-\sqrt{8}, \sqrt{8})$

② Solve $\dfrac{dy}{dx} = \dfrac{xy - 3x - y + 3}{xy - 2x + 4y - 8}$, $y \neq 2$, $y \neq -3$

☆ Homogeneous D.E's "Excercises 2.2"

consider the 1^{st} order d.e

$$\boxed{\frac{dy}{dx} = f(x,y)} \text{ --- } (☆)$$

If we can write $f(x,y) = F\left(\frac{y}{x}\right)$, then (☆) becomes:

$$\boxed{\frac{dy}{dx} = F\left(\frac{y}{x}\right)} \text{ --- } (☆☆)$$

and this is called Homogeneous D.E

let $\frac{y}{x} = V$ or $y = Vx$ --- (i)

$$\frac{dy}{dx} = V \cdot 1 + x\frac{dV}{dx} \text{ --- (ii)}$$

(i) and (ii) into (☆☆):-

$$V + x\frac{dV}{dx} = F(V)$$

and this is a sep. d.e and then we Know how to solve it.

Ex : Solve the D.E :- $\frac{dy}{dx} = \frac{y-x}{y+x}$ ---- (A)

Sol :- $\frac{dy}{dx} = \frac{x\left(-1+\frac{y}{x}\right)}{x\left(1+\frac{y}{x}\right)}$

$= \frac{\frac{y}{x}-1}{\frac{y}{x}+1} = F\left(\frac{y}{x}\right)$

Now, let $y = Vx$ ---- (1)

$\frac{dy}{dx} = V + x\frac{dV}{dx}$ ---- (2)

(1) and (2) into (A) :-

$V + x\frac{dV}{dx} = \frac{Vx - x}{Vx + x}$

$\rightarrow V + x\frac{dV}{dx} = \frac{V-1}{V+1}$

$x\frac{dV}{dx} = \frac{V-1}{V+1} - V$

$\rightarrow x\,dV = \left(\frac{\cancel{V}-1-V^2-\cancel{V}}{V+1}\right)dx$

$\int\left(\frac{V+1}{1+V^2}\right)dV = \int -\frac{1}{x}\,dx$

$\rightarrow \int\frac{dV}{1+V^2} + \int\frac{V}{1+V^2}\,dV = -\int\frac{1}{x}\,dx$

$$\tan^{-1} v + \frac{1}{2}\ln(1+v^2) = -\ln|x| + C$$

$$\tan^{-1}\left(\frac{y}{x}\right) + \frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) = -\ln|x| + C$$

is an implicit sol.

Ex:- Solve the IVP: $\begin{cases} y' = \dfrac{3y^2 - x^2}{2xy} & \text{--- (B)} \\ y(1) = 2 \end{cases}$

write y as a fun. of x and find the interval in which the solution is defined.

Sol:- $$\frac{dy}{dx} = \frac{3y^2}{2xy} - \frac{x^2}{2xy}$$

$$= \frac{3}{2}\frac{y}{x} - \frac{1}{2}\frac{1}{\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right)$$

$\therefore$ homog.

let $y = vx$ --- (1)

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \text{ --- (2)}$$

~~[illegible]~~

1 and 2 into B :-

$$V + x\frac{dV}{dx} = \frac{3}{2}V - \frac{1}{2V}$$

$$\rightarrow x\frac{dV}{dx} = \frac{3}{2}V - \frac{1}{2V} - V$$

$$\rightarrow x\frac{dV}{dx} = \frac{2V^2-2}{4V}$$

$$\rightarrow \int \frac{4V}{2V^2-2}\,dV = \int \frac{1}{x}\,dx$$

$$\ln|V^2-1| = \ln|x| + C \quad \cancel{y(1)=2}$$

$$\ln\left|\frac{y^2}{x^2}-1\right| = \ln|x| + C \;,\; y(1)=2$$

$$\ln|4-1| = \ln 1 + C$$

$$\boxed{C = \ln 3}$$

$$\rightarrow \frac{y^2}{x^2} - 1 = \pm 3x$$

since $y(1)=2 \rightarrow \frac{y^2}{x^2} - 1 = +3x$

$$\cancel{y^2 = 3x^3 + x^2}$$

$$y = +x\sqrt{3x+1}$$

$$3x+1 \geq 0 \rightarrow x \geq -\frac{1}{3}$$

$\rightarrow \quad y = x\sqrt{3x+1} \quad , x \geq \frac{-1}{3}$

Homework: (1) Solve :-

$$x\,dy = \left(x e^{\frac{y}{x}} + y + x\right) dx$$

(2) Solve :-

$$(y \ln y - y \ln x + y)\,dx = x\,dy$$

2.2: Seperable of homog. D.E's (Done) ✗

2.1: Linear D.E's (1^{st} order):-

The general form of 1^{st} order linear d.e's is :-

$$\frac{dy}{dt} + P(t)\,y = q(t) \quad \text{---} \ (*)$$

If $P(t)$ and $q(t)$ are constants, $(*)$ becomes seperable D.E.

but we need to solve $(*)$ for any fun. $P(t)$ and $q(t)$

Multiply both sides of $(*)$ by $M(t) \geq 0$

$$\longrightarrow \quad M(t)\frac{dy}{dt} + P(t)M(t)\,y = M(t)\,q(t) \quad \text{---} \ (1)$$

and we Know:

$$\frac{d}{dt}\big(M(t)\,y(t)\big) = M(t)\frac{dy}{dt} + y\frac{dM}{dt} \quad \text{---} \ (2)$$

comparing R.H.S of (2) with L.H.S of (1), we get

$$\boxed{\frac{dM}{dt} = P(t)\,M(t)} \quad \text{---} \ (3)$$

③ $\Rightarrow \int \frac{dM}{M} = \int P(t)\, dt$

$$\ln(M(t)) = \int P(t)\, dt \quad \text{"} C=0 \text{"}$$

$$\longrightarrow \boxed{M(t) = e^{\int P(t)\, dt}}$$ $\longrightarrow$ مهم / حفظ

Thus Eq ① becomes :-

$$\frac{d}{dt}\left(M(t)\, y(t)\right) = M(t)\, q(t)$$

$$\longrightarrow M(t)\, y(t) = \int M(t)\, q(t)\, dt + C$$

$$\longrightarrow y(t) = \frac{1}{M(t)}\left[\int M(t)\, q(t)\, dt + C\right]$$

① ~~where $M(t) = e$~~

$\longrightarrow$ where $M(t) = e^{\int P(t) dt}$ is the general sol. of equation (*)

Ex: Solve the initial value problem (IVP) :-

$$\begin{cases} t\dfrac{dy}{dt} + 2y = 4t^2 \\ y(1) = 2 \end{cases}$$

Find the interval of this d.e :

Sol:- * Standard :-

$\dfrac{dy}{dt} + \dfrac{2}{t} y = 4t$, is linear in y with $P(t) = \dfrac{2}{t}$, $q(t) = 4t$

$$M(t) = e^{\int P(t)\,dt} = e^{\int \frac{2}{t}\,dt}$$

$$= e^{2\ln(t)} = e^{\ln t^2} = t^2 .$$

$$y = \frac{1}{M}\left[\int M.q \,dt + C\right]$$

$$y = \frac{1}{t^2}\left[\int 4t^3\,dt + C\right]$$

$$y = \frac{1}{t^2}\left(t^4 + C\right)$$

$$y = t^2 + \frac{c}{t^2} .$$

$y(1) = 2 \rightarrow 2 = 1 + \frac{c}{1}$

$$\boxed{C = 1}$$

$$\therefore \boxed{y = t^2 + \frac{1}{t^2}}$$

Domain:-

$(-\infty, 0) \cup (0, \infty)$

initial condition "$y(1)=2$" in the $(0, \infty)$ interval

$t_0 = 1$

0

$\rightarrow$

* The largest interval in which the solution is certain to exist is $(0, \infty)$

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$\therefore$ therefore, $y = t^2 + \frac{1}{t^2}$, $t > 0$

Ex:- $\frac{dy}{dx} = \frac{y}{ye^y - 2x}$

notice that this equation is not linear in y, but it is linear in x since;-

$$\frac{dx}{dy} = \frac{ye^y - 2x}{y}$$

$$\frac{dx}{dy} = e^y - \frac{2x}{y}$$

$$\frac{dx}{dy} + \frac{2}{y}x = e^y \quad \text{(Standard)}$$

$P(y) = \frac{2}{y}$, $q(y) = e^y$

$M(y) = e^{\int \frac{2}{y} dy} = y^2$

$$x(y) = \frac{1}{M(y)} \left[\int M(y)\, q(y)\, dy + C \right]$$

$$= \frac{1}{y^2} \left[\int y^2 e^y \, dy + C \right]$$

اشتقاق		تكامل
y^2		e^y
$2y$	$(+)$	e^y
2	$(-)$	e^y
0	$(+)$	e^y

$$\therefore x = \frac{1}{y^2} \left[y^2 e^y - 2y e^y + 2e^y + c \right]$$

$$\therefore x = e^y - \frac{2e^y}{y} + \frac{2e^y}{y^2} + \frac{c}{y^2}$$

☆ Bernoulli's Eq. :- (Part of 2.4)

Exc. → [27 − 31]

$$\frac{dy}{dt} + P(t) y = q(t) y^n$$

if $n = 0$ ⟶ linear

if $n = 1$ ⟶ seperable

for $n \neq 0$, $n \neq 1$ ⟶

$$y^{-n}\frac{dy}{dt} + P(t)\,y^{1-n} = q(t)$$

let $V = y^{1-n}$ --- (1)

$$\frac{dV}{dt} = (1-n)\,y^{-n}\frac{dy}{dt} \quad \text{--- (2)}$$

(1) and (2) into

$$\frac{1}{1-n}\frac{dV}{dt} + P(t)V = q(t)$$

$$\boxed{\frac{dV}{dt} + (1-n)P(t)V = (1-n)q(t)}$$

↳ which is linear in V

Ex:- Solve : $t^2\frac{dy}{dt} + 2ty - y^3 = 0$, $t > 0$

$$\frac{dy}{dt} + \frac{2}{t}y = \frac{1}{t^2}(y^3)$$ Bernoulli's Eq. with $n=3$

let $V = y^{-2}$

$$y = V^{-\frac{1}{2}} \quad \text{-- (1)}$$

$$\boxed{\frac{dy}{dt} = \frac{-1}{2}V^{-\frac{3}{2}}\frac{dV}{dt}} \quad \text{— (2)}$$

① and ② into Ⓑ :-

$$-\frac{1}{2} V^{-\frac{3}{2}} \frac{dV}{dt} + \frac{2}{t} V^{-\frac{1}{2}} = \frac{1}{t^2} V^{-\frac{3}{2}}$$

$$\boxed{\frac{dV}{dt} + \frac{-4}{t} V = -\frac{2}{t^2}}$$ linear in V

$$\mu = e^{\int -\frac{4}{t} dt} = t^{-4}$$

$$V = \frac{1}{t^{-4}} \left[\int t^{-4} \cdot \frac{-2}{t^2} \, dt + C \right]$$

$$V = \frac{1}{t^{-4}} \left[- \int 2t^{-6} \, dt + C \right]$$

$$V = \frac{1}{t^{-4}} \left[\frac{-2}{-5} t^{-5} + C \right]$$

$$V = \frac{2}{5} t^{-1} + C t^4$$

$$\rightarrow V = \frac{2}{5t} + C t^4$$

$$\rightarrow y^{-2} = \frac{2}{5t} + C t^4$$

Ex) Solve $\boxed{\frac{dy}{dx} + \frac{2y}{6x+1} = \frac{-3x^2}{(6x+1)y^2}}$ (*)

is Bernolli's eq. with $\boxed{n=-2}$

let $v = y^{1-n} = y^3$ or $\boxed{y = v^{\frac{1}{3}}}$ — ①

$$\boxed{\frac{dy}{dx} = \frac{1}{3} v^{\frac{-2}{3}} \frac{dv}{dx}} \quad ②$$

Setting ① & ② into (*)

$$\frac{1}{3} v^{\frac{-2}{3}} \frac{dv}{dx} + \frac{2 v^{\frac{1}{3}}}{6x+1} = \frac{-3x^2}{6x+1} v^{\frac{-2}{3}}$$

divide by $\frac{1}{3} v^{\frac{-2}{3}}$ ⇒

$$\frac{dv}{dt} + \frac{6}{6x+1} v = \frac{-9x^2}{6x+1} \quad \text{lin. in } v.$$

$$\mu(x) = e^{\int \frac{6}{6x+1} dx} = e^{\ln|6x+1|} = 6x+1$$

$$v = \frac{1}{6x+1}\left[\int \left(\frac{-9x^2}{6x+1}\right)(6x+1)\,dx + C\right]$$

$$y^3 = \frac{\left[-\frac{9x^3}{3} + C\right]}{6x+1} \Rightarrow y = \sqrt[3]{\frac{-3x^3+C}{6x+1}}$$

H.W's ① Solve $3(1+x^2)\frac{dy}{dx} = 2xy(y^3-1)$

Bernolli + seperable.

② Solve the IVP $\begin{cases} \frac{dy}{dx} = \frac{x^2+y^2}{xy} \\ y(e) = 2e \end{cases}$

homog. & Bernolli $(n=-1)$

③ $(t^2+1)\frac{dy}{dt} = 4ty + 4t\sqrt{y}$

seperable & Bernoulli's eq. $(n=\frac{1}{2})$

The end of Quiz #1+2