Eq③

$\psi_x = e^y \cdot \frac{1}{x} - y \cdot \frac{1}{1+x^2} + g'(x)$

Eq①

$= \frac{e^y}{x} - \frac{y}{1+x^2}$

$\Rightarrow g'(x) = 0$

$\Rightarrow \boxed{g(x) = C_1}$ — ④

page ③:

Finally, ④ into ③ gives:

$C_2 = e^y \ln x - y \tan^{-1} x + 2y + C_1$

or $e^y \ln x - y \tan^{-1} x + 2y = C$

is the solution of our problem.

Non exact Made exact

Thm 2.6.2 Consider $M(x,y)\,dx + N(x,y)\,dy = 0$ (*)

Suppose $M_y \neq N_x$ (not exact)

- If $\frac{M_y - N_x}{N(x,y)} = f(x)$ "x alone"

$\Rightarrow I(x) = e^{\int f(x)\,dx}$

at a rate of 3 L/min. of the well-stirred mixture flows out at a rate of 1 L/min. Find the quantity of salt, $Q(t)$

Sol. $\frac{dQ}{dt}$ = rate in - rate out

$= (1)(3) - \left(\frac{Q}{10+3t-1t}\right)(1)$

$$\frac{dQ}{dt} = 3 - \frac{Q}{10+2t}, \quad Q(0)=10,$$

$$\frac{dQ}{dt} + \frac{1}{10+2t}Q = 3, \quad Q(0)=10$$

$$\mu(t) = e^{\int \frac{1}{10+2t}dt} = \sqrt{10+2t}$$

$$Q(t) = \frac{1}{\sqrt{10+2t}}\left[\int 3\sqrt{10+2t}\,dt + C\right]$$

(page 5)

$$= \frac{1}{\sqrt{10+2t}}\left[\frac{3(10+2t)^{\frac{3}{2}}}{\frac{3}{2}(2)} + C\right]$$

$$\boxed{Q(t) = 10+2t + \frac{C}{\sqrt{10+2t}}}$$

$$Q(0) = 10 + \frac{C}{\sqrt{10}} = 10$$

$$\Rightarrow \boxed{C=0}$$

$$\therefore \boxed{Q(t) = 10+2t}$$

20

Ex 3. A ball with mass 0.15K is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance.

$\frac{dQ}{dt} + \frac{3}{100}Q = \frac{3}{4}$ lin. in Q

$\mu = e^{\int \frac{3}{100}dt} = e^{\frac{3}{100}t}$

$Q(t) = e^{-\frac{3}{100}t}\left[\int \frac{3}{4} e^{\frac{3}{100}t} dt + C\right]$

$= e^{-\frac{3}{100}t}\left[\frac{3}{4}\cdot\frac{100}{3} e^{\frac{3}{100}t} + C\right]$

$\boxed{Q(t) = 25 + C e^{-\frac{3}{100}t}}$

$Q(0) = 25 + C = 50 \Rightarrow C = 25$

$\therefore \boxed{Q(t) = 25 + 25 e^{-\frac{3}{100}t}}$ (a)

(b) $Q_L = \lim_{t\to\infty} Q(t) = 25$

(c) $Q(t) = 25.5$

(page 4)

$25 + 25 e^{-\frac{3}{100}t} = 25.5$

$e^{-\frac{3}{100}t} = 0.02$

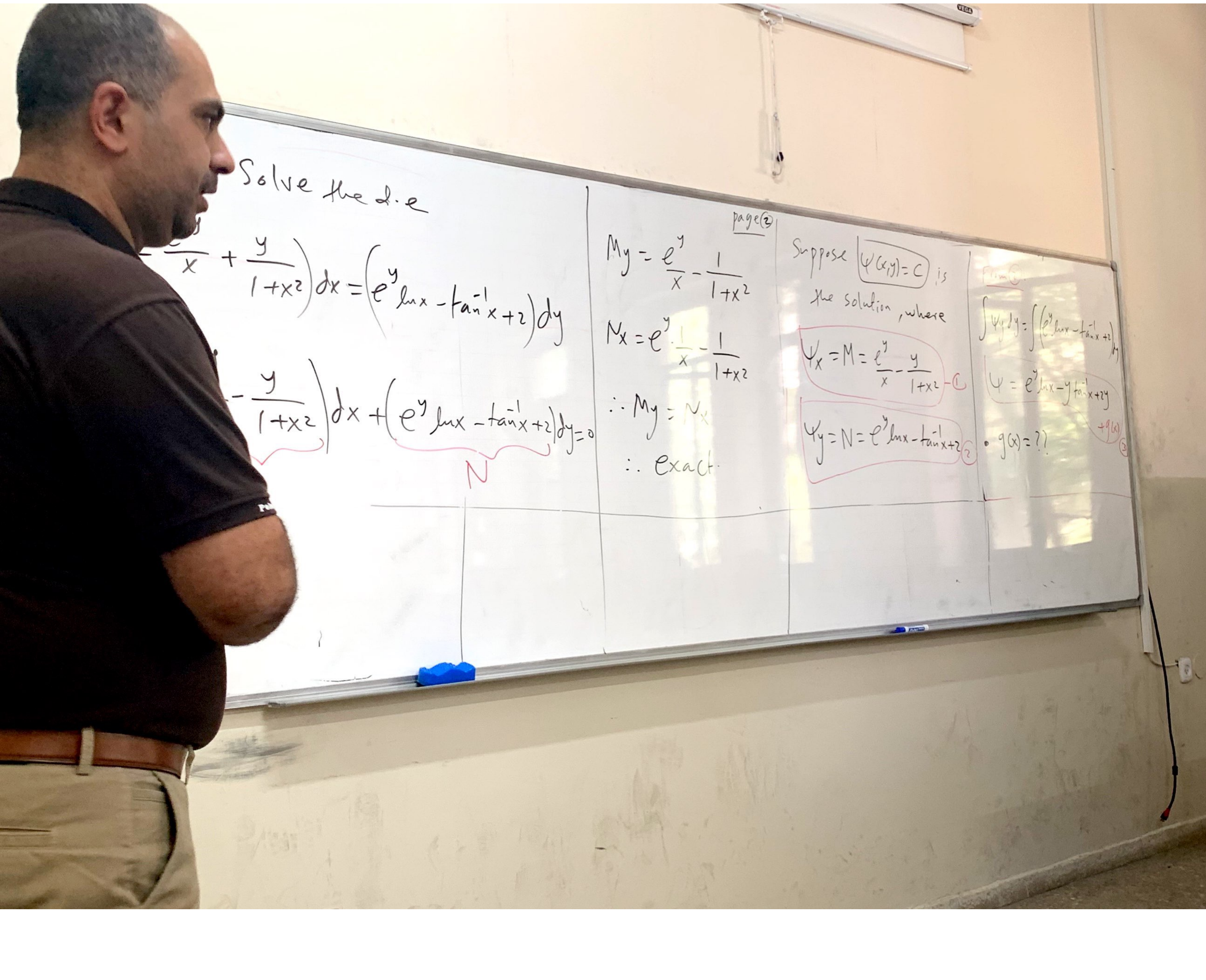
$\Rightarrow -\frac{3}{100}t = \ln(0.02)$

$\therefore \boxed{t = -\frac{100}{3}\ln(0.02)}$

$\approx \ldots$

Ex (2). (old exam)

A tank contains initially 10 g of salt dissolved in 10L of water. Water that contains 1g/L of salt flows into the tank


Solve the d.e
$\left(\frac{y}{1+x^2}\right)dx = \left(e^y \ln x - \tan^{-1}x + 2\right)dy$
$\left(- \frac{y}{1+x^2}\right)dx + \left(e^y \ln x - \tan^{-1}x + 2\right)dy = 0$
N
page 2
$M_y = \frac{e^y}{x} - \frac{1}{1+x^2}$
$N_x = e^y \cdot \frac{1}{x} - \frac{1}{1+x^2}$
$\therefore M_y = N_x$
$\therefore$ exact
Suppose $\psi(x,y) = C$ is the solution, where
$\psi_x = M = \frac{e^y}{x} - \frac{y}{1+x^2}$ — 1
$\psi_y = N = e^y \ln x - \tan^{-1}x + 2$ — 2
$\int \psi_y \, dy = \int \left(e^y \ln x - \tan^{-1}x + 2\right) dy$
$\psi = e^y \ln x - y \tan^{-1}x + 2y + g(x)$
• $g(x) = ??$

• If $\dfrac{M_y - N_x}{M(x,y)} = g(y)$ " y alone",

$$I(y) = e^{-\int g(y)\,dy}$$

In each case, we multiply (*) by $I(x)$ or $I(y)$, then the new

page (3):

eq. must be exact

ex. Solve the d.e

$$\boxed{(x+2)\sin y + (x\cos y)\,y' = 0}$$

Sol. $\underbrace{(x+2)\sin y}_{M}\,dx + \underbrace{(x\cos y)}_{N}\,dy = 0$

$\boxed{M_y = (x+2)\cos y}$, $\boxed{N_x = \cos y}$

$\Rightarrow M_y \neq N_x$ (not exact)

• $\dfrac{M_y - N_x}{N} = \dfrac{(x+2)\cos y - \cos y}{x\cos y} = \dfrac{x+1}{x}$

$\therefore I(x) = e^{\int \frac{x+1}{x}dx} = e^{\int (1+\frac{1}{x})dx} = xe^x, x>$

2.3 Modeling with First order d.e

(see also section 1.1 + 1.2)

Ex1. At time $t=0$ tank contains 50 bound of Salt dissolved in 100 gal of water, Assume that the water containing $\frac{1}{4}$ bound of Salt/gal is entering the tank at a rate of 3 gal/min & leave it at the same rate.

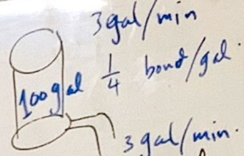
(page 3)

a) Find the amount of Salt $Q(t)$ in the tank at any time t.

b) Find the limiting amount of salt Q_L in the tank after every long time.

c) Find the time T when $Q(t)=25.5$

Sol.



Let $Q(t)$ be the amount of Salt in the tank at any time t.

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$= \left(\frac{1}{4}\right)(3) - \left(\frac{Q}{100}\right)(3)$$

$$\frac{dQ}{dt} = \frac{3}{4} - \frac{3}{100}Q, \quad Q(0)=50$$

Multiply (A) by $I(x) = xe^x$,

$$\boxed{x(x+2)e^x \sin y \, dx + x^2 e^x \cos y \, dy = 0} \quad (B)$$

let $\Psi(x,y) = C$

$$\boxed{\Psi_x = x(x+2)e^x \sin y} \quad - (1)$$

$$\boxed{\Psi_y = x^2 e^x \cos y} \quad - (2)$$

page (4)

Eq (2): $\int \Psi_y \, dy = \int x^2 e^x \cos y \, dy$

$$\boxed{\Psi = x^2 e^x \sin y + h(x)} \quad - (3)$$

$$\Psi_x \overset{Eq (3)}{=} (2x e^x + x^2 e^x) \sin y + h'(x)$$

$$\overset{eq (1)}{=} x^2 e^x \sin y + 2x e^x \sin y$$

$$h'(x) = 0 \Rightarrow \boxed{h(x) = c} \quad - (4)$$

(4) in (3): $x^2 e^x \sin y = C \quad , x > 0$

$$y = \sin^{-1}\left(\frac{c}{x^2 e^x}\right)$$

H.w do ex_3 in another method (Seperable)

Ex (4). Solve $(3x^2 y - 8x) y' = 4y - 2xy^2$

There is $\boxed{\psi(x,y) = C}$

Such that

$$\boxed{\psi_x = y\cos x + 2xe^y} - (1)$$

$$\boxed{\psi_y = \sin x + x^2 e^y - 1} - (2)$$

page (2)

Consider Eq (1):

$$\int \psi_x \, dx = \int \left(y\cos x + 2xe^y\right) dx$$

$$\boxed{\psi(x,y) = y\sin x + x^2 e^y + g(y)} \quad (3)$$

Now to find $g(y)$, we differentiate Eq (3) w.r.to y:

$$\psi_y = \cancel{\sin x} + \cancel{x^2 e^y} + g'(y) \overset{Eq(2)}{=} \cancel{\sin x} + \cancel{x^2 e^y} - 1$$

$$\Rightarrow g'(y) = -1$$

$$\therefore \boxed{g(y) = -y} - (4)$$

Put (4) into (3):

the solution is

$$\boxed{y\sin x + x^2 e^y - y = C}$$

2.6 Exact Equation and Integrating Factors

Th... Consider a d.e ... the form

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} \quad (1)$$

where M, N, M_y, N_x are all continuous on the Region $R: \alpha < x < \beta,\ \gamma < y < \delta$

page (1)

Then Eq (1) is Exact iff

$$M_y = N_x$$

that is there exists a function ψ satisfying

$$\boxed{\psi_x = M} \text{ and } \boxed{\psi_y = N}$$

iff $M_y = N_x$.

Pf. See the book.

Ex. (1) Solve the d.e

$$(y\cos x + 2xe^y)\,dx + (\sin x + x^2 e^y - 1)\,dy = 0$$

Sol. $M = y\cos x + 2xe^y$

$N = \sin x + x^2 e^y - 1$

$M_y = \cos x + 2xe^y$

$N_x = \cos x + 2xe^y$

$\Rightarrow M_y = N_x$ exact

2.6 (More examples)

Ex. 4 . Solve the d.e

$$(3x^2y - 8x)y' = 4y - 2xy^2$$

Sol. $\boxed{(2xy^2 - 4y)dx + (3x^2y - 8x)dy = 0}$ (*)

$M = 2xy^2 - 4y$

$N = 3x^2y - 8x$

$M_y = 4xy - 4$

$N_x = 6xy - 8$ $\Rightarrow M_y \neq N_x$ not exact.

- $\dfrac{M_y - N_x}{M} = \dfrac{4xy - 4 - 6xy + 8}{2xy^2 - 4y}$

$= \dfrac{-2xy + 4}{2xy^2 - 4y}$

page 11

$= \dfrac{-2\cancel{(xy-2)}}{2y\cancel{(xy-2)}} = \dfrac{-1}{y}$ "y alone"

$\therefore I(y) = e^{-\int -\frac{1}{y}dy} = y, \; y > 0$

multiply (*) by $I(y) = y$,

$\boxed{(2xy^3 - 4y^2)dx + (3x^2y^2 - 8xy)dy = 0}$ (**)

there exists $\psi(x,y)$ such that

$\boxed{\psi_x = M = 2xy^3 - 4y^2}$ — 1

$\boxed{\psi_y = N = 3x^2y^2 - 8xy}$ — 2

From 1, $\int \psi_x dx = \int (2xy^3 - 4y^2)dx$

$\boxed{\psi(x,y) = x^2y^3 - 4y^2x + g(y)}$ — 3

$+g'(y)$

xy

$\boxed{(y) = c}$

c

H.w's (old exams)

(5) $\begin{cases} y' = \dfrac{3y^2 - x^2}{2xy} \\ y(1) = 2 \end{cases}$.

Write y as a fun. of x.

Ans. $y = x\sqrt{3x+1}$

Homog. or Bernoulli's, nonexact made exact.

page (2)

(6) Solve the IVP

$\begin{cases} (2x^2+y) + (x^2y - x)y' = 0 \\ y(1) = 1 \end{cases}$.

(non exact made exact).

there

$\boxed{\psi_x = M}$

$\boxed{\psi_y = N}$

From (1),

$\psi(x,y)$

(More examples)

Newton's law of cooling

...ates that the temperature ... changes at rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings.

That is

$$\boxed{\frac{du}{dt} = -k(u-T)}$$

page (1)

$u(t)$: temperature of an object

T: " " surrounding (ambient)

k: positive constant

Ex. Spse that the temperature of a cup of coffee obeys Newton's law of cooling. If the coffee has a temp. of $90^\circ C$ when freshly poured, and 1 min. later has cooled to $85^\circ C$ in a room at $20^\circ C$, determine when the coffee reaches a temp. of 65°c.

Sol.

$$\begin{cases} \dfrac{du}{dt} = -k(u-20) \\ u(0) = 90, \; u(1) = 85 \end{cases}$$

We need $t = ??$ when $u(t) = 65$.

Sol.

$$\frac{du}{dt} + ku = 20k$$

lin. $p(t) = k$, $q(t) = 20k$

$$\mu(t) = e^{\int k\,dt} = e^{kt}$$

$$u(t) = \frac{1}{e^{kt}}\left[\int 20k\, e^{kt}\, dt + c\right]$$

page ②

$$= e^{-kt}\left[20e^{kt} + c\right]$$

$$\boxed{u(t) = 20 + c\,e^{-kt}}$$

$$u(0) = 20 + c = 90 \Rightarrow \boxed{c = 70}$$

$$u(1) = 20 + 70e^{-k} = 85$$

$$e^{-k} = \frac{65}{70} \Rightarrow \boxed{k = \ln\left(\frac{70}{65}\right)}$$

$$\therefore u(t) = 20 + 70\,e^{\left[-\ln\left(\frac{70}{65}\right)\right]t}$$

$$\boxed{u(t) = 20 + 70\left(\frac{70}{65}\right)^{-t}}$$

now,

$$65 = 20 + 70\left(\frac{70}{65}\right)^{-t}$$

$$\frac{45}{70} = \left(\frac{70}{65}\right)^{-t}$$

page(4)

$$\int \psi_y \, dy = \int x^2 e^x \cos y \, dy$$

$$= \boxed{x^2 e^x \sin y + h(x)} - ③$$

$$③ \quad (2x e^x + x^2 e^x) \sin y + h'(x)$$

$$\underset{=}{eq①} \; x^2 e^x \sin y + 2x e^x \sin y$$

$$h'(x) = 0 \Rightarrow \boxed{h(x) = c} - ④$$

④ in ③: $x^2 e^x \sin y = c \quad , x > 0$

$$y = \sin^{-1}\left(\frac{c}{x^2 e^x}\right)$$

H.w do ex_3 in another method (Seperable)

Ex④. Solve $(3x^2y - 8x)y' = 4y - 2xy^2$