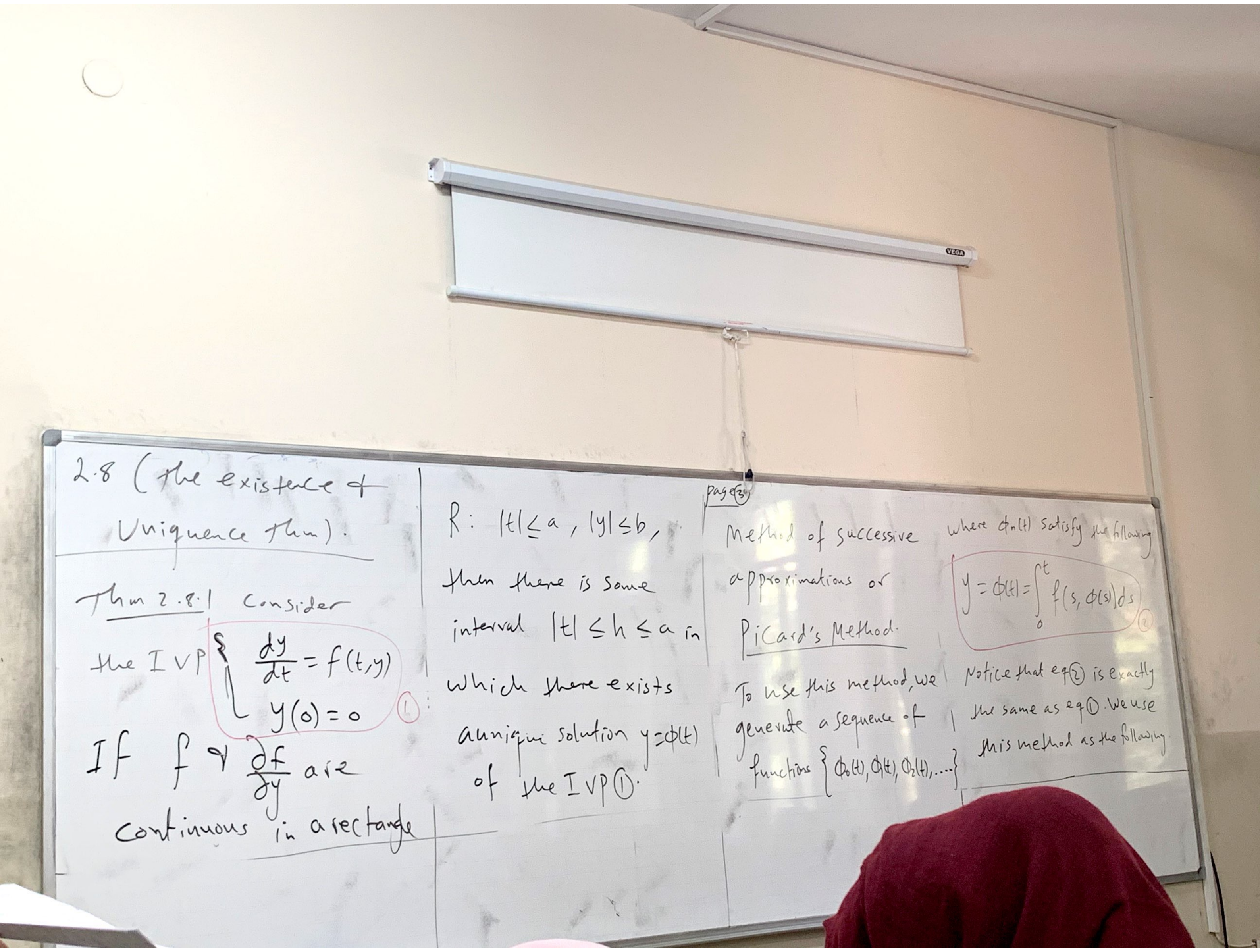


2.8 (the existence & Uniquence thm).

Thm 2.8.1 Consider the IVP $\begin{cases} \frac{dy}{dt} = f(t,y) \\ y(0) = 0 \end{cases}$ ①:

If f & $\frac{\partial f}{\partial y}$ are continuous in a rectangle

$R: |t| \leq a, |y| \leq b,$

then there is some interval $|t| \leq h \leq a$ in which there exists a unique solution $y = \phi(t)$ of the IVP ①.

page ③

Method of successive approximations or

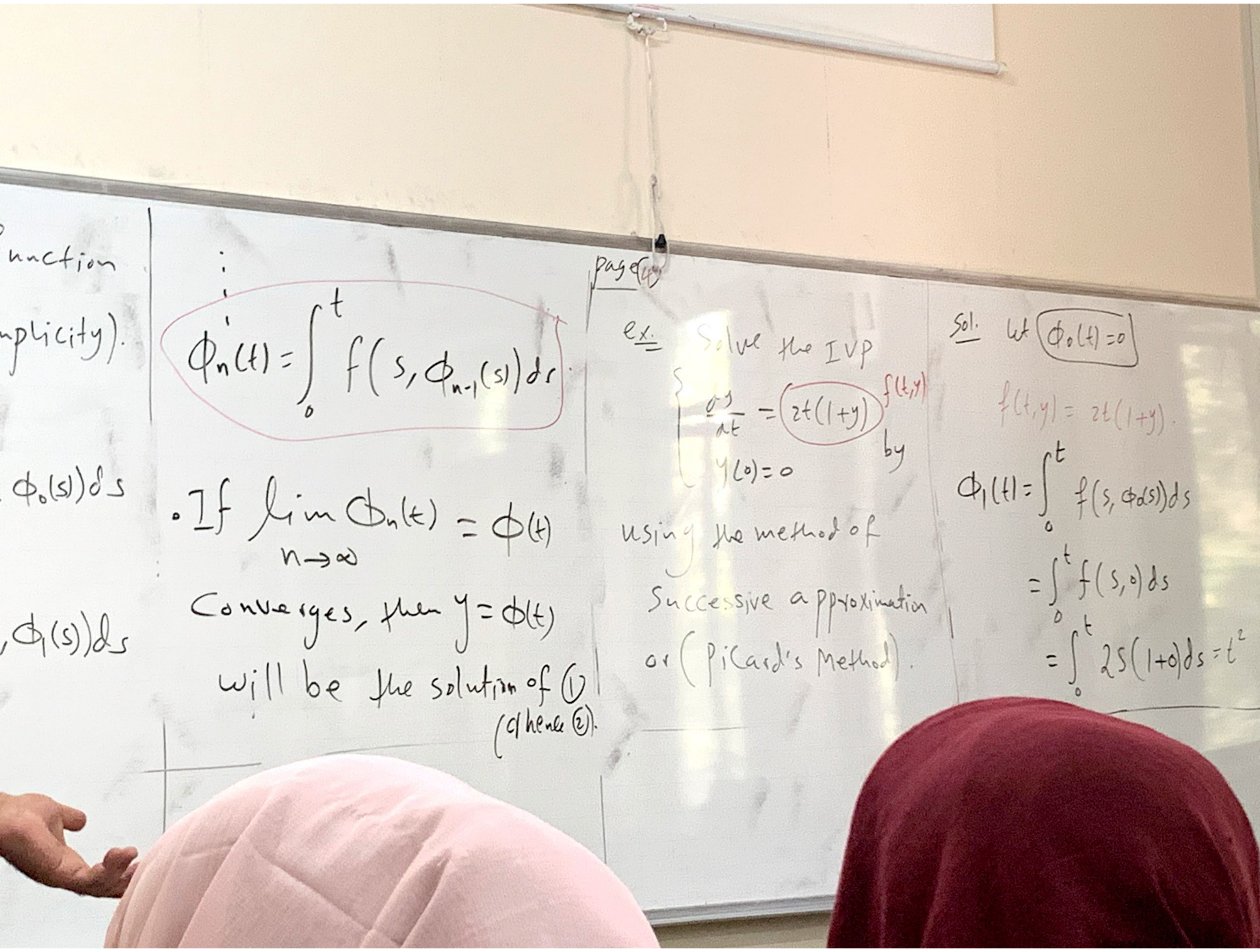
Picard's Method.

To use this method, we generate a sequence of functions $\{\phi_0(t), \phi_1(t), \phi_2(t), \dots\}$

where $\phi_n(t)$ satisfy the following

$$y = \phi(t) = \int_0^t f(s, \phi(s))\,ds \quad ②$$

Notice that eq② is exactly the same as eq①. We use this method as the following



unction

nplicity).

$\phi_0(s))ds$

$,\phi_1(s))ds$

⋮

$$\boxed{\phi_n(t) = \int_0^t f(s, \phi_{n-1}(s))ds}$$

- If $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ Converges, then $y = \phi(t)$ will be the solution of ① (&hence ②).

page (4)

ex. Solve the IVP

$$\begin{cases} \frac{dy}{dt} = \overset{f(t,y)}{\boxed{2t(1+y)}} \\ y(0) = 0 \end{cases}$$

by using the method of Successive approximation or (Picard's Method).

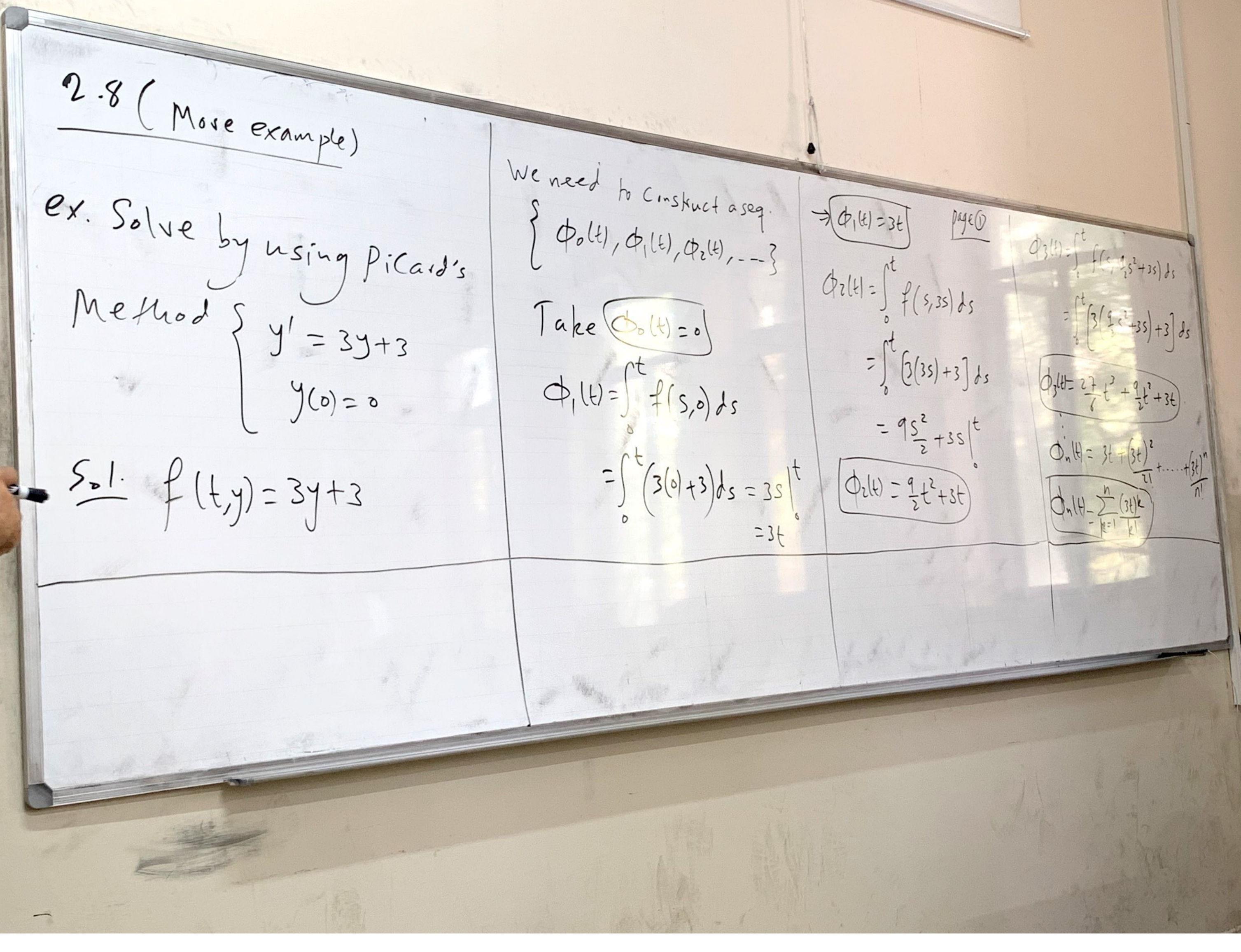
Sol. let $\boxed{\phi_0(t) = 0}$

$f(t,y) = 2t(1+y)$.

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))ds$$

$$= \int_0^t f(s,0)ds$$

$$= \int_0^t 2s(1+0)ds = t^2$$



2.8 (More example)

ex. Solve by using Picard's Method $\begin{cases} y' = 3y+3 \\ y(0) = 0 \end{cases}$

Sol. $f(t,y) = 3y+3$

We need to construct a seq. $\{\phi_0(t), \phi_1(t), \phi_2(t), \dots\}$

Take $\boxed{\phi_0(t) = 0}$

$\phi_1(t) = \int_0^t f(s,0)\,ds$

$= \int_0^t (3(0)+3)\,ds = 3s\Big|_0^t$

$= 3t$

$\rightarrow \boxed{\phi_1(t) = 3t}$ page ①

$\phi_2(t) = \int_0^t f(s,3s)\,ds$

$= \int_0^t [3(3s)+3]\,ds$

$= \frac{9s^2}{2} + 3s\Big|_0^t$

$\boxed{\phi_2(t) = \frac{9}{2}t^2 + 3t}$

$\phi_3(t) = \int_0^t f(s, \frac{9}{2}s^2 + 3s)\,ds$

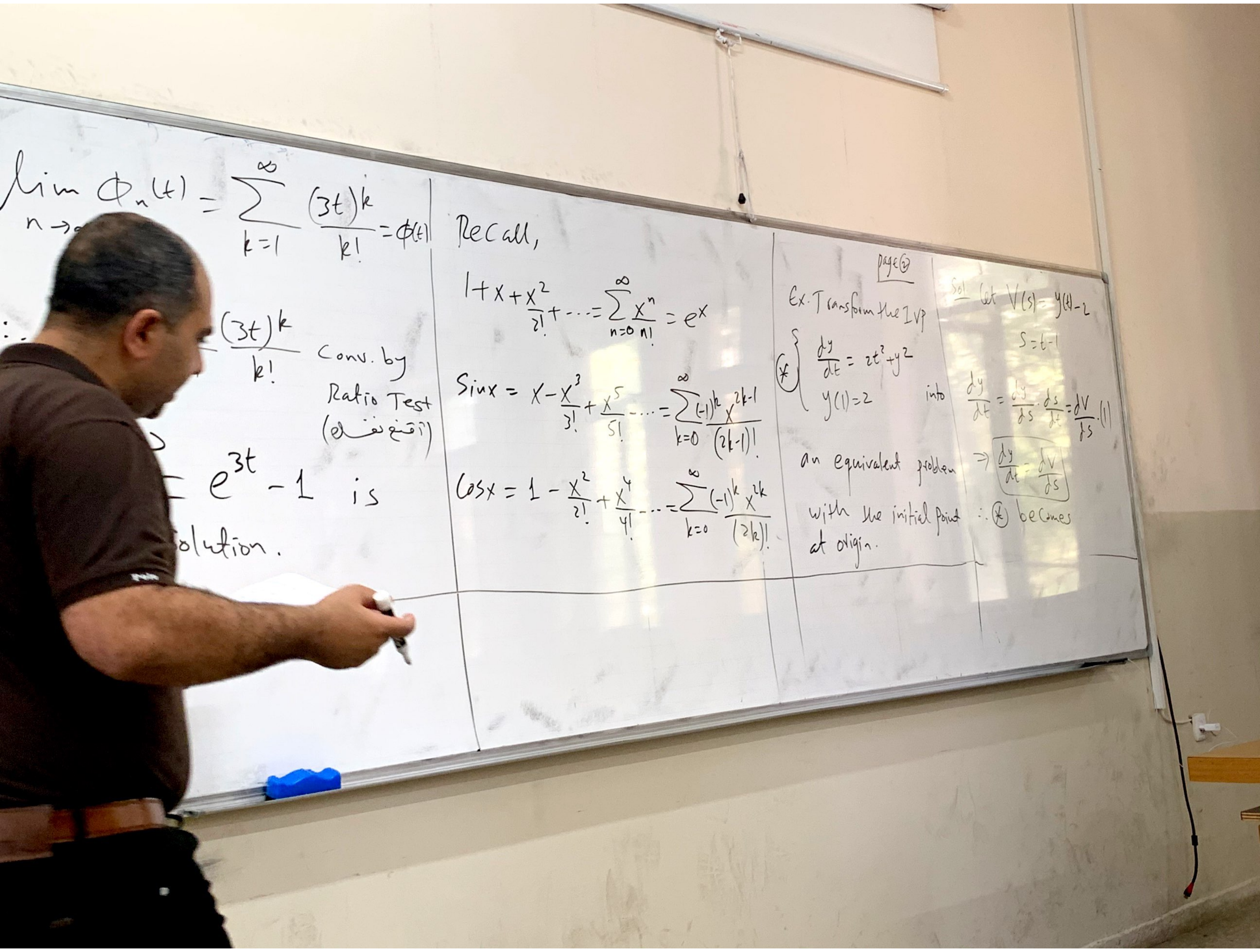
$= \int_0^t [3(\frac{9}{2}s^2 + 3s) + 3]\,ds$

$\boxed{\phi_3(t) = \frac{27}{6}t^3 + \frac{9}{2}t^2 + 3t}$

$\vdots$

$\phi_n(t) = 3t + \frac{(3t)^2}{2!} + \dots + \frac{(3t)^n}{n!}$

$\boxed{\phi_n(t) = \sum_{k=1}^{n} \frac{(3t)^k}{k!}}$



$$\lim_{n \to} \Phi_n(t) = \sum_{k=1}^{\infty} \frac{(3t)^k}{k!} = \phi(t)$$

$\therefore$ $\frac{(3t)^k}{k!}$ Conv. by Ratio Test (أقنع نفسك)

$= e^{3t} - 1$ is

olution.

Recall,

$$1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k-1}}{(2k-1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

page ②

Ex. Transform the IVP

$$(*) \begin{cases} \frac{dy}{dt} = 2t^2 + y^2 \\ y(1) = 2 \end{cases}$$

into an equivalent problem with the initial point at origin.

Sol let $V(s) = y(t) - 2$

$s = t - 1$

$$\frac{dy}{dt} = \frac{dy}{ds} \cdot \frac{ds}{dt} = \frac{dV}{ds} \cdot (1)$$

$$\Rightarrow \boxed{\frac{dy}{dt} = \frac{dV}{ds}}$$

$\therefore$ (*) becomes

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page ②

Ex. Transform the IVP

$$(*)\begin{cases} \dfrac{dy}{dt} = 2t^2 + y^2 \\ y(1) = 2 \end{cases}$$

into an equivalent problem with the initial point at origin.

Sol. let $\boxed{V(s) = y(t) - 2}$ ①

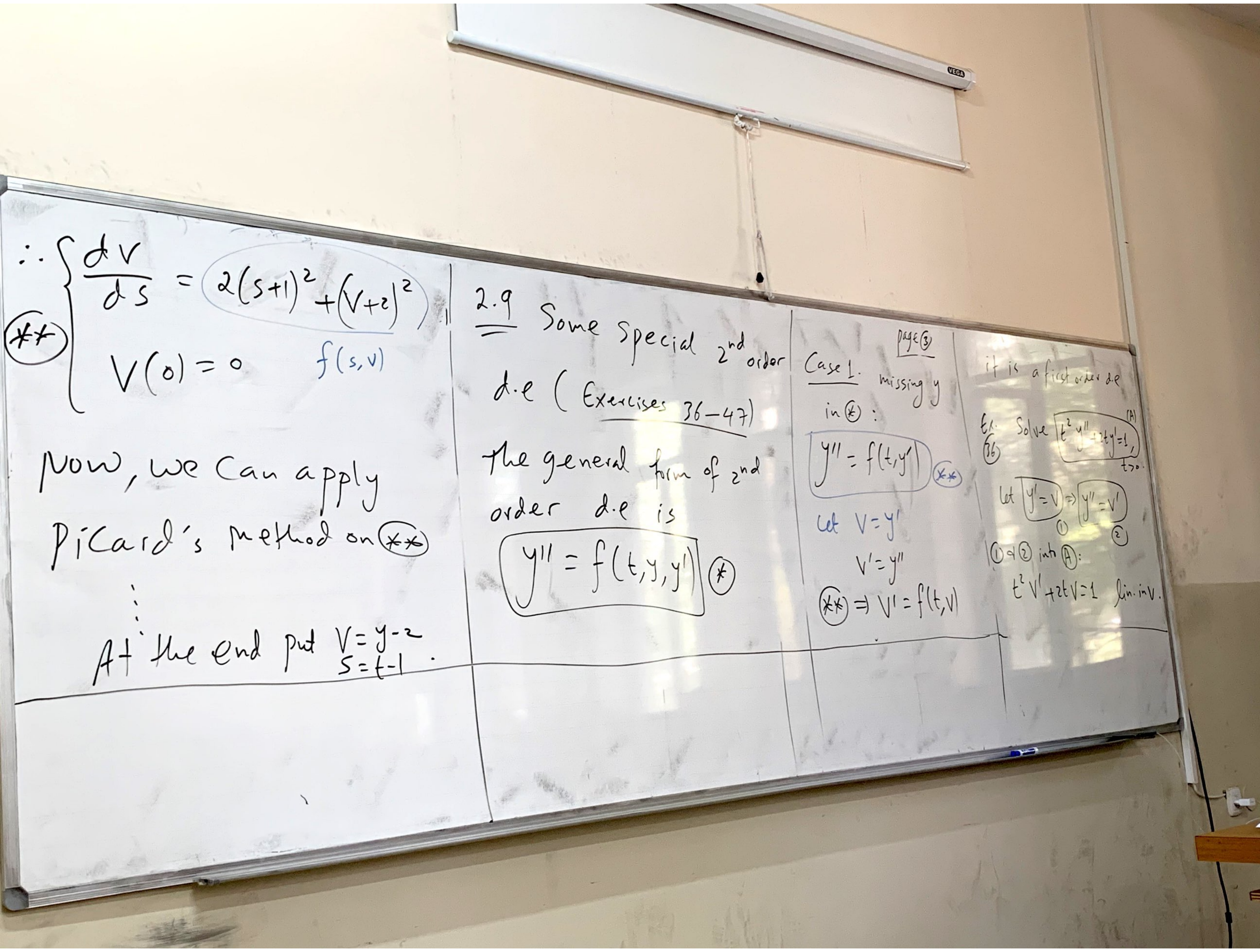
$\boxed{s = t - 1}$ ②

$$\frac{dy}{dt} = \frac{dy}{ds}\cdot\frac{ds}{dt} = \frac{dV}{ds}\cdot(1)$$

$$\Rightarrow \boxed{\frac{dy}{dt} = \frac{dV}{ds}} \text{ ③}$$

①, ②, ③ into (*),

∴ (*) becomes



$$\therefore \circledast\!\circledast \begin{cases} \dfrac{dv}{ds} = \underbrace{2(s+1)^2 + (v+2)^2}_{f(s,v)} \\ v(0) = 0 \end{cases}$$

Now, we can apply Picard's method on (**)

⋮

At the end put $V = y-2$, $s = t-1$.

2.9 Some special 2^{nd} order d.e (Exercises 36–47)

The general form of 2^{nd} order d.e is

$$\boxed{y'' = f(t, y, y')} \quad (*)$$

page ③

Case 1. missing y in (*):

$$\boxed{y'' = f(t, y')} \quad (**)$$

let $v = y'$

$v' = y''$

$(**) \Rightarrow v' = f(t, v)$

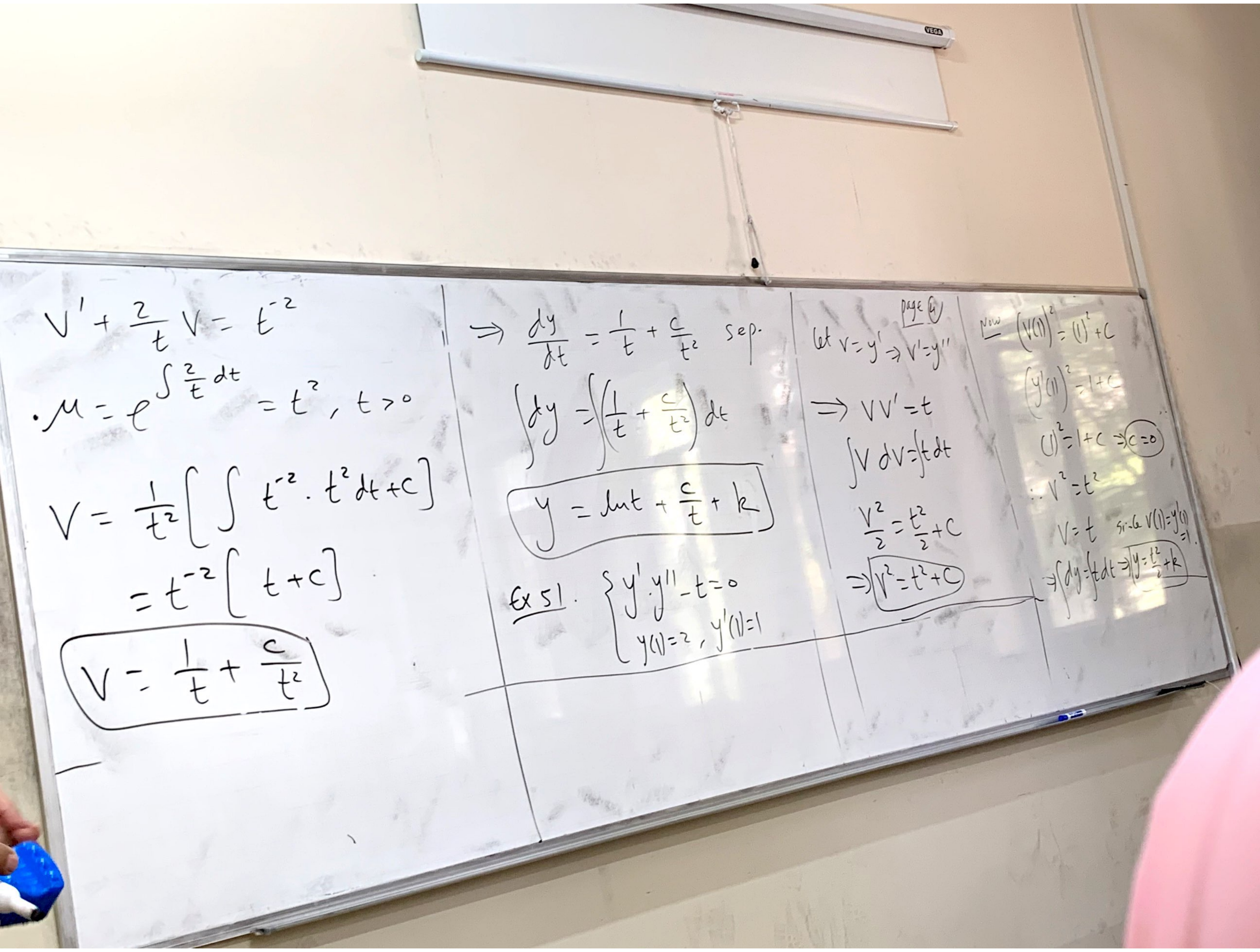
it is a first order d.e

Ex. ㊱ Solve $\boxed{t^2 y'' + 2ty' = 1}$ (A), $t > 0$.

let $\underbrace{\boxed{y' = v}}_{①} \Rightarrow \underbrace{\boxed{y'' = v'}}_{②}$

① & ② into (A):

$t^2 v' + 2t v = 1$ lin. in v.



$V' + \frac{2}{t} V = t^{-2}$

$\cdot \mu = e^{\int \frac{2}{t} dt} = t^2, \ t > 0$

$V = \frac{1}{t^2}\left[\int t^{-2} \cdot t^2 dt + c\right]$

$= t^{-2}\left[t + c\right]$

$\boxed{V = \frac{1}{t} + \frac{c}{t^2}}$

$\Rightarrow \frac{dy}{dt} = \frac{1}{t} + \frac{c}{t^2}$ sep.

$\int dy = \int\left(\frac{1}{t} + \frac{c}{t^2}\right) dt$

$\boxed{y = \ln t + \frac{c}{t} + k}$

Ex 51. $\begin{cases} y' \cdot y'' - t = 0 \\ y(1) = 2, \ y'(1) = 1 \end{cases}$

page ④

let $v = y' \Rightarrow v' = y''$

$\Rightarrow v v' = t$

$\int v \, dv = \int t \, dt$

$\frac{v^2}{2} = \frac{t^2}{2} + C$

$\Rightarrow \boxed{v^2 = t^2 + C}$

Now $(v(1))^2 = (1)^2 + C$

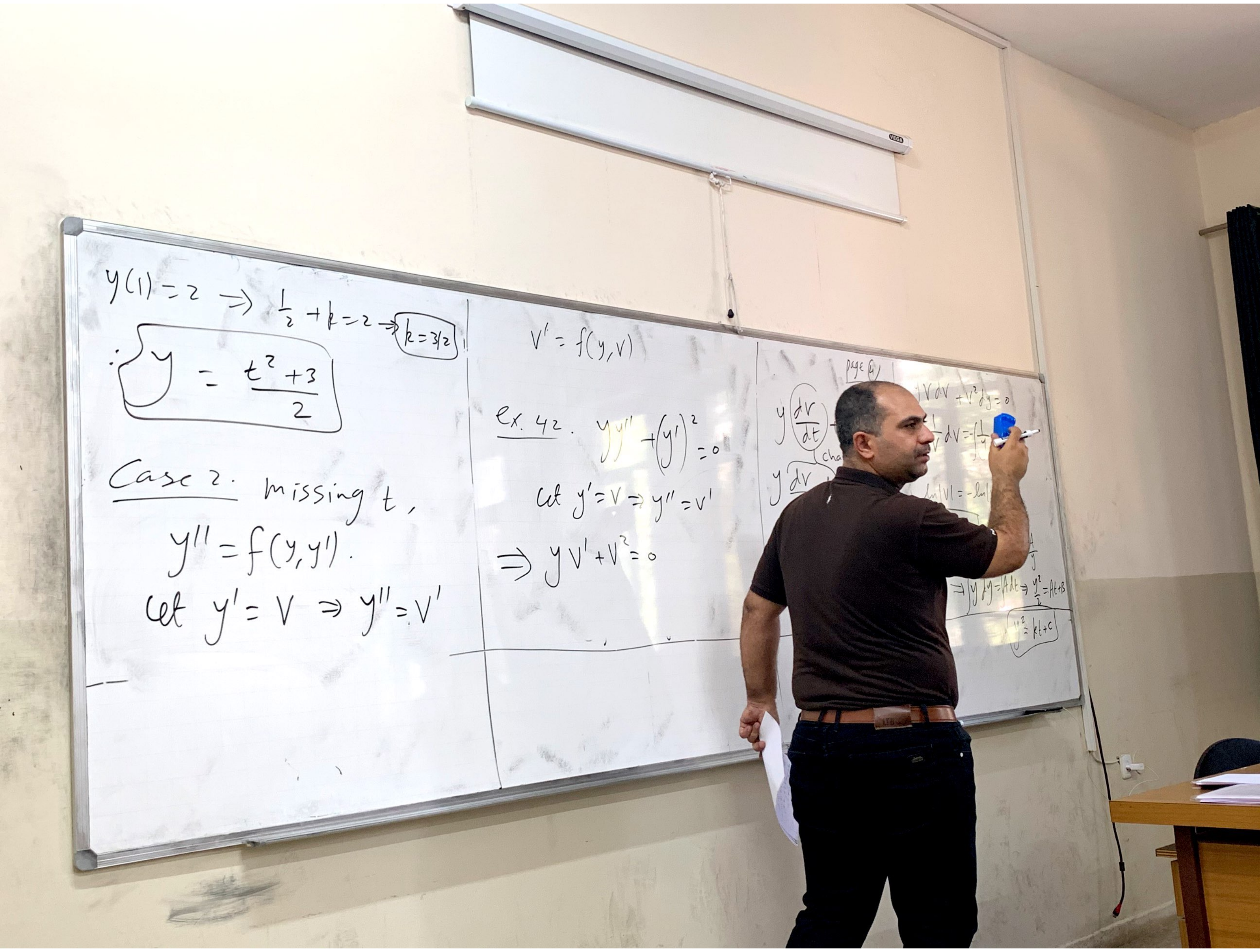
$(y'(1))^2 = 1 + C$

$(1)^2 = 1 + C \Rightarrow \boxed{C = 0}$

$\therefore v^2 = t^2$

$v = t$ since $v(1) = y'(1) = 1$.

$\Rightarrow \int dy = \int t \, dt \Rightarrow \boxed{y = \frac{t^2}{2} + k}$



$y(1) = 2 \Rightarrow \frac{1}{2} + k = 2 \Rightarrow \boxed{k = 3/2}$

$\therefore \boxed{y = \frac{t^2+3}{2}}$

Case 2. missing t,

$y'' = f(y, y')$.

let $y' = V \Rightarrow y'' = V'$

$V' = f(y, V)$

ex. 42. $y y'' + (y')^2 = 0$

let $y' = V \Rightarrow y'' = V'$

$\Rightarrow y V' + V^2 = 0$

page (4)

$y \frac{dv}{dt}$

cha

$y \frac{dv}{}$

$y v\, dv + v^2 dy = 0$

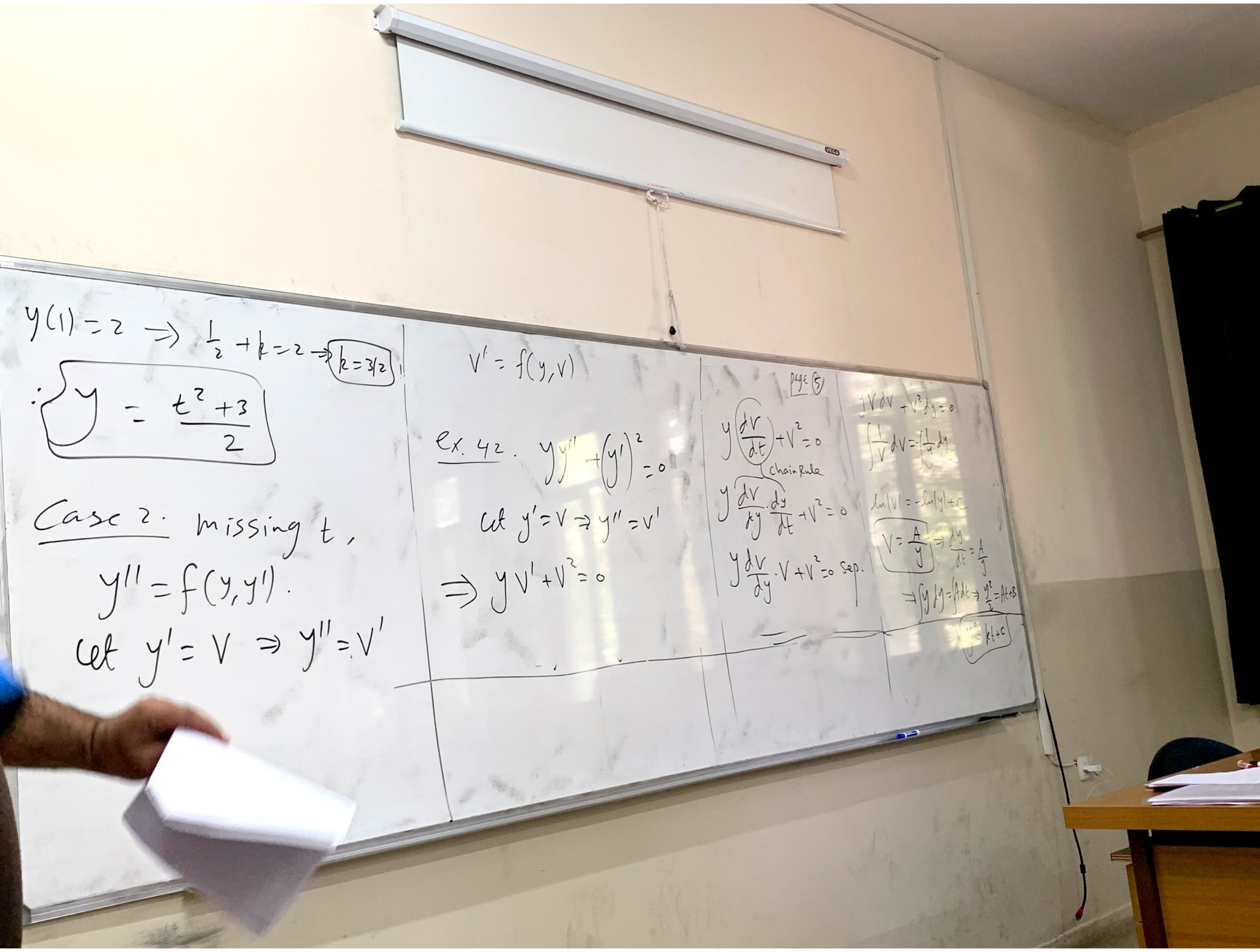
$\int \frac{1}{v} dv = -\int \frac{1}{y}$

$\ln|v| = -\ln|$

$\frac{A}{y}$

$\Rightarrow \int y\, dy = \int A\, dt \Rightarrow \frac{y^2}{2} = At + B$

$\boxed{y^2 = kt + c}$



$y(1)=2 \Rightarrow \frac{1}{2}+k=2 \Rightarrow \boxed{k=3/2}$

$$\therefore \boxed{y=\frac{t^2+3}{2}}$$

Case 2. missing t,

$y''=f(y,y')$.

let $y'=v \Rightarrow y''=v'$

$v'=f(y,v)$

ex. 42. $yy''+(y')^2=0$

let $y'=v \Rightarrow y''=v'$

$\Rightarrow yv'+v^2=0$

page ⑤

$y\underbrace{\frac{dv}{dt}}_{\text{Chain Rule}}+v^2=0$

$y\frac{dv}{dy}\cdot\frac{dy}{dt}+v^2=0$

$y\frac{dv}{dy}\cdot v+v^2=0$ Sep.

$yv\,dv+v^2dy=0$

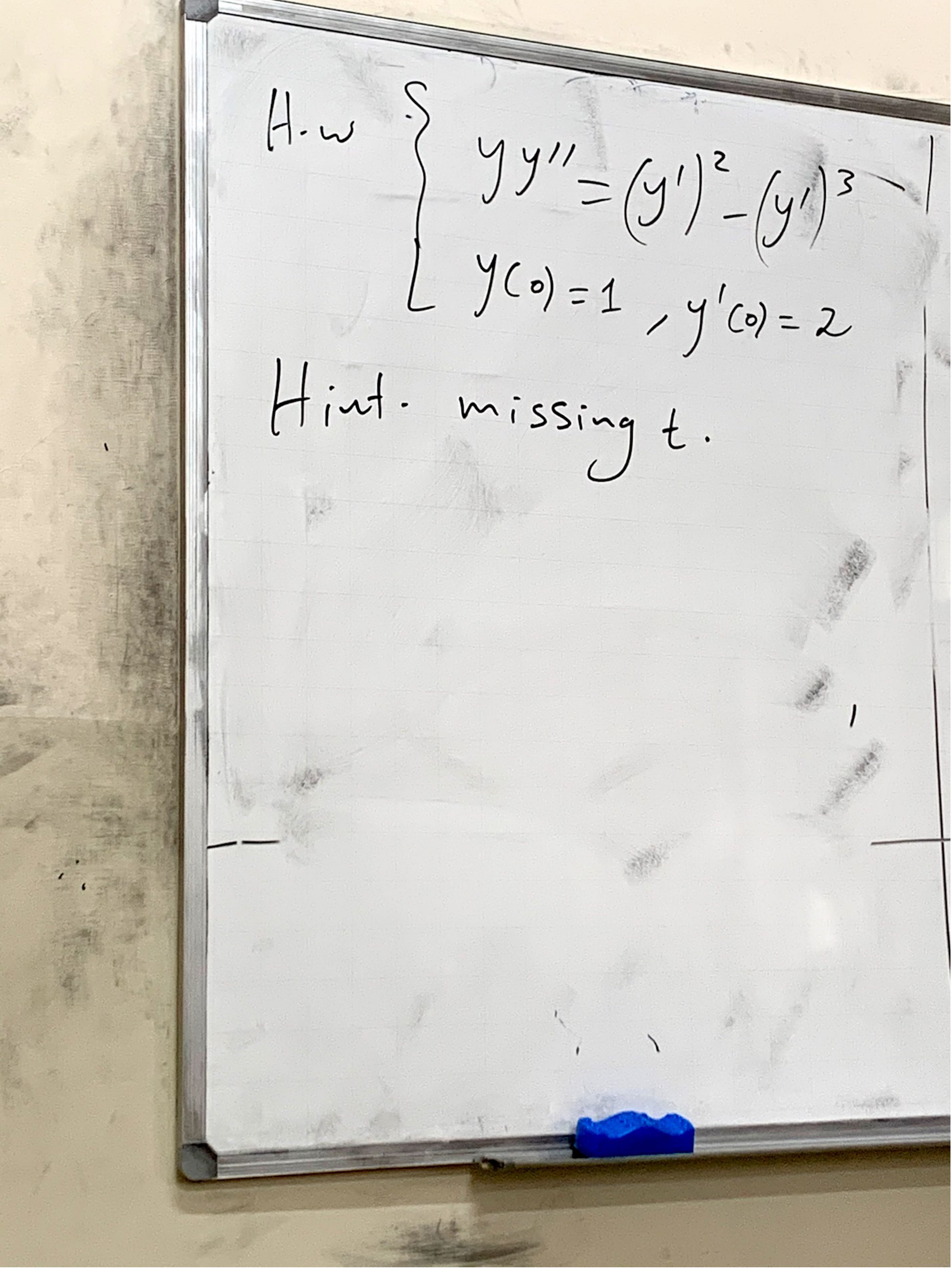
$\int\frac{1}{v}dv=-\int\frac{1}{y}dy$

$\ln|v|=-\ln|y|+c$

$\boxed{v=\frac{A}{y}} \Rightarrow \frac{dy}{dt}=\frac{A}{y}$

$\Rightarrow \int y\,dy=\int A\,dt \Rightarrow \frac{y^2}{2}=At+B$

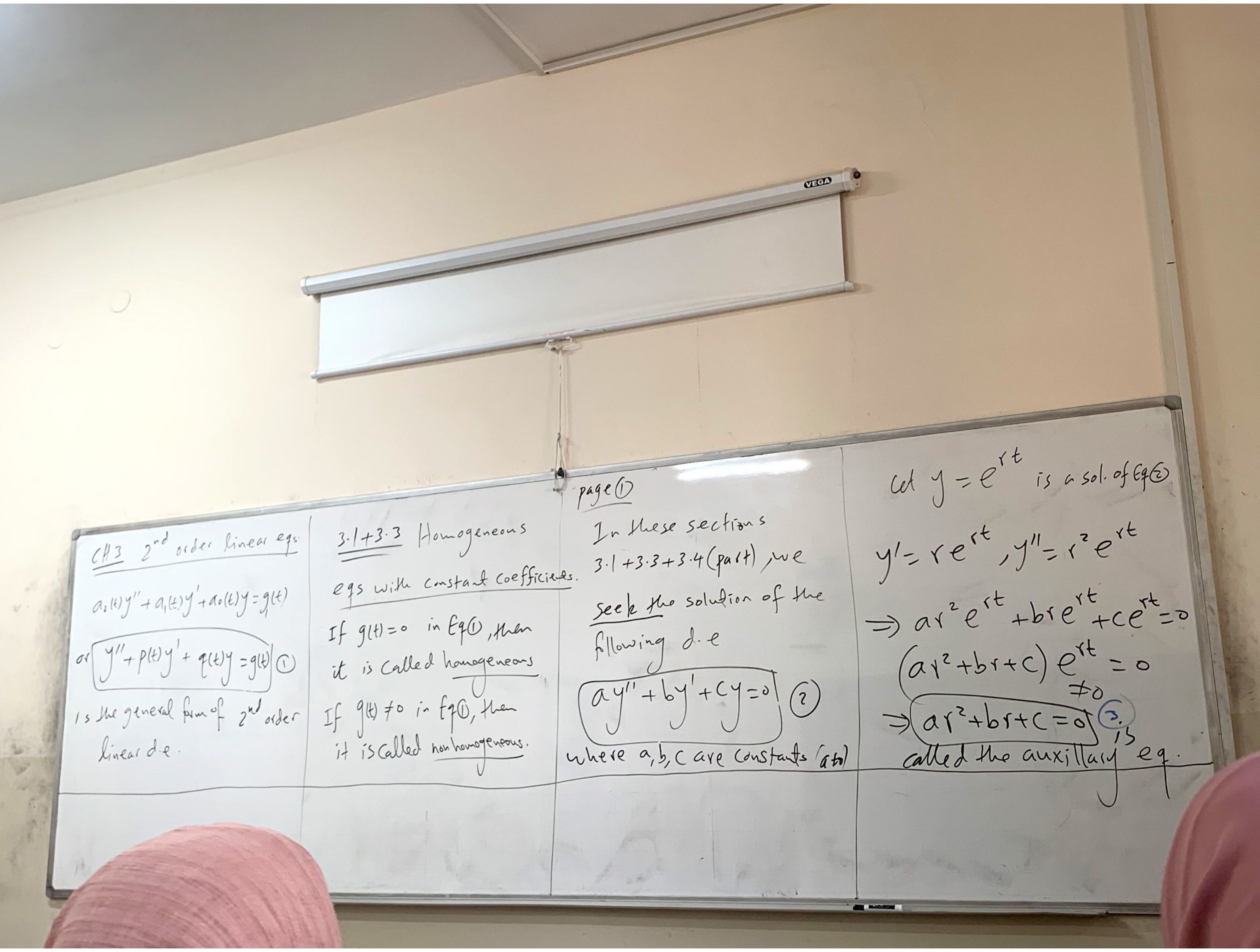
$\boxed{y^2=kt+c}$



H.w

$$\begin{cases} yy'' = (y')^2 - (y')^3 \\ y(0) = 1 \;, \; y'(0) = 2 \end{cases}$$

Hint. missing t.



CH3 2nd order linear eqs.

$a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t)$

or $\boxed{y'' + p(t)y' + q(t)y = g(t)}$ ①

is the general form of 2nd order linear d.e.

3.1 + 3.3 Homogeneous eqs with constant coefficients.

If $g(t) = 0$ in Eq ①, then it is called homogeneous

If $g(t) \neq 0$ in Eq ①, then it is called non homogeneous.

page ①

In these sections 3.1 + 3.3 + 3.4 (part), we seek the solution of the following d.e

$\boxed{ay'' + by' + cy = 0}$ ②

where a, b, c are constants ($a \neq 0$)

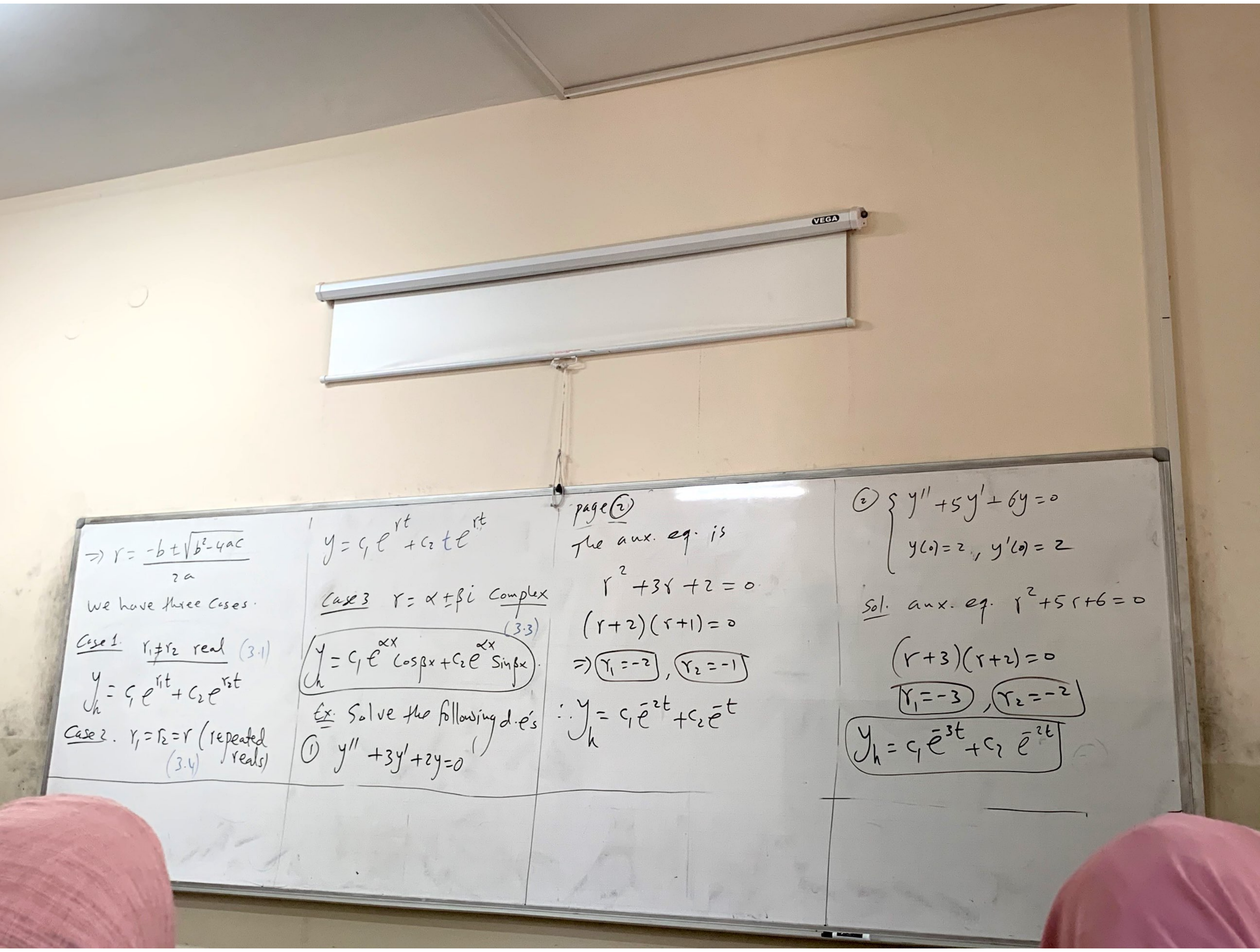
Let $y = e^{rt}$ is a sol. of Eq ②

$y' = re^{rt}$, $y'' = r^2 e^{rt}$

$\Rightarrow ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$

$(ar^2 + br + c)\underset{\neq 0}{e^{rt}} = 0$

$\Rightarrow \boxed{ar^2 + br + c = 0}$ ③ is called the auxillary eq.



$\Rightarrow r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

we have three cases.

Case 1. $r_1 \neq r_2$ real (3.1)

$y_h = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Case 2. $r_1 = r_2 = r$ (repeated reals) (3.4)

$y = c_1 e^{rt} + c_2 t e^{rt}$

Case 3 $r = \alpha \pm \beta i$ Complex (3.3)

$$\boxed{y_h = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x}.$$

Ex. Solve the following d.e's

① $y'' + 3y' + 2y = 0$

page ②

The aux. eq. is

$r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$\Rightarrow \boxed{r_1 = -2}, \boxed{r_2 = -1}$

$\therefore y_h = c_1 e^{-2t} + c_2 e^{-t}$

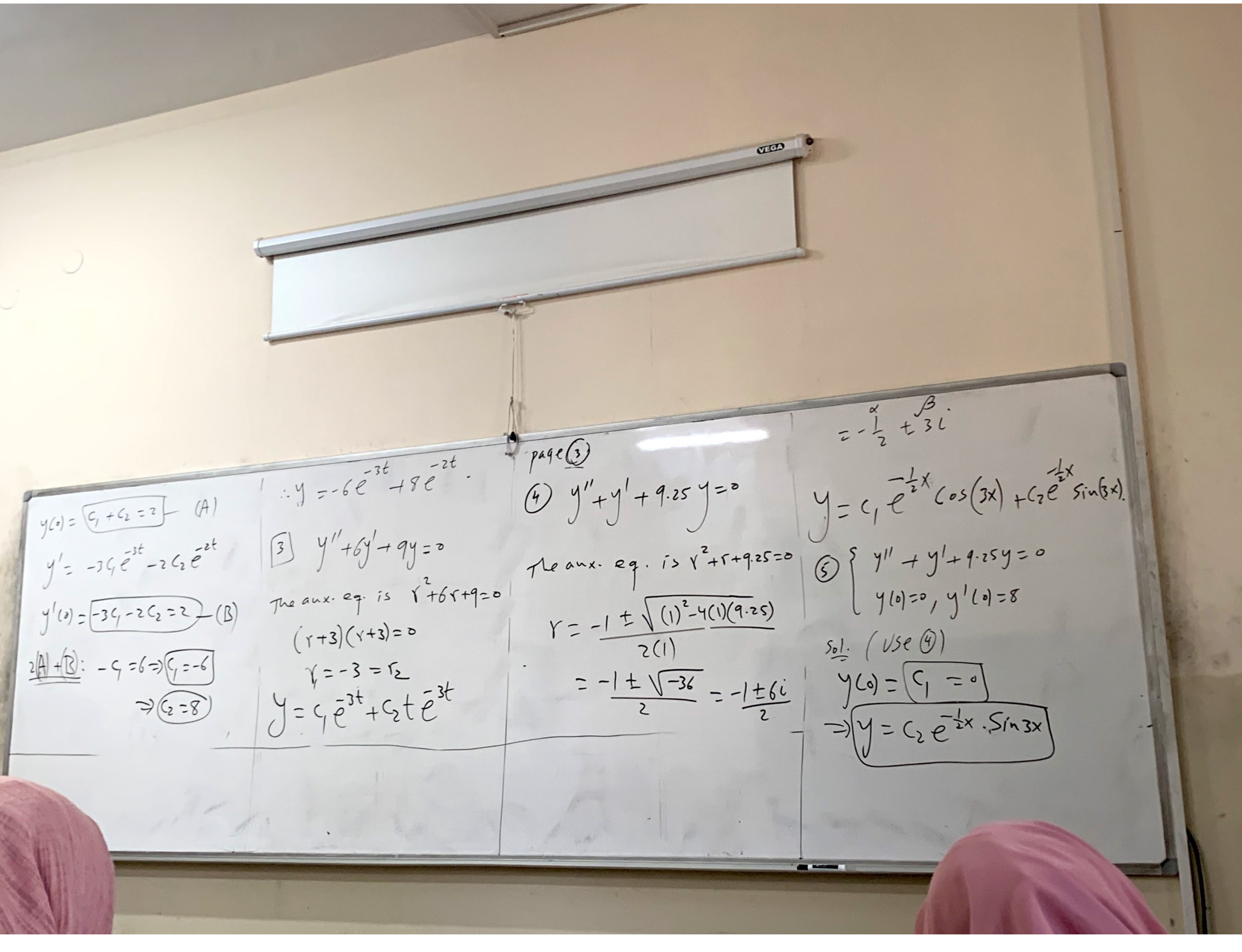
② $\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 2, \; y'(0) = 2 \end{cases}$

Sol. aux. eq. $r^2 + 5r + 6 = 0$

$(r+3)(r+2) = 0$

$\boxed{r_1 = -3}, \boxed{r_2 = -2}$

$$\boxed{y_h = c_1 e^{-3t} + c_2 e^{-2t}}$$



$y(0) = \boxed{c_1 + c_2 = 2}$ — (A)

$y' = -3c_1 e^{-3t} - 2c_2 e^{-2t}$

$y'(0) = \boxed{-3c_1 - 2c_2 = 2}$ — (B)

$\underline{2\boxed{A} + \boxed{B}}$: $-c_1 = 6 \Rightarrow \boxed{c_1 = -6}$

$\Rightarrow \boxed{c_2 = 8}$

$\therefore y = -6e^{-3t} + 8e^{-2t}$.

$\boxed{3}$ $y'' + 6y' + 9y = 0$

The aux. eq. is $r^2 + 6r + 9 = 0$

$(r+3)(r+3) = 0$

$r_1 = -3 = r_2$

$y = c_1 e^{-3t} + c_2 t e^{-3t}$

page ③

④ $y'' + y' + 9.25y = 0$

The aux. eq. is $r^2 + r + 9.25 = 0$

$$r = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(9.25)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-36}}{2} = \frac{-1 \pm 6i}{2}$$

$$= \overset{\alpha}{-\frac{1}{2}} \pm \overset{\beta}{3}i.$$

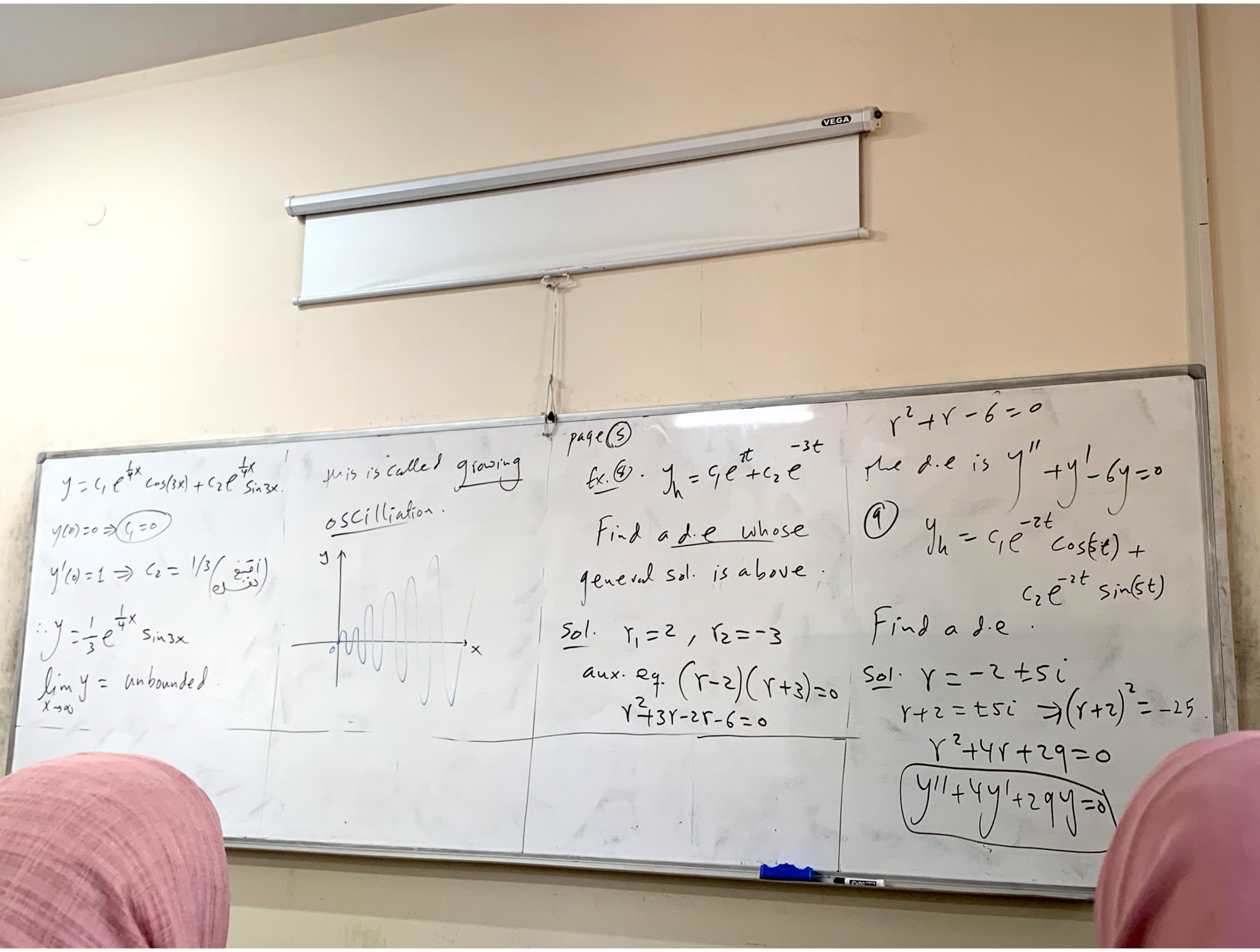
$$y = c_1 e^{-\frac{1}{2}x} \cos(3x) + c_2 e^{-\frac{1}{2}x} \sin(3x).$$

⑤ $\begin{cases} y'' + y' + 9.25y = 0 \\ y(0) = 0,\ y'(0) = 8 \end{cases}$

Sol. (Use ④)

$y(0) = \boxed{c_1 = 0}$

$\Rightarrow \boxed{y = c_2 e^{-\frac{1}{2}x} \cdot \sin 3x}$



$y = c_1 e^{\frac{1}{4}x} \cos(3x) + c_2 e^{\frac{1}{4}x} \sin 3x.$

$y(0) = 0 \Rightarrow \boxed{c_1 = 0}$

$y'(0) = 1 \Rightarrow c_2 = 1/3$ (اقتنع دفعه)

$\therefore y = \frac{1}{3} e^{\frac{1}{4}x} \sin 3x$

$\lim_{x \to \infty} y =$ unbounded.

this is called growing oscilliation.

page ⑤

Ex. ⑧. $y_h = c_1 e^{2t} + c_2 e^{-3t}$

Find a d.e whose general sol. is above.

Sol. $r_1 = 2$, $r_2 = -3$

aux. eq. $(r-2)(r+3) = 0$

$r^2 + 3r - 2r - 6 = 0$

$r^2 + r - 6 = 0$

the d.e is $y'' + y' - 6y = 0$

⑨ $y_h = c_1 e^{-2t} \cos(5t) + c_2 e^{-2t} \sin(5t)$

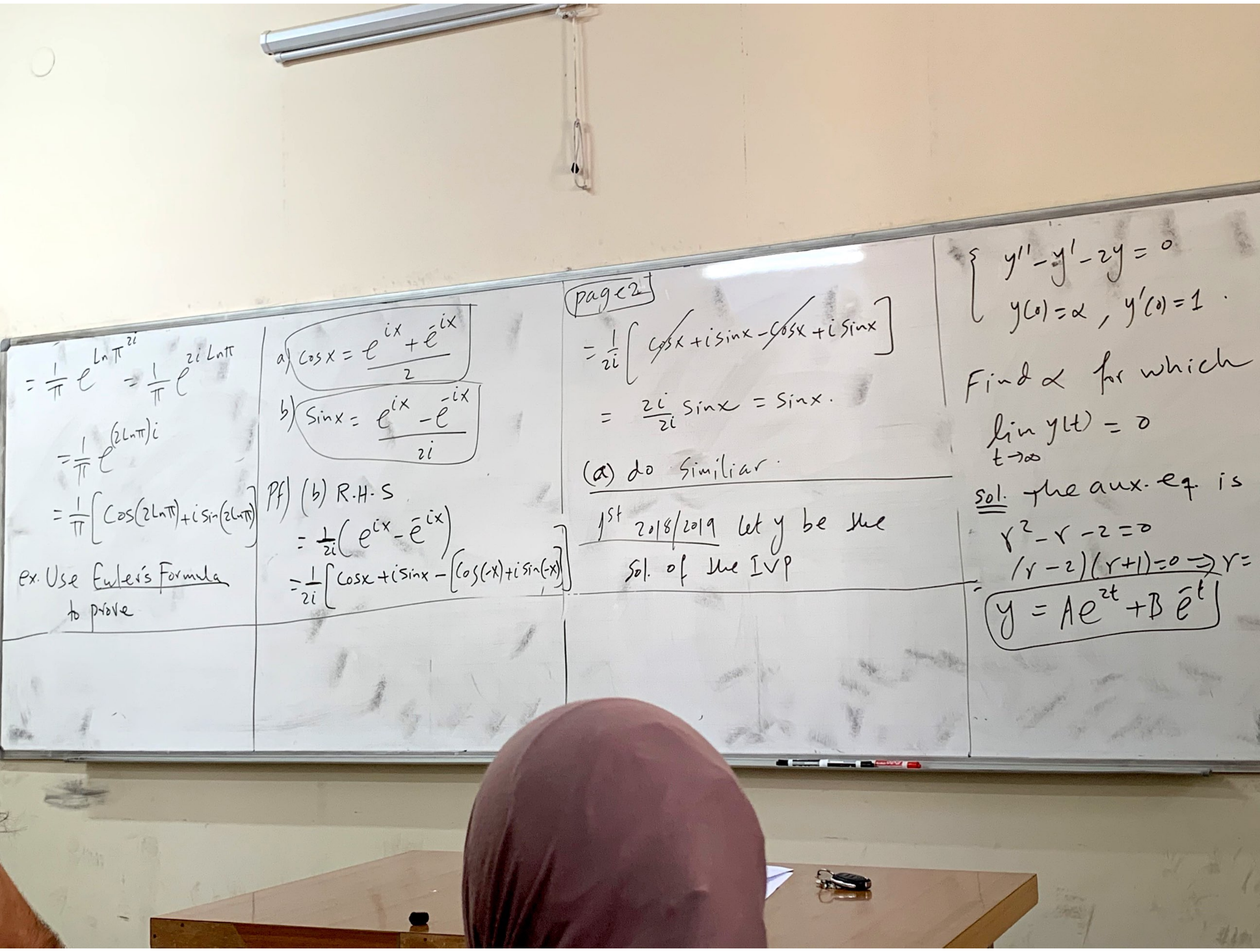
Find a d.e.

Sol. $r = -2 \pm 5i$

$r + 2 = \pm 5i \Rightarrow (r+2)^2 = -25$

$r^2 + 4r + 29 = 0$

$\boxed{y'' + 4y' + 29y = 0}$



$= \frac{1}{\pi} e^{\ln \pi^{2i}} = \frac{1}{\pi} e^{2i \ln \pi}$

$= \frac{1}{\pi} e^{(2\ln \pi)i}$

$= \frac{1}{\pi}\left[\cos(2\ln \pi) + i\sin(2\ln \pi)\right]$

ex. Use <u>Euler's Formula</u> to prove

a) $\boxed{\cos x = \frac{e^{ix} + \bar{e}^{ix}}{2}}$

b) $\boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$

Pf) (b) R.H.S

$= \frac{1}{2i}\left(e^{ix} - \bar{e}^{ix}\right)$

$= \frac{1}{2i}\left[\cos x + i\sin x - [\cos(-x) + i\sin(-x)]\right]$

$\boxed{\text{page 2}}$

$= \frac{1}{2i}\left[\cancel{\cos x} + i\sin x - \cancel{\cos x} + i\sin x\right]$

$= \frac{2i}{2i}\sin x = \sin x.$

(a) do Similiar.

<u>1st 2018/2019</u> let y be the sol. of the IVP

$$\begin{cases} y'' - y' - 2y = 0 \\ y(0) = \alpha, \quad y'(0) = 1. \end{cases}$$

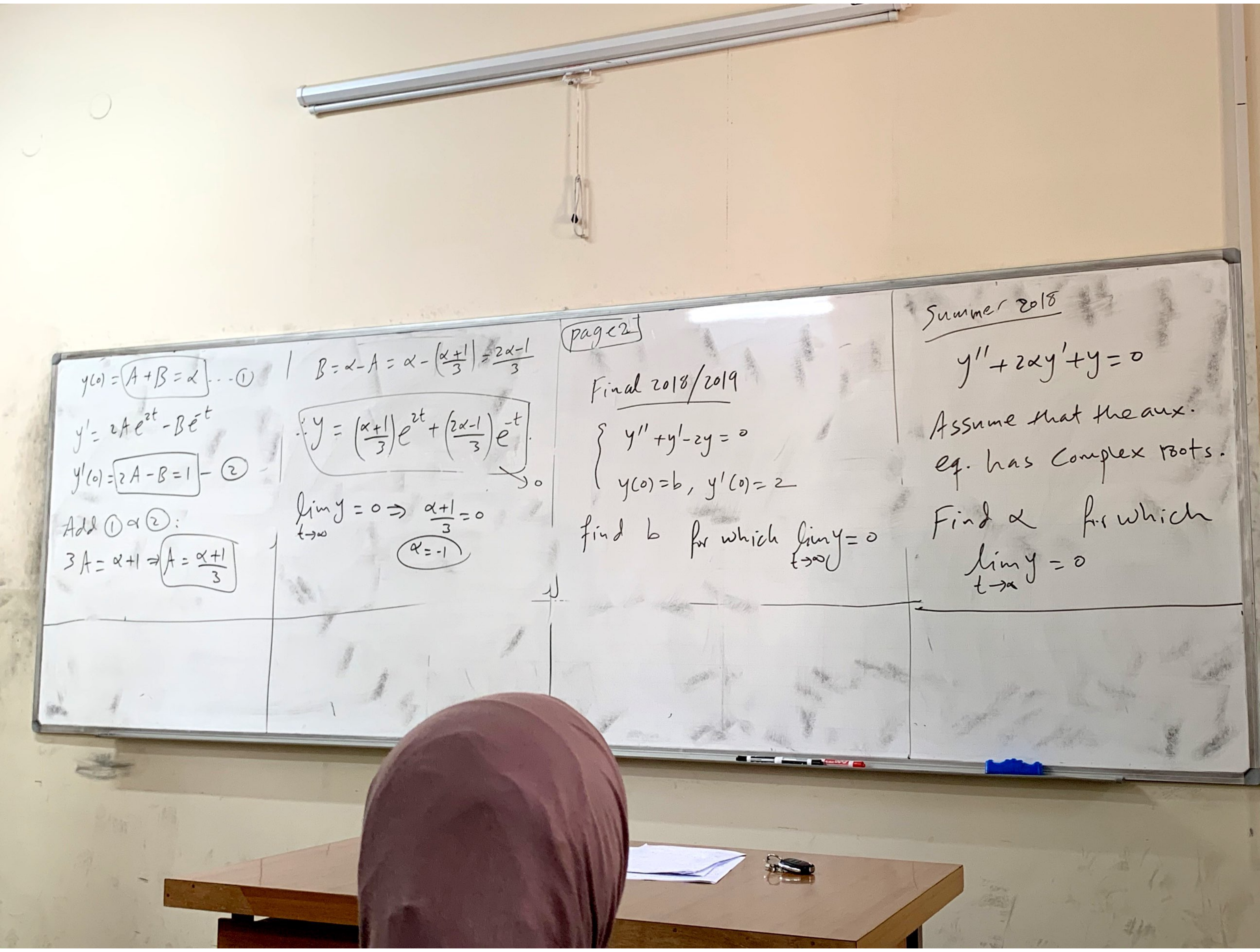
Find α for which $\lim_{t \to \infty} y(t) = 0$

<u>Sol.</u> the aux. eq. is

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \Rightarrow r =$

$\boxed{y = Ae^{2t} + B\bar{e}^{t}}$



$y(0) = \boxed{A + B = \alpha} \ldots$ ①

$y' = 2Ae^{2t} - Be^{-t}$

$y'(0) = \boxed{2A - B = 1}$ – ②

Add ① & ②:

$3A = \alpha + 1 \Rightarrow \boxed{A = \frac{\alpha+1}{3}}$

$B = \alpha - A = \alpha - \left(\frac{\alpha+1}{3}\right) = \frac{2\alpha-1}{3}$

$$\boxed{\therefore y = \left(\frac{\alpha+1}{3}\right)e^{2t} + \left(\frac{2\alpha-1}{3}\right)\underbrace{e^{-t}}_{\to 0}}$$

$\lim_{t\to\infty} y = 0 \Rightarrow \frac{\alpha+1}{3} = 0$

$\boxed{\alpha = -1}$

page 2

Final 2018/2019

$$\begin{cases} y'' + y' - 2y = 0 \\ y(0) = b, \ y'(0) = 2 \end{cases}$$

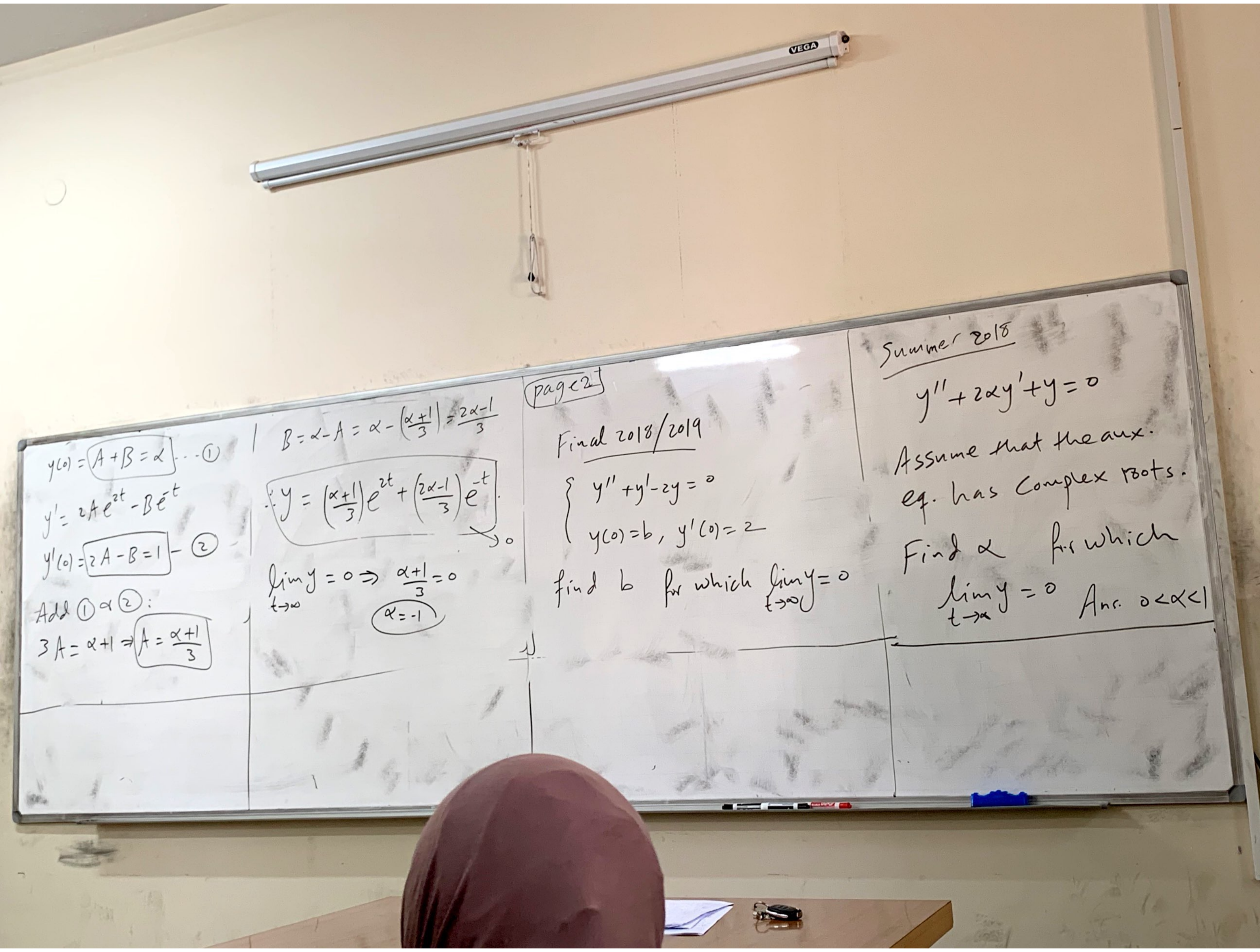
find b for which $\lim_{t\to\infty} y = 0$

Summer 2018

$y'' + 2\alpha y' + y = 0$

Assume that the aux. eq. has complex roots.

Find α for which $\lim_{t\to\infty} y = 0$



$y(0) = \boxed{A + B = \alpha} \cdots ①$

$y' = 2Ae^{2t} - Be^{-t}$

$y'(0) = \boxed{2A - B = 1} - ②$

Add ① & ②:

$3A = \alpha + 1 \Rightarrow \boxed{A = \frac{\alpha+1}{3}}$

$B = \alpha - A = \alpha - \left(\frac{\alpha+1}{3}\right) = \frac{2\alpha-1}{3}$

$\therefore \boxed{y = \left(\frac{\alpha+1}{3}\right)e^{2t} + \left(\frac{2\alpha-1}{3}\right)e^{-t}}$ $\to 0$

$\lim_{t\to\infty} y = 0 \Rightarrow \frac{\alpha+1}{3} = 0$

$\alpha = -1$

page (2)

Final 2018/2019

$$\begin{cases} y'' + y' - 2y = 0 \\ y(0) = b, \; y'(0) = 2 \end{cases}$$

find b for which $\lim_{t\to\infty} y = 0$

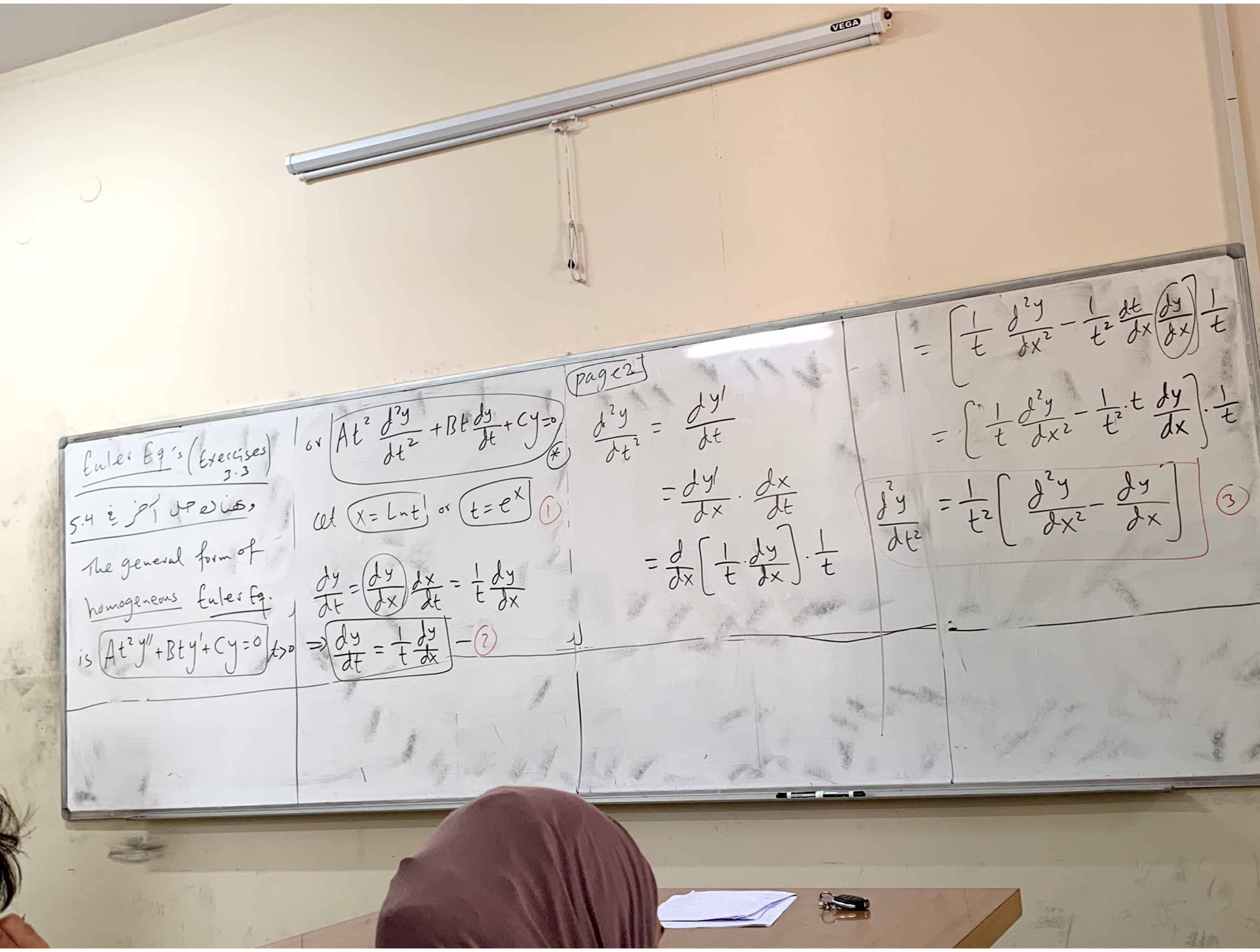
Summer 2018

$y'' + 2\alpha y' + y = 0$

Assume that the aux. eq. has complex roots.

Find α for which $\lim_{t\to\infty} y = 0$

Ans. $0 < \alpha < 1$



Euler Eq's (Exercises 3.3

محاضرة آخر في 5.4,

The general form of homogeneous Euler Eq. is $\boxed{At^2y''+Bty'+Cy=0}$ $t>0$

or $$\boxed{At^2\frac{d^2y}{dt^2}+Bt\frac{dy}{dt}+Cy=0} \quad (*)$$

let $\boxed{x=\ln t}$ or $\boxed{t=e^x}$ ①

$$\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}=\frac{1}{t}\frac{dy}{dx}$$

$$\Rightarrow \boxed{\frac{dy}{dt}=\frac{1}{t}\frac{dy}{dx}} \text{ — } ②$$

page 2

$$\frac{d^2y}{dt^2}=\frac{dy'}{dt}$$

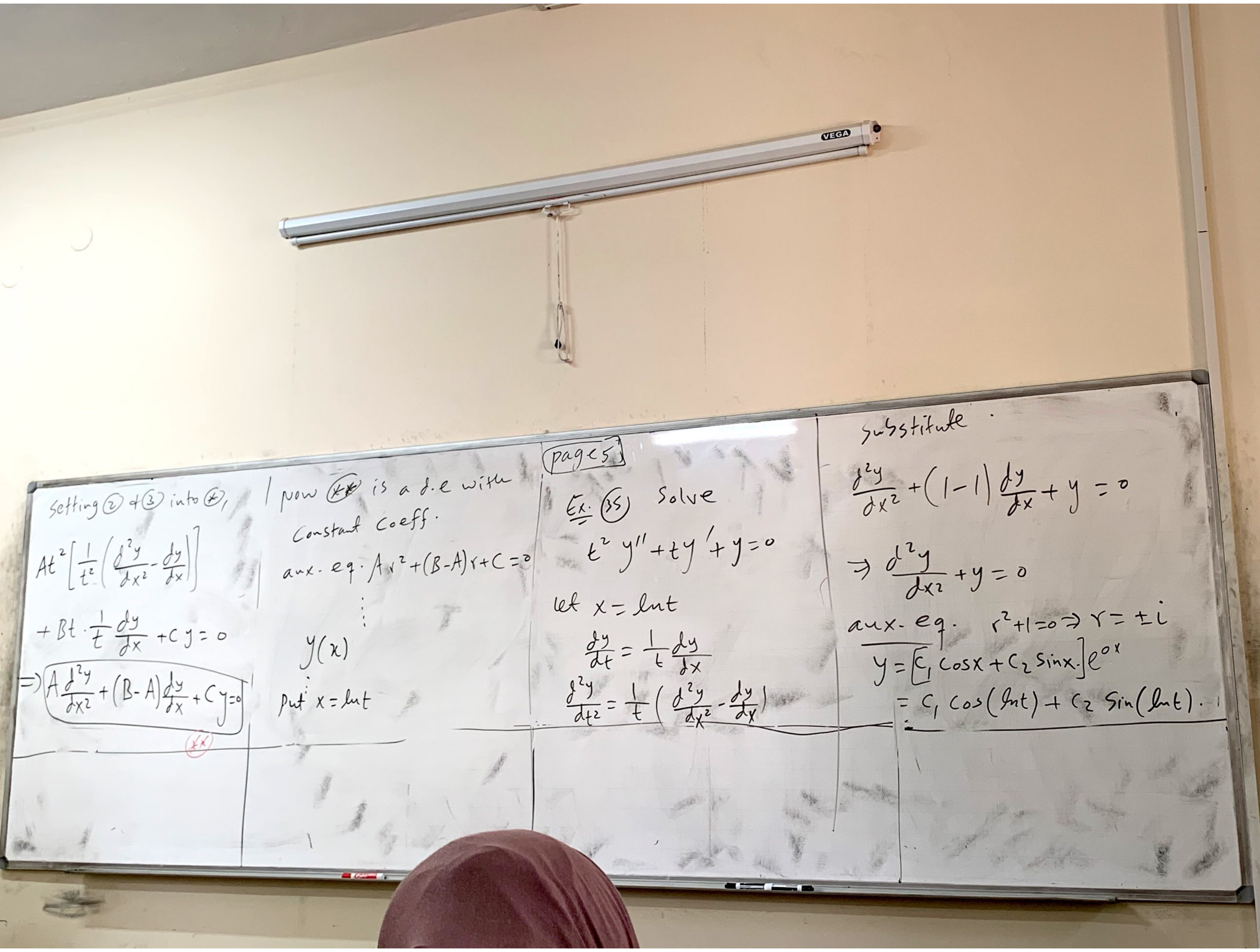
$$=\frac{dy'}{dx}\cdot\frac{dx}{dt}$$

$$=\frac{d}{dx}\left[\frac{1}{t}\cdot\frac{dy}{dx}\right]\cdot\frac{1}{t}$$

$$=\left[\frac{1}{t}\frac{d^2y}{dx^2}-\frac{1}{t^2}\frac{dt}{dx}\frac{dy}{dx}\right]\frac{1}{t}$$

$$=\left[\frac{1}{t}\frac{d^2y}{dx^2}-\frac{1}{t^2}\cdot t\frac{dy}{dx}\right]\cdot\frac{1}{t}$$

$$\boxed{\frac{d^2y}{dt^2}=\frac{1}{t^2}\left[\frac{d^2y}{dx^2}-\frac{dy}{dx}\right]} \quad ③$$



setting ② + ③ into ⊛,

$$At^2\left[\frac{1}{t^2}\left(\frac{d^2y}{dx^2}-\frac{dy}{dx}\right)\right]$$

$$+Bt\cdot\frac{1}{t}\frac{dy}{dx}+Cy=0$$

$$\Rightarrow \boxed{A\frac{d^2y}{dx^2}+(B-A)\frac{dy}{dx}+Cy=0}$$

(**)

now (**) is a d.e with constant coeff.

aux. eq. $Ar^2+(B-A)r+C=0$

⋮

$y(x)$

⋮

put $x=\ln t$

page 57

Ex. ㉟ Solve

$$t^2y''+ty'+y=0$$

let $x=\ln t$

$$\frac{dy}{dt}=\frac{1}{t}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2}=\frac{1}{t}\left(\frac{d^2y}{dx^2}-\frac{dy}{dx}\right)$$

substitute.

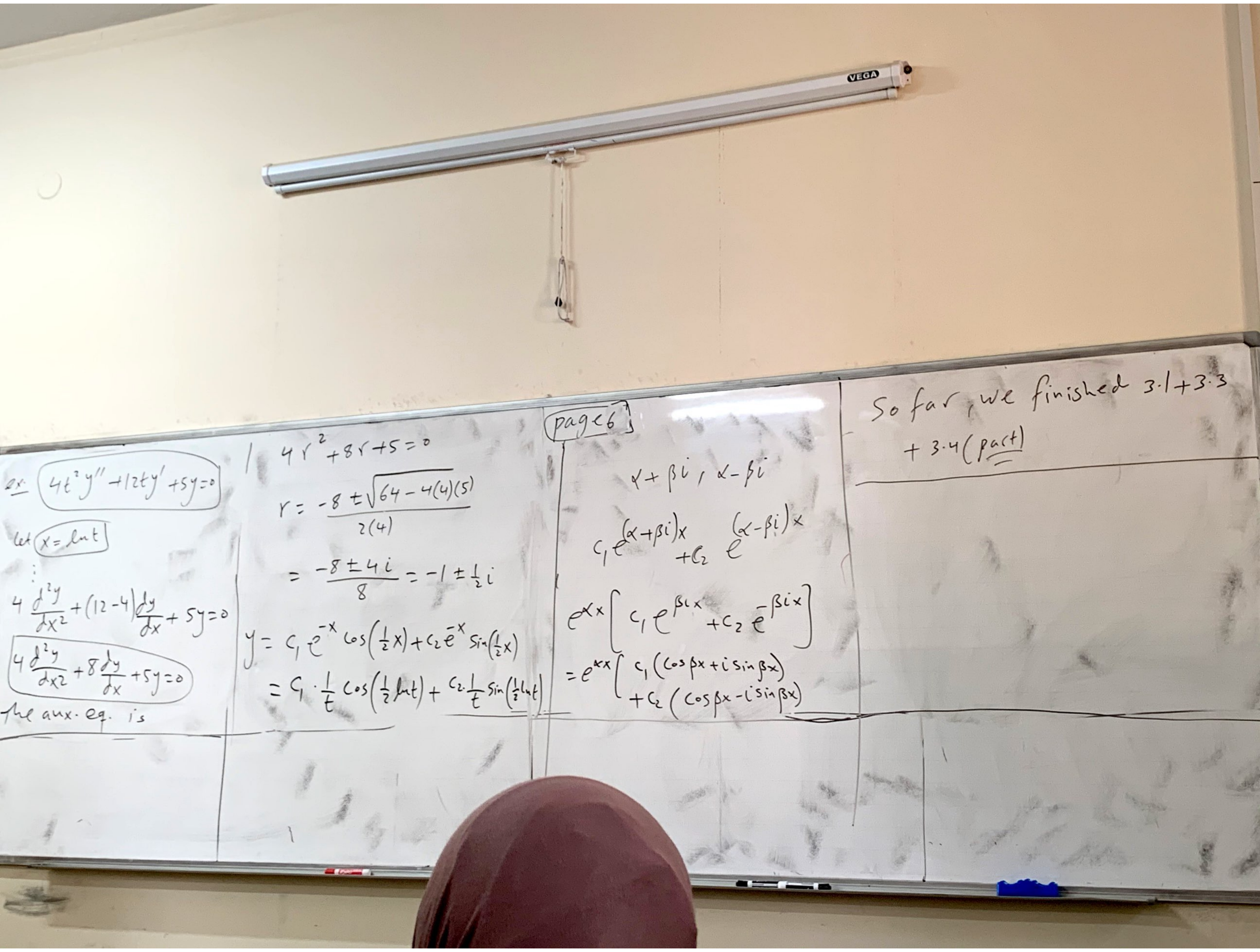
$$\frac{d^2y}{dx^2}+(1-1)\frac{dy}{dx}+y=0$$

$$\Rightarrow \frac{d^2y}{dx^2}+y=0$$

aux. eq. $r^2+1=0 \Rightarrow r=\pm i$

$$y=\left[c_1\cos x+c_2\sin x\right]e^{0x}$$

$$=c_1\cos(\ln t)+c_2\sin(\ln t).$$



ex. $\boxed{4t^2y''+12ty'+5y=0}$

let $\boxed{x=\ln t}$

⋮

$$4\frac{d^2y}{dx^2}+(12-4)\frac{dy}{dx}+5y=0$$

$$\boxed{4\frac{d^2y}{dx2}+8\frac{dy}{dx}+5y=0}$$

The aux. eq. is

$$4r^2+8r+5=0$$

$$r=\frac{-8\pm\sqrt{64-4(4)(5)}}{2(4)}$$

$$=\frac{-8\pm4i}{8}=-1\pm\frac{1}{2}i$$

$$y=c_1e^{-x}\cos\left(\frac{1}{2}x\right)+c_2e^{-x}\sin\left(\frac{1}{2}x\right)$$

$$=c_1\cdot\frac{1}{t}\cos\left(\frac{1}{2}\ln t\right)+c_2\cdot\frac{1}{t}\sin\left(\frac{1}{2}\ln t\right)$$

$\boxed{\text{page 6}}$

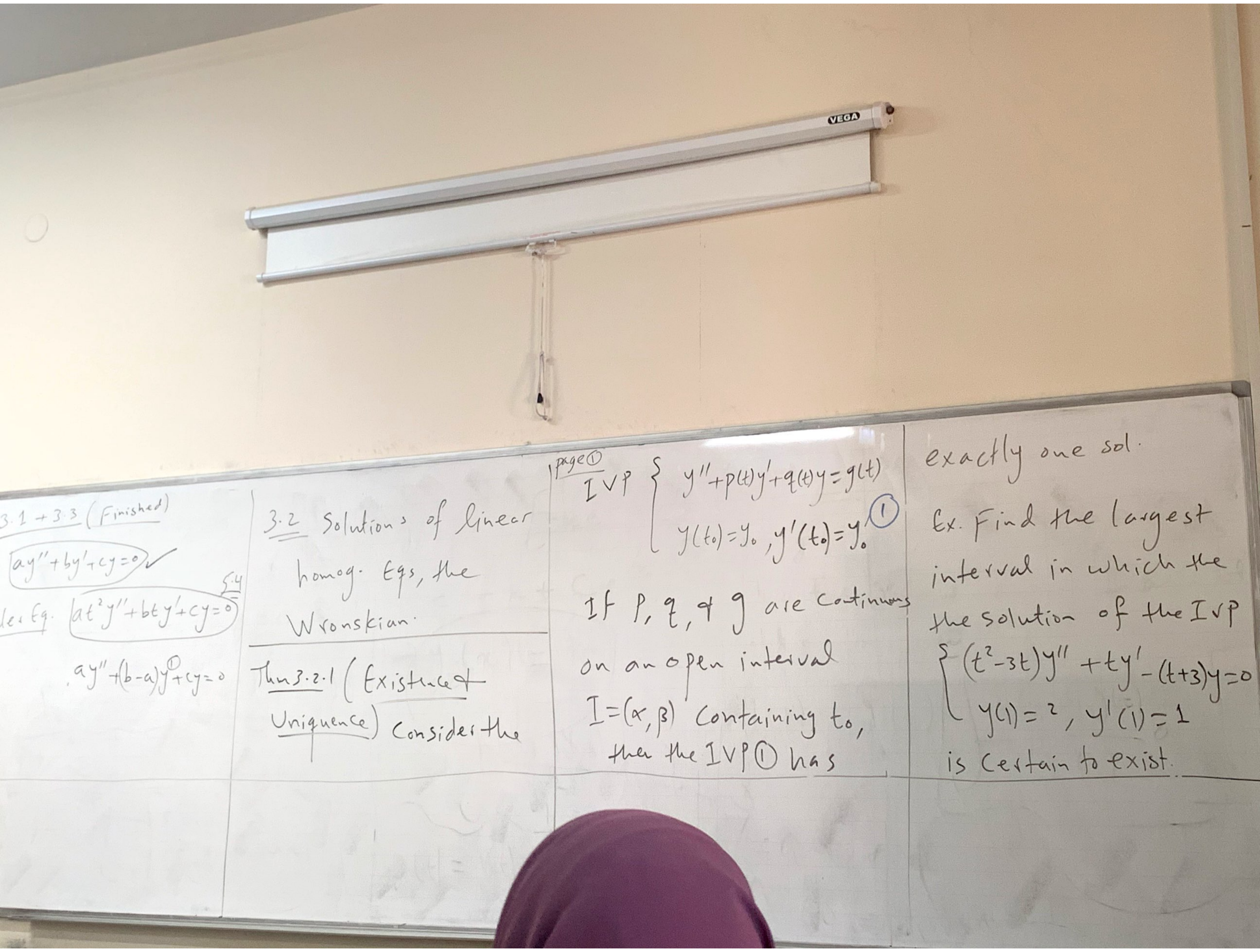
$\alpha+\beta i,\ \alpha-\beta i$

$$c_1e^{(\alpha+\beta i)x}+c_2e^{(\alpha-\beta i)x}$$

$$e^{\alpha x}\left[c_1e^{\beta ix}+c_2e^{-\beta ix}\right]$$

$$=e^{\alpha x}\left[c_1(\cos\beta x+i\sin\beta x)+c_2(\cos\beta x-i\sin\beta x)\right.$$

So far, we finished 3.1 + 3.3
+ 3.4 (part)



3.1 + 3.3 (Finished)

$\boxed{ay''+by'+cy=0}$ ✓

5.4

ler Eq. $\boxed{at^2y''+bty'+cy=0}$

$ay''+(b-a)y'①+cy=0$

3.2 Solutions of linear homog. Eqs, the Wronskian.

Thm 3.2.1 (Existence & Uniquence) Consider the

page①

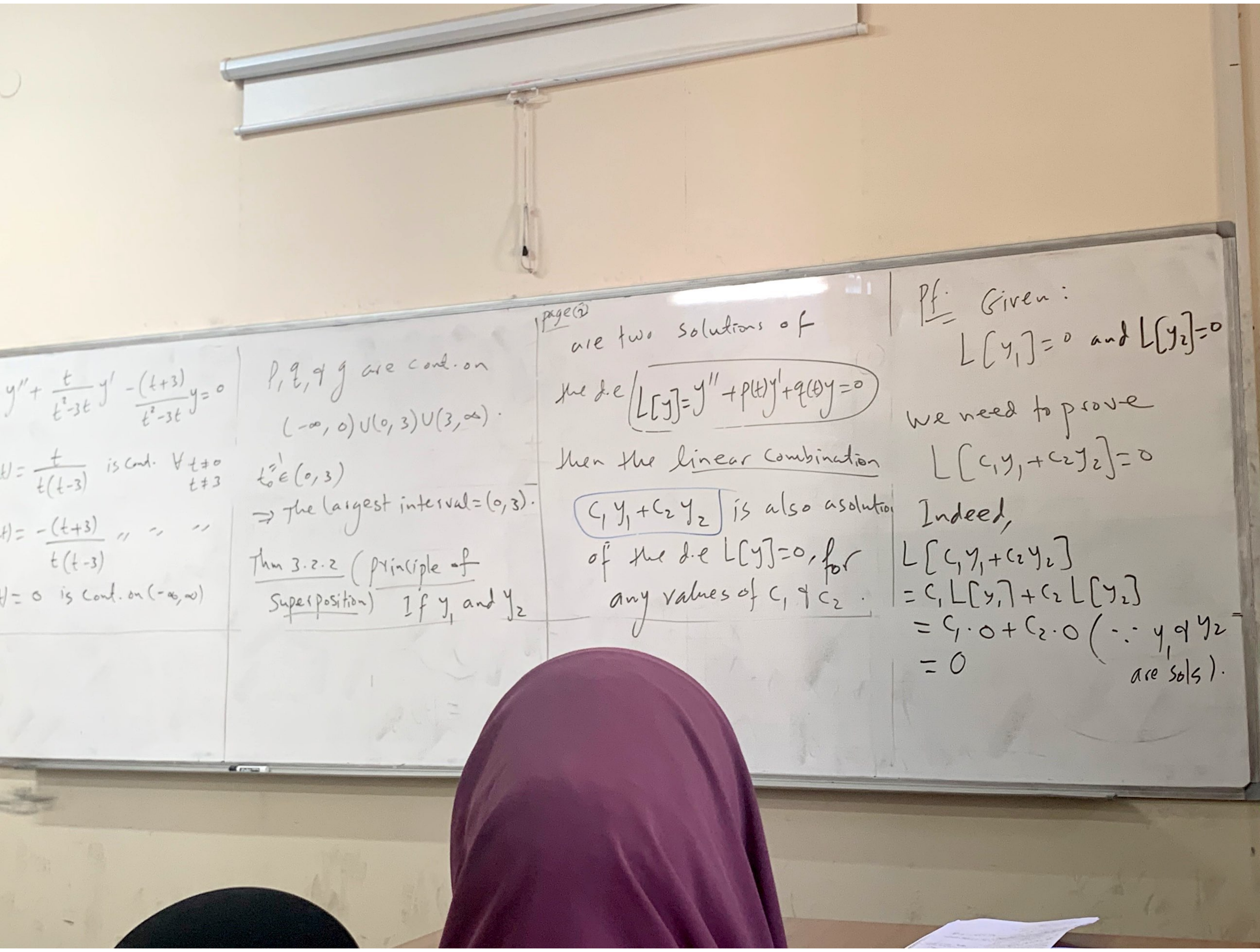
IVP $\begin{cases} y''+p(t)y'+q(t)y=g(t) \\ y(t_0)=y_0, y'(t_0)=y_0' \end{cases}$ ①

If p, q, & g are continuous on an open interval $I=(\alpha,\beta)$ containing t_0, then the IVP ① has exactly one sol.

Ex. Find the largest interval in which the the solution of the IVP

$$\begin{cases} (t^2-3t)y''+ty'-(t+3)y=0 \\ y(1)=2, y'(1)=1 \end{cases}$$

is Certain to exist.



$y'' + \frac{t}{t^2-3t} y' - \frac{(t+3)}{t^2-3t} y = 0$

$t) = \frac{t}{t(t-3)}$ is Cont. $\forall\, t \neq 0$, $t \neq 3$

$t) = -\frac{(t+3)}{t(t-3)}$ 〃 〃 〃

$t) = 0$ is Cont. on $(-\infty, \infty)$

p, q, & g are cont. on $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$.

$t_0^{=1} \in (0, 3)$

$\Rightarrow$ The largest interval $= (0, 3)$.

Thm 3.2.2 (Principle of Superposition) If y_1 and y_2

page ②

are two solutions of the d.e $\boxed{L[y] = y'' + p(t)y' + q(t)y = 0}$

then the linear combination $\boxed{c_1 y_1 + c_2 y_2}$ is also a solution of the d.e $L[y] = 0$, for any values of c_1 & c_2.

Pf: Given:

$L[y_1] = 0$ and $L[y_2] = 0$

we need to prove

$L[c_1 y_1 + c_2 y_2] = 0$

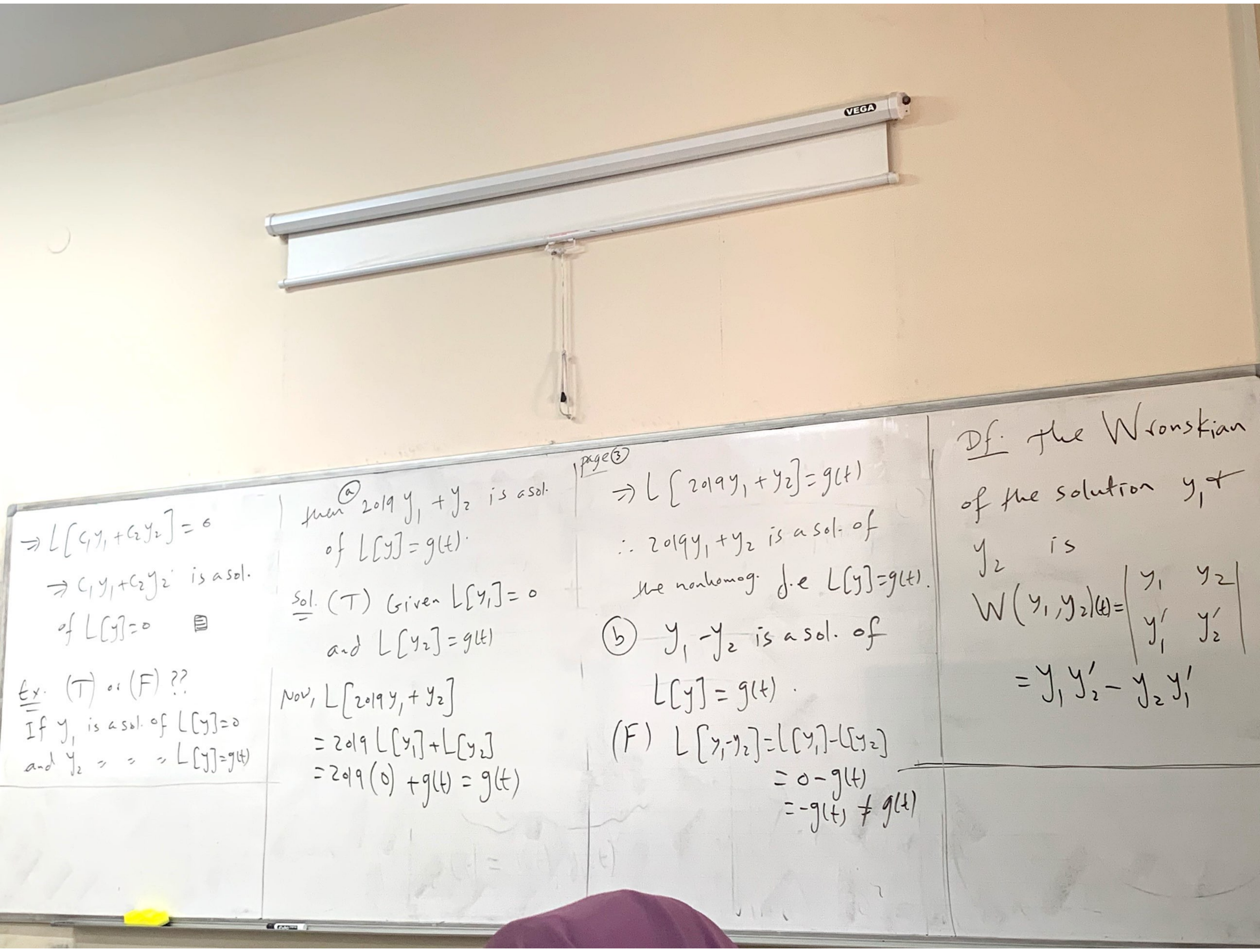
Indeed,

$L[c_1 y_1 + c_2 y_2]$

$= c_1 L[y_1] + c_2 L[y_2]$

$= c_1 \cdot 0 + c_2 \cdot 0$ ($\because y_1$ & y_2 are sols).

$= 0$



$\Rightarrow L[c_1y_1+c_2y_2]=0$

$\Rightarrow c_1y_1+c_2y_2$ is a sol. of $L[y]=0$

Ex. (T) or (F) ??

If y_1 is a sol. of $L[y]=0$
and y_2 〃 〃 〃 $L[y]=g(t)$

then ⓐ $2019y_1+y_2$ is a sol. of $L[y]=g(t)$.

Sol. (T) Given $L[y_1]=0$
and $L[y_2]=g(t)$

Now, $L[2019y_1+y_2]$
$=2019L[y_1]+L[y_2]$
$=2019(0)+g(t)=g(t)$

page③

$\Rightarrow L[2019y_1+y_2]=g(t)$

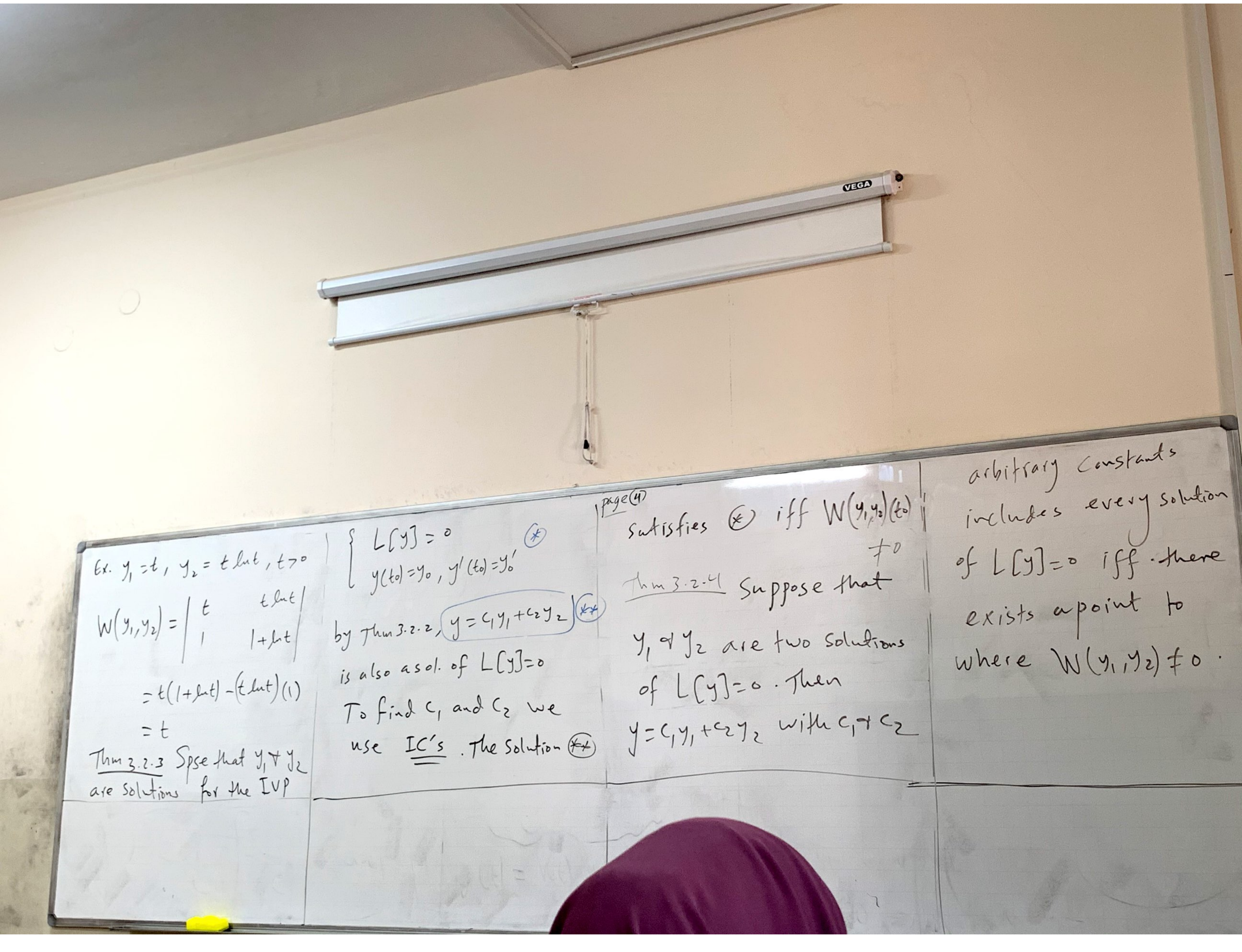
$\therefore 2019y_1+y_2$ is a sol. of the nonhomog. d.e $L[y]=g(t)$.

ⓑ y_1-y_2 is a sol. of $L[y]=g(t)$.

(F) $L[y_1-y_2]=L[y_1]-L[y_2]$
$=0-g(t)$
$=-g(t)\neq g(t)$

Df. the Wronskian of the solution y_1 & y_2 is

$$W(y_1,y_2)(t)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$=y_1y_2'-y_2y_1'$$



Ex. $y_1 = t$, $y_2 = t\ln t$, $t > 0$

$$W(y_1, y_2) = \begin{vmatrix} t & t\ln t \\ 1 & 1+\ln t \end{vmatrix}$$

$$= t(1+\ln t) - (t\ln t)(1)$$

$$= t$$

Thm 3.2.3 Spse that y_1 & y_2 are solutions for the IVP

$$\begin{cases} L[y] = 0 \\ y(t_0) = y_0,\ y'(t_0) = y_0' \end{cases} \quad (*)$$

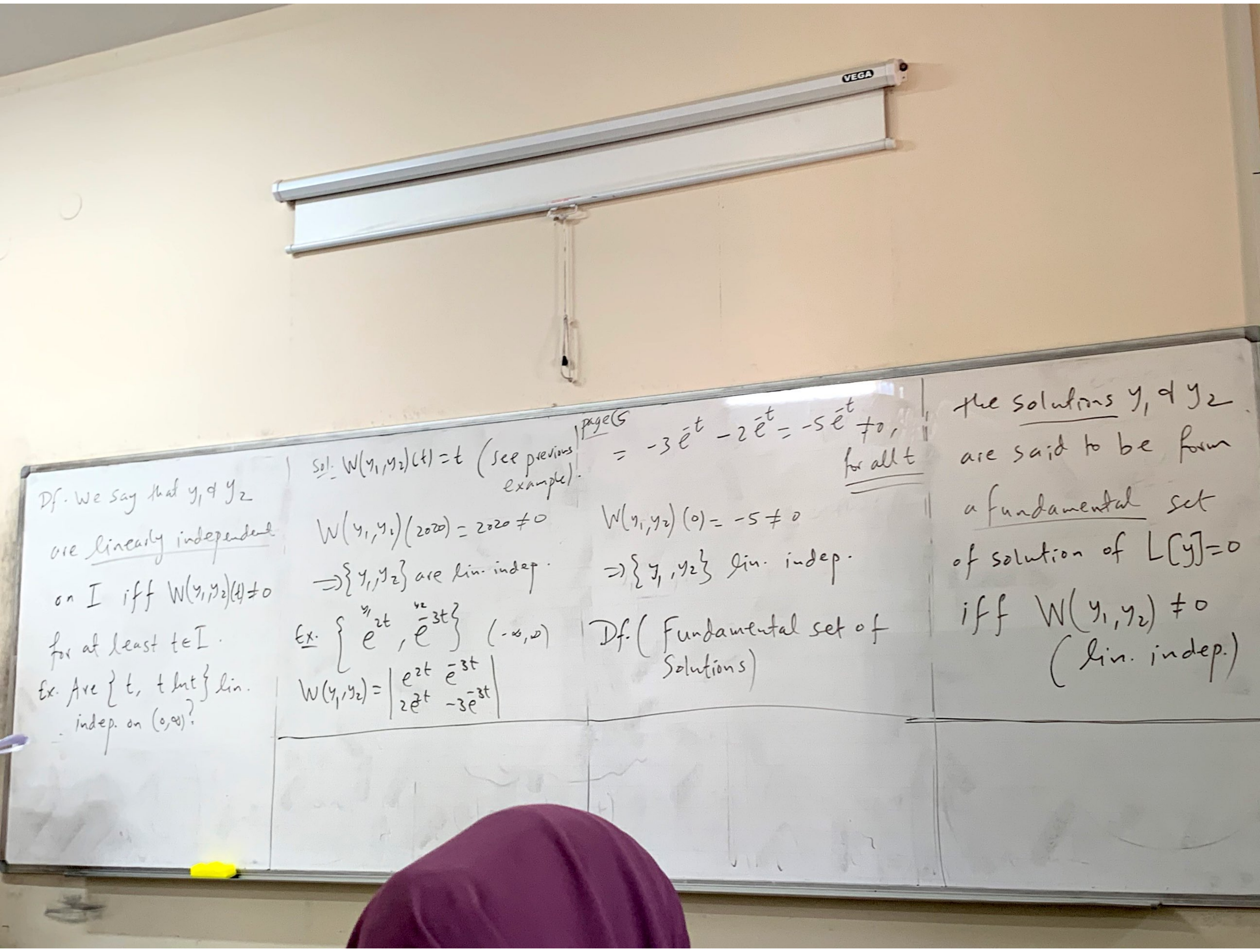
by Thm 3.2.2, $\boxed{y = c_1 y_1 + c_2 y_2}$ (**) is also a sol. of $L[y]=0$

To find c_1 and c_2 we use IC's. The solution (**)

page ④

satisfies (*) iff $W(y_1,y_2)(t_0) \neq 0$

Thm 3.2.4 Suppose that y_1 & y_2 are two solutions of $L[y]=0$. Then $y = c_1 y_1 + c_2 y_2$ with c_1 & c_2 arbitrary constants includes every solution of $L[y]=0$ iff there exists a point t_0 where $W(y_1, y_2) \neq 0$.



Df. We say that y_1 & y_2 are linearly independent on I iff $W(y_1,y_2)(t) \neq 0$ for at least $t \in I$.

Ex. Are $\{t,\ t\ln t\}$ lin. indep. on $(0,\infty)$?

Sol. $W(y_1,y_2)(t) = t$ (see previous example).

$W(y_1,y_1)(2020) = 2020 \neq 0$

$\Rightarrow \{y_1,y_2\}$ are lin. indep.

Ex. $\left\{ \overset{y_1}{e^{2t}}, \overset{y_2}{e^{-3t}} \right\}$ $(-\infty,\infty)$

$W(y_1,y_2) = \begin{vmatrix} e^{2t} & e^{-3t} \\ 2e^{2t} & -3e^{-3t} \end{vmatrix}$

page(5

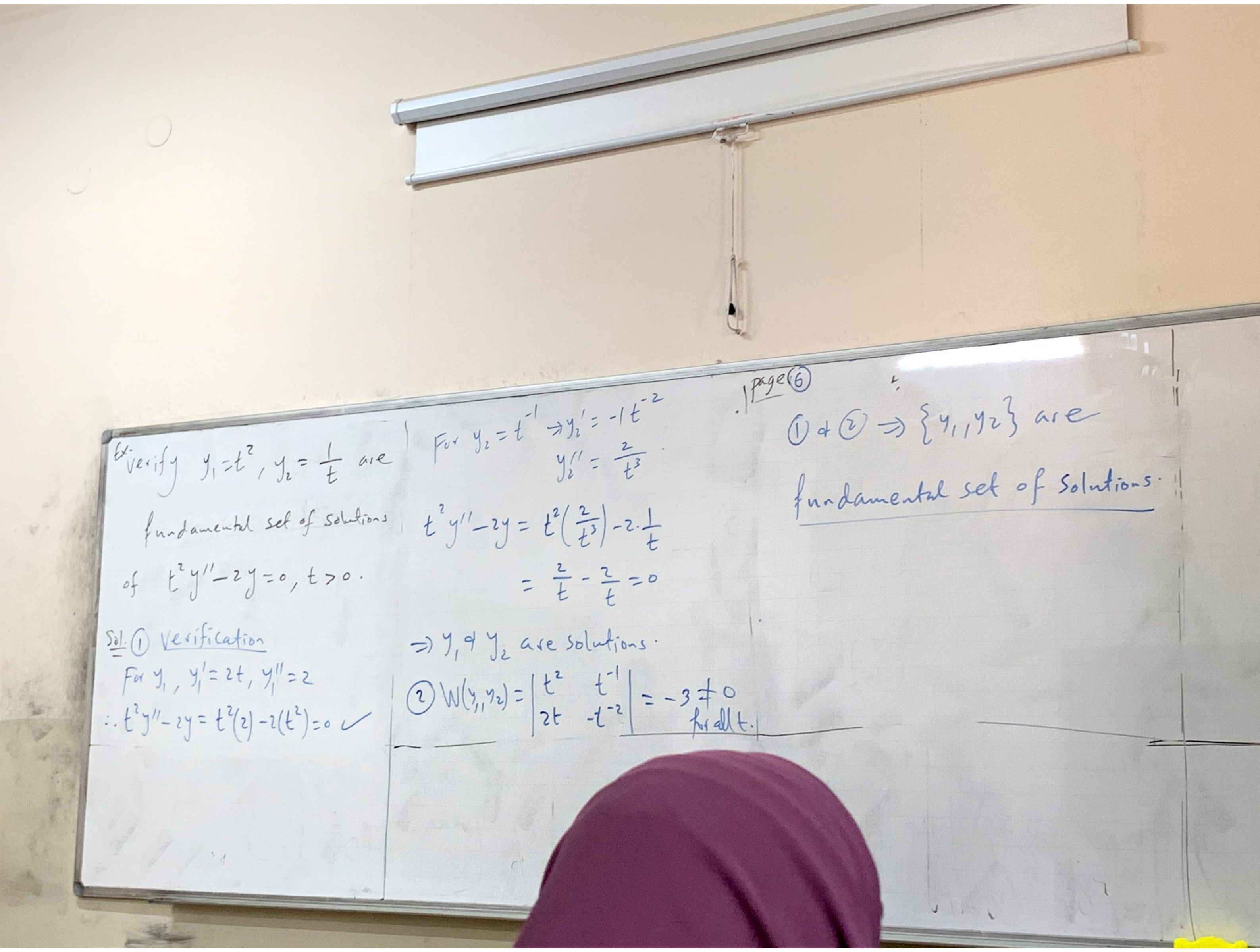
$= -3e^{-t} - 2e^{-t} = -5e^{-t} \neq 0,$ for all t

$W(y_1,y_2)(0) = -5 \neq 0$

$\Rightarrow \{y_1,y_2\}$ lin. indep.

Df. (Fundamental set of Solutions)

the solutions y_1 & y_2 are said to be form a fundamental set of solution of $L[y]=0$ iff $W(y_1,y_2) \neq 0$ (lin. indep.)



Ex. Verify $y_1 = t^2$, $y_2 = \frac{1}{t}$ are fundamental set of solutions of $t^2 y'' - 2y = 0$, $t > 0$.

Sol. ① Verification

For y_1, $y_1' = 2t$, $y_1'' = 2$

$\therefore t^2 y'' - 2y = t^2(2) - 2(t^2) = 0$ ✓

For $y_2 = t^{-1} \Rightarrow y_2' = -1t^{-2}$

$y_2'' = \frac{2}{t^3}$.

$$t^2 y'' - 2y = t^2\left(\frac{2}{t^3}\right) - 2\cdot\frac{1}{t}$$

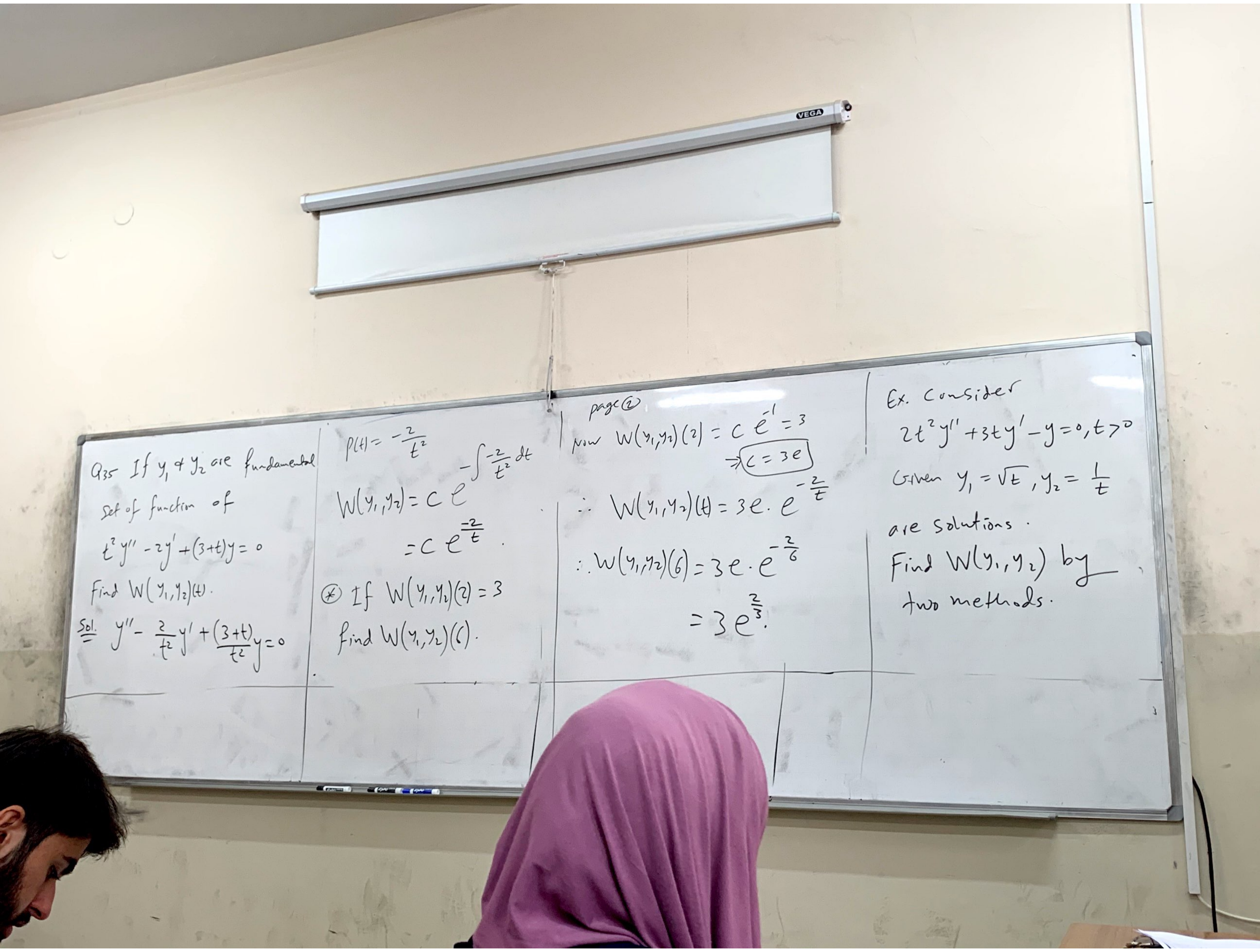
$$= \frac{2}{t} - \frac{2}{t} = 0$$

$\Rightarrow y_1$ & y_2 are solutions.

② $W(y_1, y_2) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -3 \neq 0$ for all t.

page ⑥

① & ② $\Rightarrow \{y_1, y_2\}$ are fundamental set of solutions.



Q35 If y_1 & y_2 are fundamental set of function of

$t^2y'' - 2y' + (3+t)y = 0$

Find $W(y_1, y_2)(t)$.

Sol. $y'' - \frac{2}{t^2}y' + \frac{(3+t)}{t^2}y = 0$

$p(t) = -\frac{2}{t^2}$

$$W(y_1, y_2) = c\,e^{-\int \frac{-2}{t^2}dt}$$

$$= c\,e^{\frac{-2}{t}}.$$

⊛ If $W(y_1, y_2)(2) = 3$

find $W(y_1, y_2)(6)$.

page ②

Now $W(y_1, y_2)(2) = c\,e^{-1} = 3$

$\Rightarrow \boxed{c = 3e}$

$\therefore W(y_1, y_2)(t) = 3e \cdot e^{-\frac{2}{t}}$

$\therefore W(y_1, y_2)(6) = 3e \cdot e^{-\frac{2}{6}}$

$= 3e^{\frac{2}{3}}.$

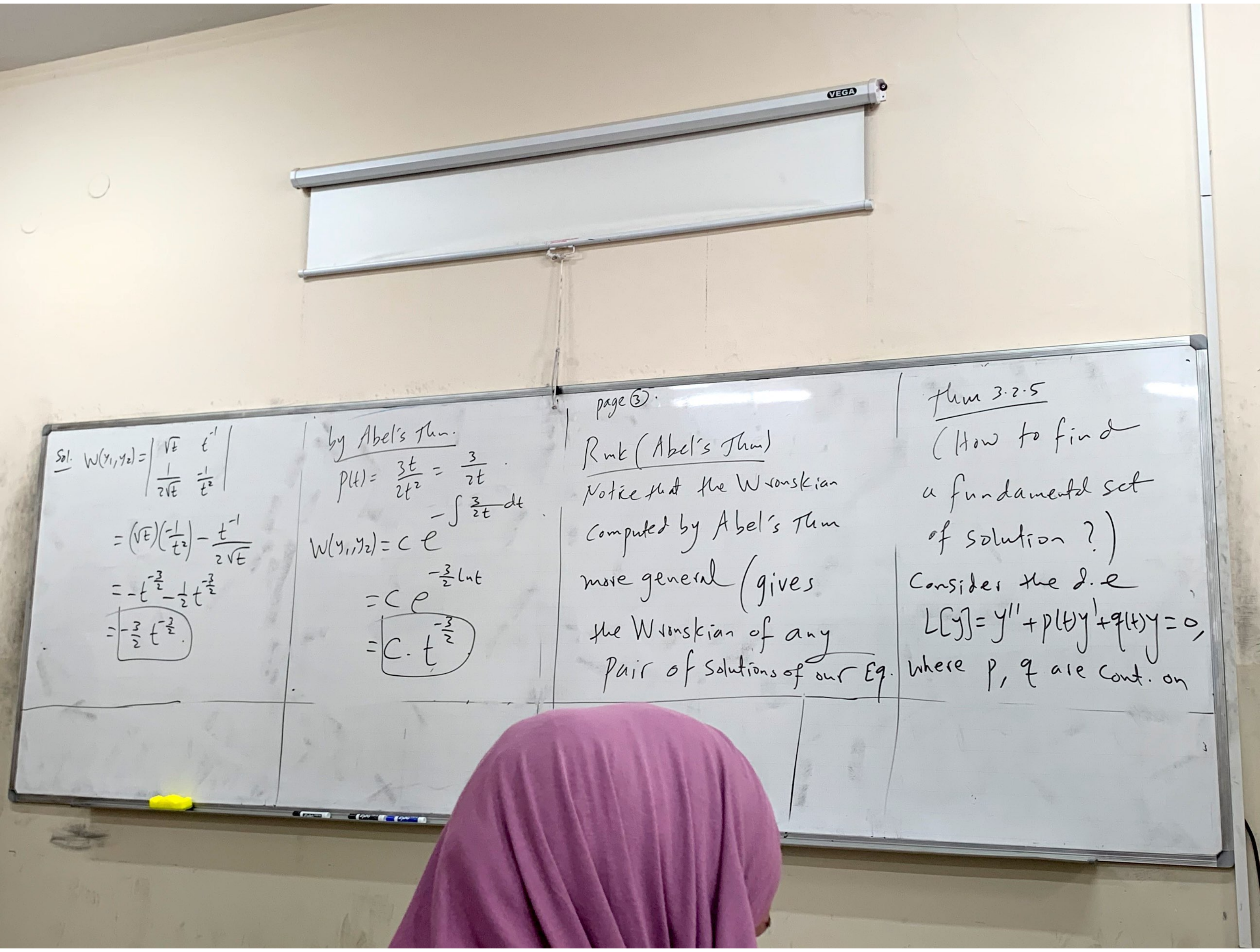
Ex. Consider

$2t^2y'' + 3ty' - y = 0,\ t > 0$

Given $y_1 = \sqrt{t}$, $y_2 = \frac{1}{t}$

are solutions.

Find $W(y_1, y_2)$ by two methods.



Sol. $W(y_1,y_2)=\begin{vmatrix} \sqrt{t} & t^{-1} \\ \frac{1}{2\sqrt{t}} & \frac{-1}{t^2} \end{vmatrix}$

$=(\sqrt{t})(\frac{-1}{t^2})-\frac{t^{-1}}{2\sqrt{t}}$

$=-t^{-\frac{3}{2}}-\frac{1}{2}t^{-\frac{3}{2}}$

$=\boxed{-\frac{3}{2}t^{-\frac{3}{2}}}.$

by Abel's Thm.

$p(t)=\frac{3t}{2t^2}=\frac{3}{2t}.$

$W(y_1,y_2)=c\,e^{-\int \frac{3}{2t}dt}.$

$=c\,e^{-\frac{3}{2}\ln t}$

$=\boxed{c\cdot t^{-\frac{3}{2}}}.$

page ③.

Rmk (Abel's Thm)

Notice that the Wronskian computed by Abel's Thm more general (gives the Wronskian of any pair of solutions of our Eq.

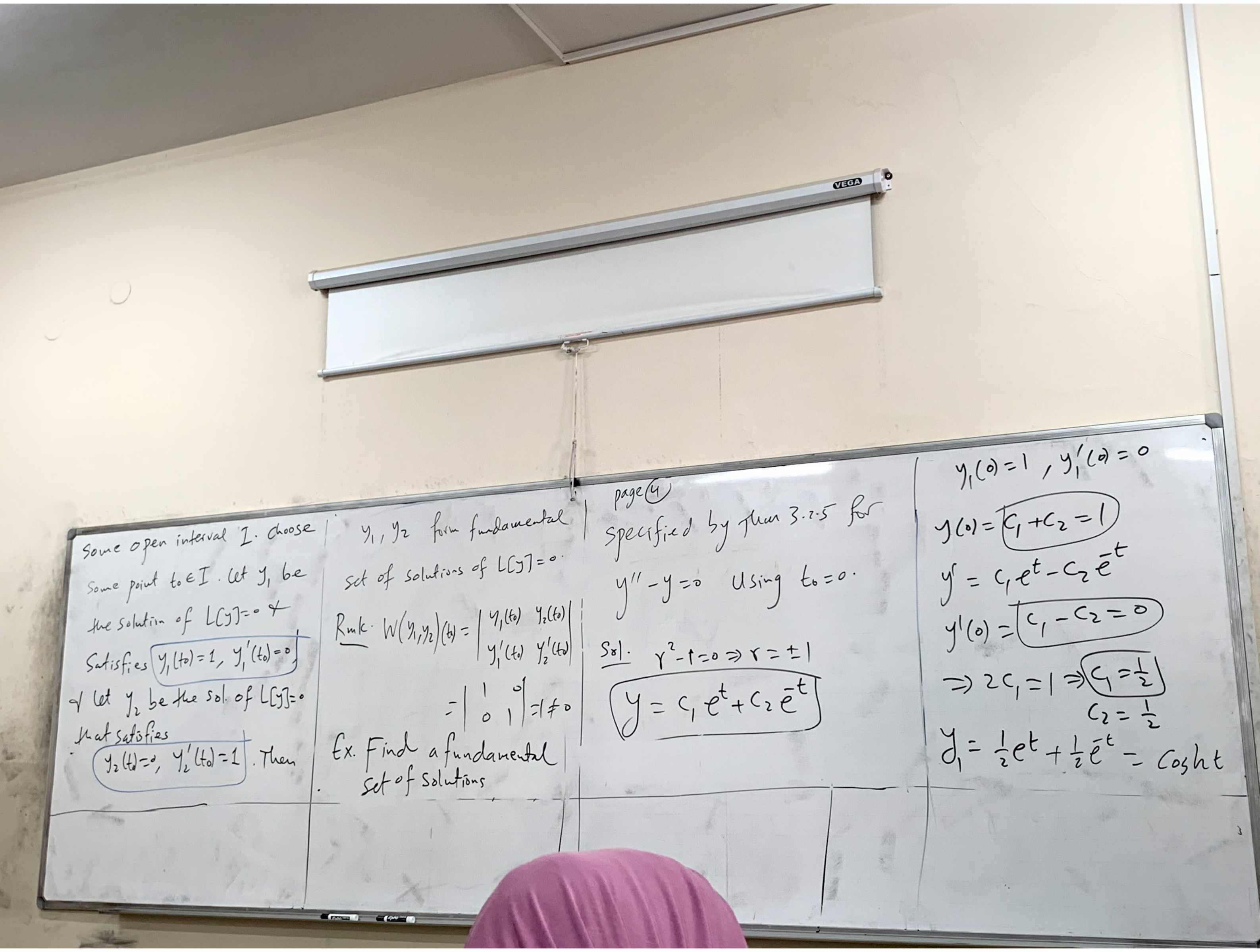
Thm 3.2.5

(How to find a fundamental set of solution ?)

Consider the d.e

$L[y]=y''+p(t)y'+q(t)y=0,$

where p, q are cont. on



Some open interval I. choose some point $t_0 \in I$. let y_1 be the solution of $L[y]=0$ & satisfies $\boxed{y_1(t_0)=1,\ y_1'(t_0)=0}$, & let y_2 be the sol. of $L[y]=0$ that satisfies $\boxed{y_2(t_0)=0,\ y_2'(t_0)=1}$. Then y_1, y_2 form fundamental set of solutions of $L[y]=0$.

Rmk. $W(y_1,y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

page ④

Ex. Find a fundamental set of solutions specified by Thm 3.2.5 for $y''-y=0$ using $t_0=0$.

Sol. $r^2-1=0 \Rightarrow r=\pm 1$

$$\boxed{y = c_1 e^t + c_2 e^{-t}}$$

$y_1(0)=1,\ y_1'(0)=0$

$y(0) = \boxed{c_1+c_2=1}$

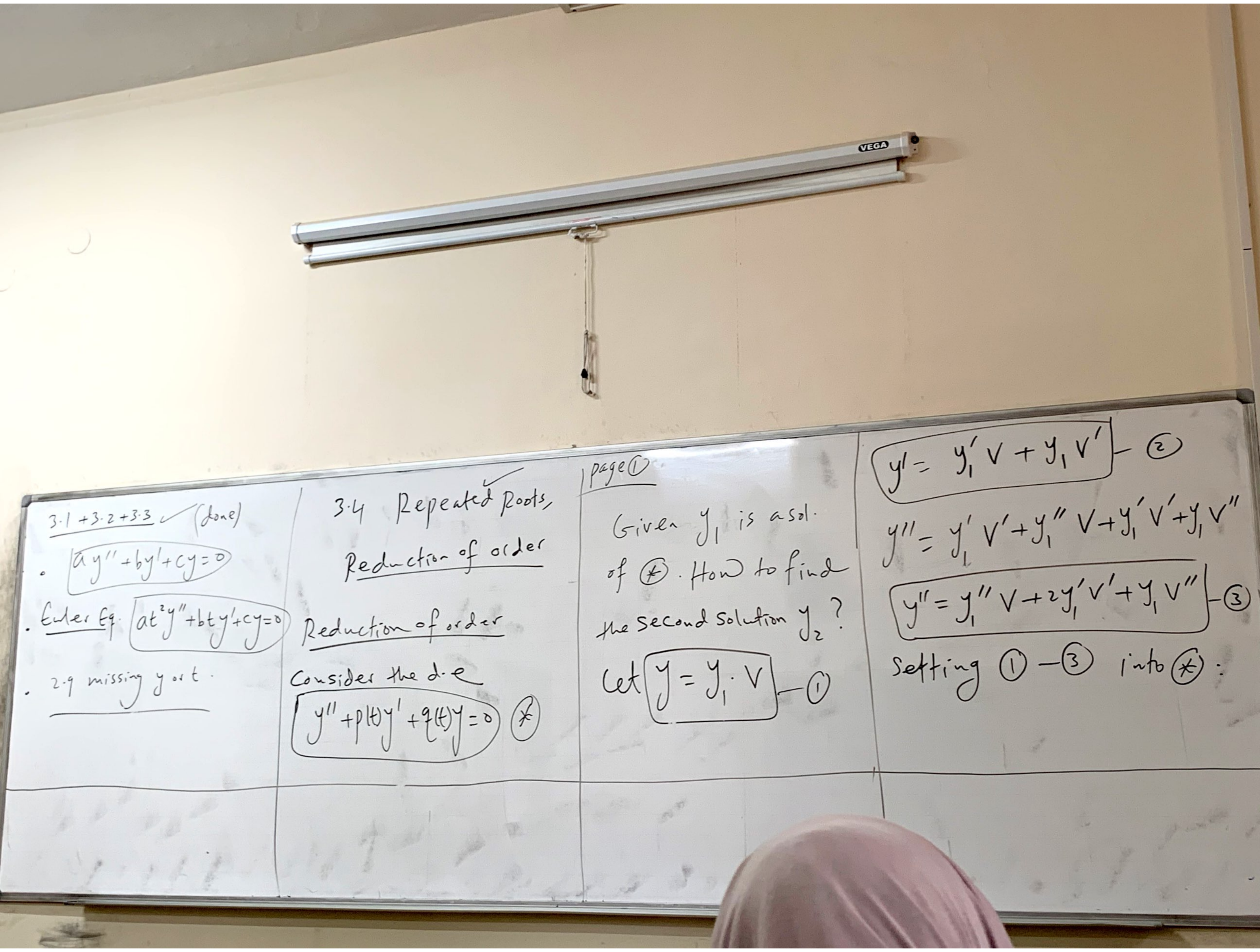
$y' = c_1 e^t - c_2 e^{-t}$

$y'(0) = \boxed{c_1-c_2=0}$

$\Rightarrow 2c_1=1 \Rightarrow \boxed{c_1=\frac{1}{2}}$

$c_2=\frac{1}{2}$

$y_1 = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh t$



$\underline{3.1 + 3.2 + 3.3}$ ✓ (done)

- $\boxed{ay'' + by' + cy = 0}$
- $\underline{\text{Euler Eq.}}$ $\boxed{at^2y'' + bty' + cy = 0}$
- $\underline{2.9 \text{ missing } y \text{ or } t.}$

3.4 Repeated Roots,

$\underline{\text{Reduction of order}}$

$\underline{\text{Reduction of order}}$

Consider the d.e

$$\boxed{y'' + p(t)y' + q(t)y = 0} \quad (*)$$

$\underline{\text{page ①}}$

Given y_1 is a sol. of $(*)$. How to find the second solution y_2?

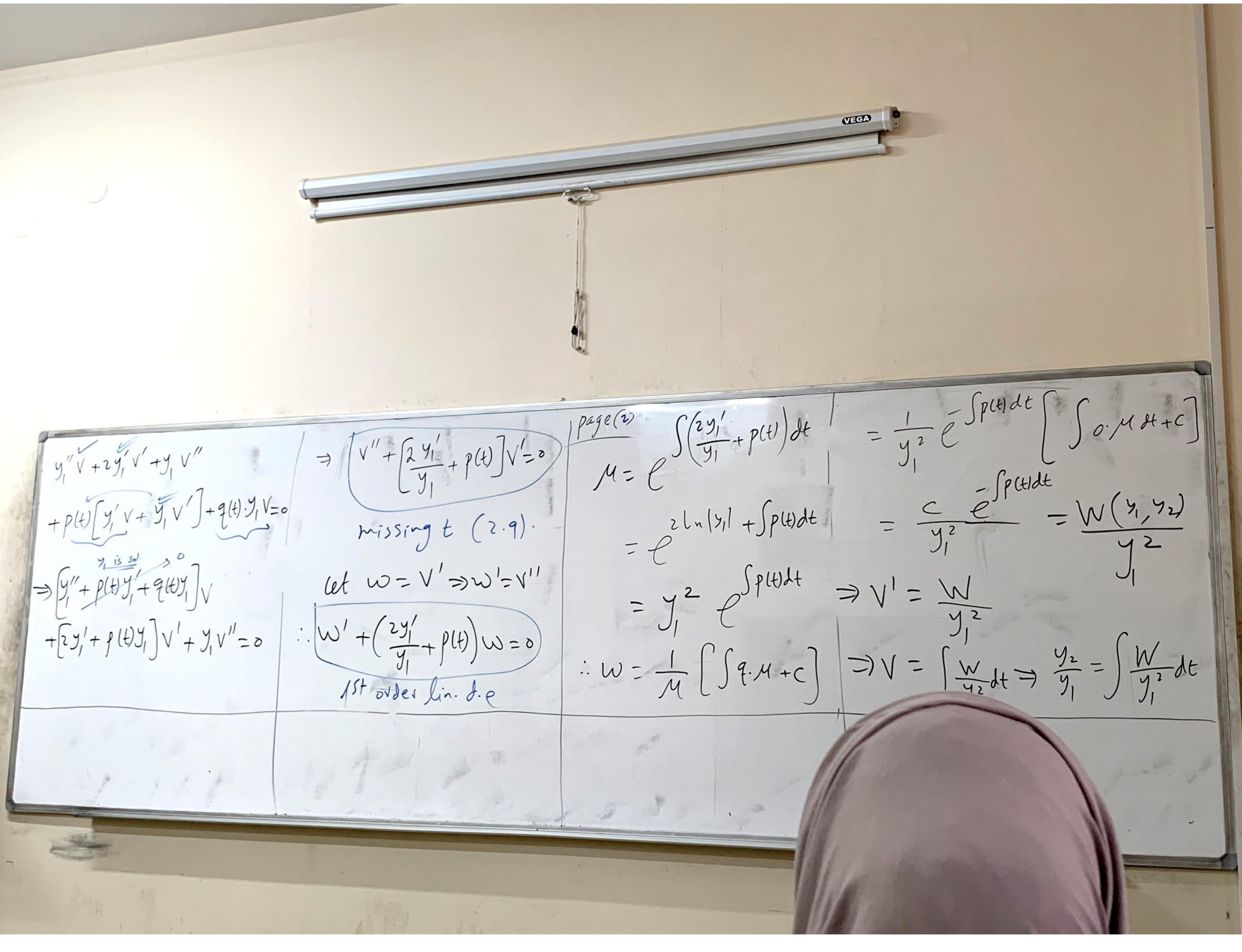
Let $\boxed{y = y_1 \cdot v}$ — ①

$$\boxed{y' = y_1' v + y_1 v'} \text{ — ②}$$

$$y'' = y_1' v' + y_1'' v + y_1' v' + y_1 v''$$

$$\boxed{y'' = y_1'' v + 2y_1' v' + y_1 v''} \text{ — ③}$$

Setting ① — ③ into $(*)$:



$y_1''v + 2y_1'v' + y_1 v''$

$+ p(t)\left[y_1'v + y_1 v'\right] + q(t)\cdot y_1 v = 0$

$\Rightarrow \overset{y_1 \text{ is sol.} \to 0}{\left[y_1'' + \cancel{p(t)y_1'} + q(t)y_1\right]}v$

$+\left[2y_1' + p(t)y_1\right]v' + y_1 v'' = 0$

$\Rightarrow \boxed{v'' + \left[\frac{2y_1'}{y_1} + p(t)\right]v' = 0}$

missing t (2.9).

let $w = v' \Rightarrow w' = v''$

$\therefore \boxed{w' + \left(\frac{2y_1'}{y_1} + p(t)\right)w = 0}$

1st order lin. d.e

page(2)

$\mu = e^{\int\left(\frac{2y_1'}{y_1} + p(t)\right)dt}$

$= e^{2\ln|y_1| + \int p(t)dt}$

$= y_1^2\, e^{\int p(t)dt}$

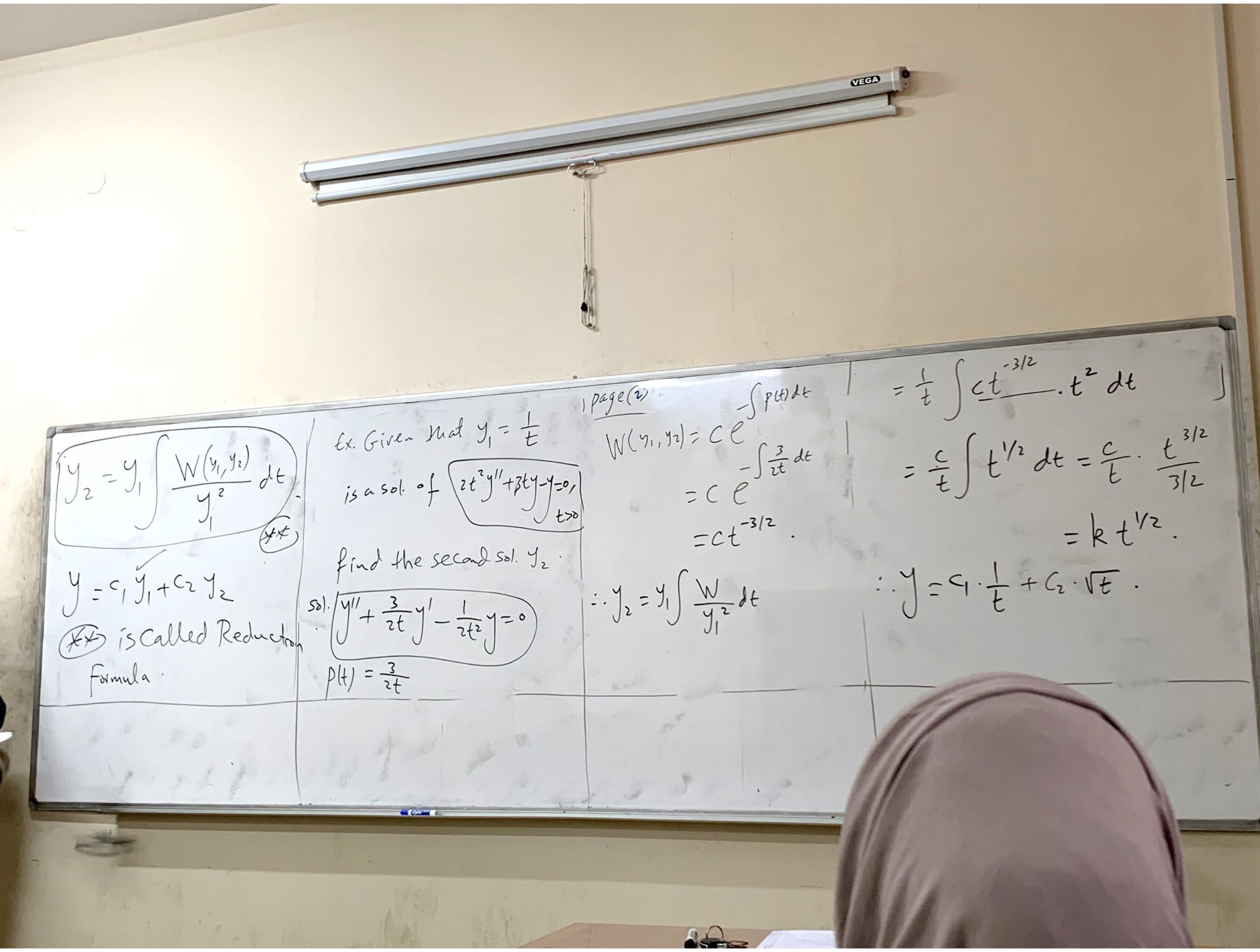
$\therefore w = \frac{1}{\mu}\left[\int q\cdot\mu + c\right]$

$= \frac{1}{y_1^2}\, e^{-\int p(t)dt}\left[\int 0\cdot\mu\, dt + c\right]$

$= \frac{c\, e^{-\int p(t)dt}}{y_1^2} = \frac{W(y_1, y_2)}{y_1^2}$

$\Rightarrow v' = \frac{W}{y_1^2}$

$\Rightarrow v = \int \frac{W}{y_2}dt \Rightarrow \frac{y_2}{y_1} = \int \frac{W}{y_1^2}dt$



$$\boxed{y_2 = y_1 \int \frac{W(y_1, y_2)}{y_1^2}\, dt} \quad (**)$$

$$y = c_1 y_1 + c_2 y_2$$

(**) is Called Reduction Formula.

Ex. Given that $y_1 = \frac{1}{t}$ is a sol. of $\boxed{2t^2y'' + 3ty' - y = 0, \ t>0}$

find the second sol. y_2.

sol. $\boxed{y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0}$

$p(t) = \frac{3}{2t}$

page(2)

$$W(y_1, y_2) = c\,e^{-\int p(t)\,dt}$$

$$= c\,e^{-\int \frac{3}{2t}\,dt}$$

$$= c\,t^{-3/2}.$$

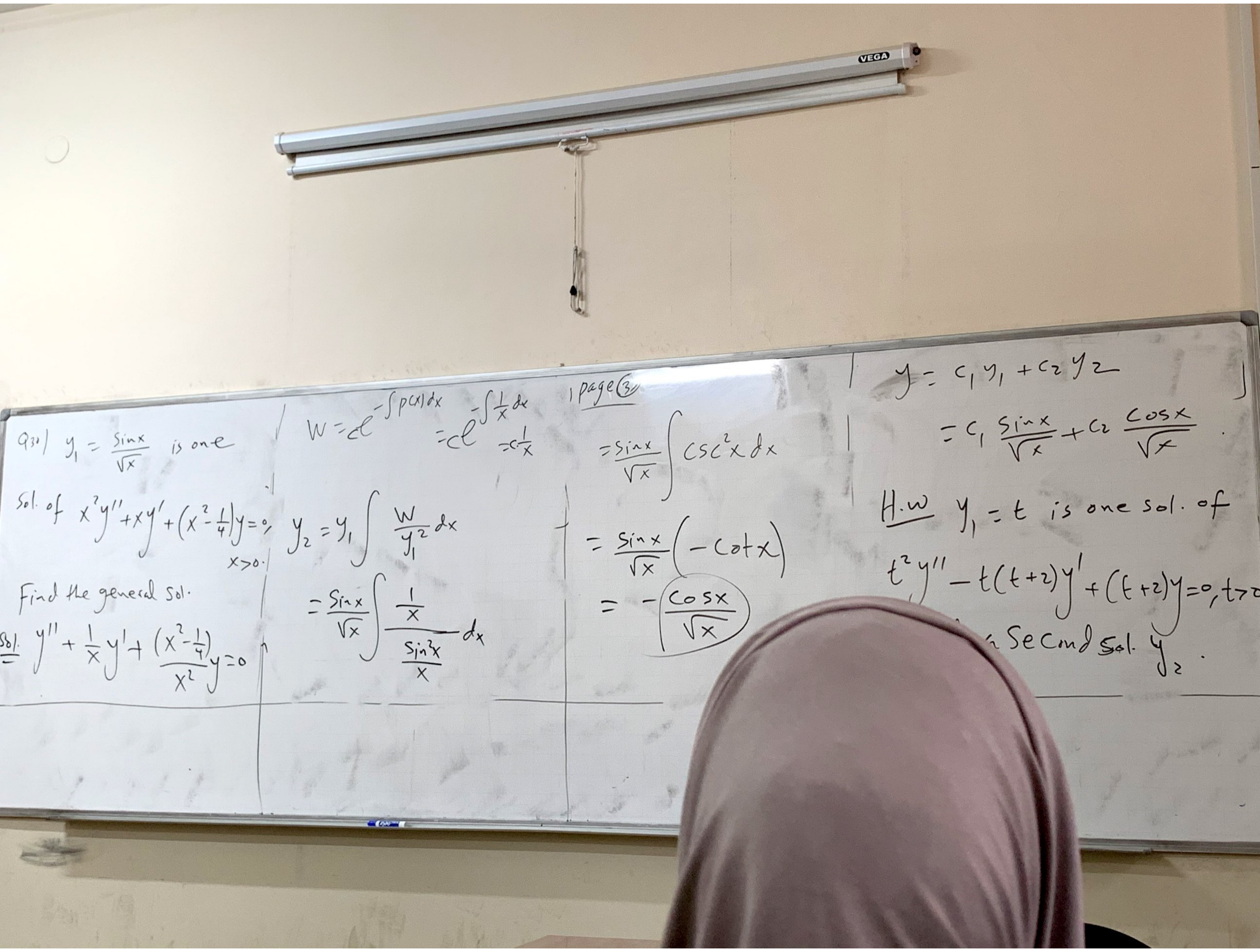
$$\therefore y_2 = y_1 \int \frac{W}{y_1^2}\,dt$$

$$= \frac{1}{t}\int \underline{c\,t^{-3/2}} \cdot t^2\,dt$$

$$= \frac{c}{t}\int t^{1/2}\,dt = \frac{c}{t}\cdot\frac{t^{3/2}}{3/2}$$

$$= k\,t^{1/2}.$$

$$\therefore y = c_1 \cdot \frac{1}{t} + c_2 \cdot \sqrt{t}.$$



Q30/ $y_1 = \frac{\sin x}{\sqrt{x}}$ is one

Sol. of $x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$, $x > 0$.

Find the general Sol.

Sol. $y'' + \frac{1}{x}y' + \frac{(x^2 - \frac{1}{4})}{x^2}y = 0$

$W = ce^{-\int p(x)dx}$

$= ce^{-\int \frac{1}{x}dx}$

$= c\frac{1}{x}$

$y_2 = y_1 \int \frac{W}{y_1^2}dx$

$= \frac{\sin x}{\sqrt{x}} \int \frac{\frac{1}{x}}{\frac{\sin^2 x}{x}} dx$

| page ③

$= \frac{\sin x}{\sqrt{x}} \int \csc^2 x \, dx$

$= \frac{\sin x}{\sqrt{x}}(-\cot x)$

$= -\boxed{\frac{\cos x}{\sqrt{x}}}$

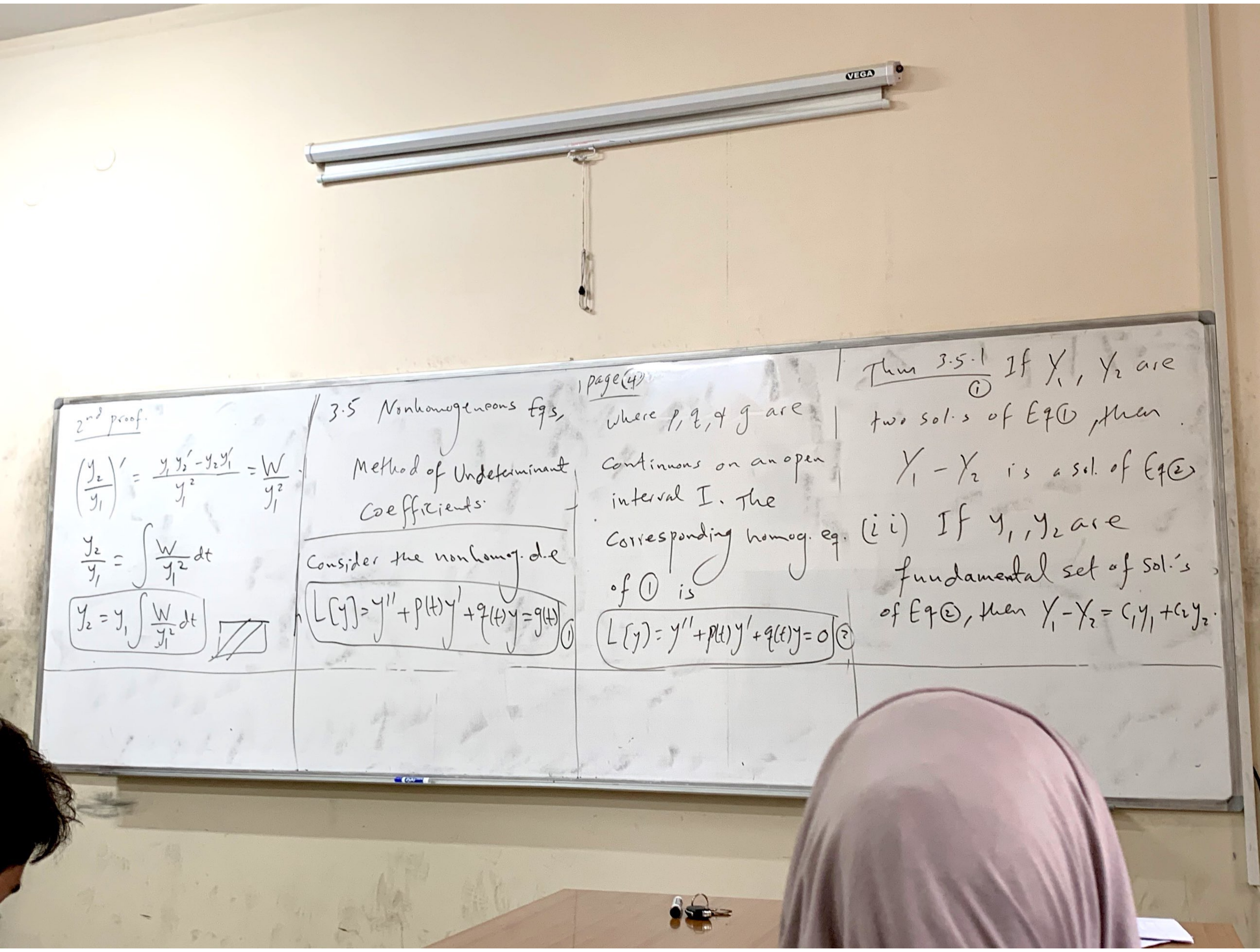
$y = c_1 y_1 + c_2 y_2$

$= c_1 \frac{\sin x}{\sqrt{x}} + c_2 \frac{\cos x}{\sqrt{x}}$.

H.W $y_1 = t$ is one sol. of

$t^2 y'' - t(t+2)y' + (t+2)y = 0$, $t > 0$

a Second Sol. y_2.



2nd proof.

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = \frac{W}{y_1^2}.$$

$$\frac{y_2}{y_1} = \int \frac{W}{y_1^2}\,dt$$

$$\boxed{y_2 = y_1 \int \frac{W}{y_1^2}\,dt}$$ ▨

3.5 Nonhomogeneous Eqs,

Method of Undeterminant Coefficients.

Consider the nonhomog. d.e

$$\boxed{L[y] = y'' + p(t)y' + q(t)y = g(t)} \quad ①$$

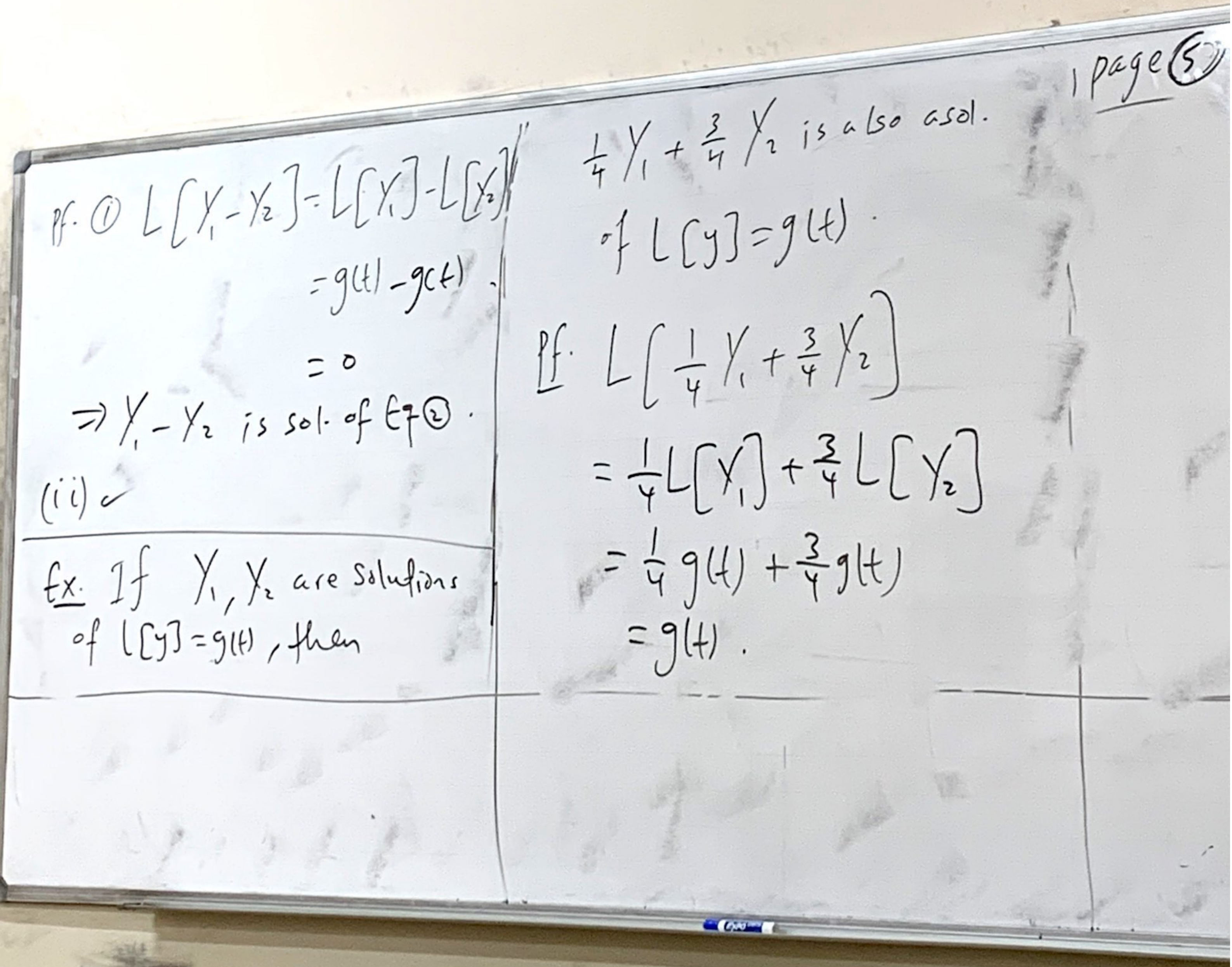
page(4)

where $p, q, \& g$ are continuous on an open interval I. The Corresponding homog. eq. of ① is

$$\boxed{L(y) = y'' + p(t)y' + q(t)y = 0} \quad ②$$

Thm 3.5.1 ⓘ If Y_1, Y_2 are two sol.s of Eq①, then $Y_1 - Y_2$ is a sol. of Eq②

(ii) If y_1, y_2 are fundamental set of sol.'s of Eq②, then $Y_1 - Y_2 = c_1 y_1 + c_2 y_2$.



page ⑤

Pf. ① $L[Y_1 - Y_2] = L[Y_1] - L[Y_2]$

$= g(t) - g(t)$

$= 0$

$\Rightarrow Y_1 - Y_2$ is sol. of Eq ②.

(ii) ✓

Ex. If Y_1, Y_2 are solutions of $L[y] = g(t)$, then

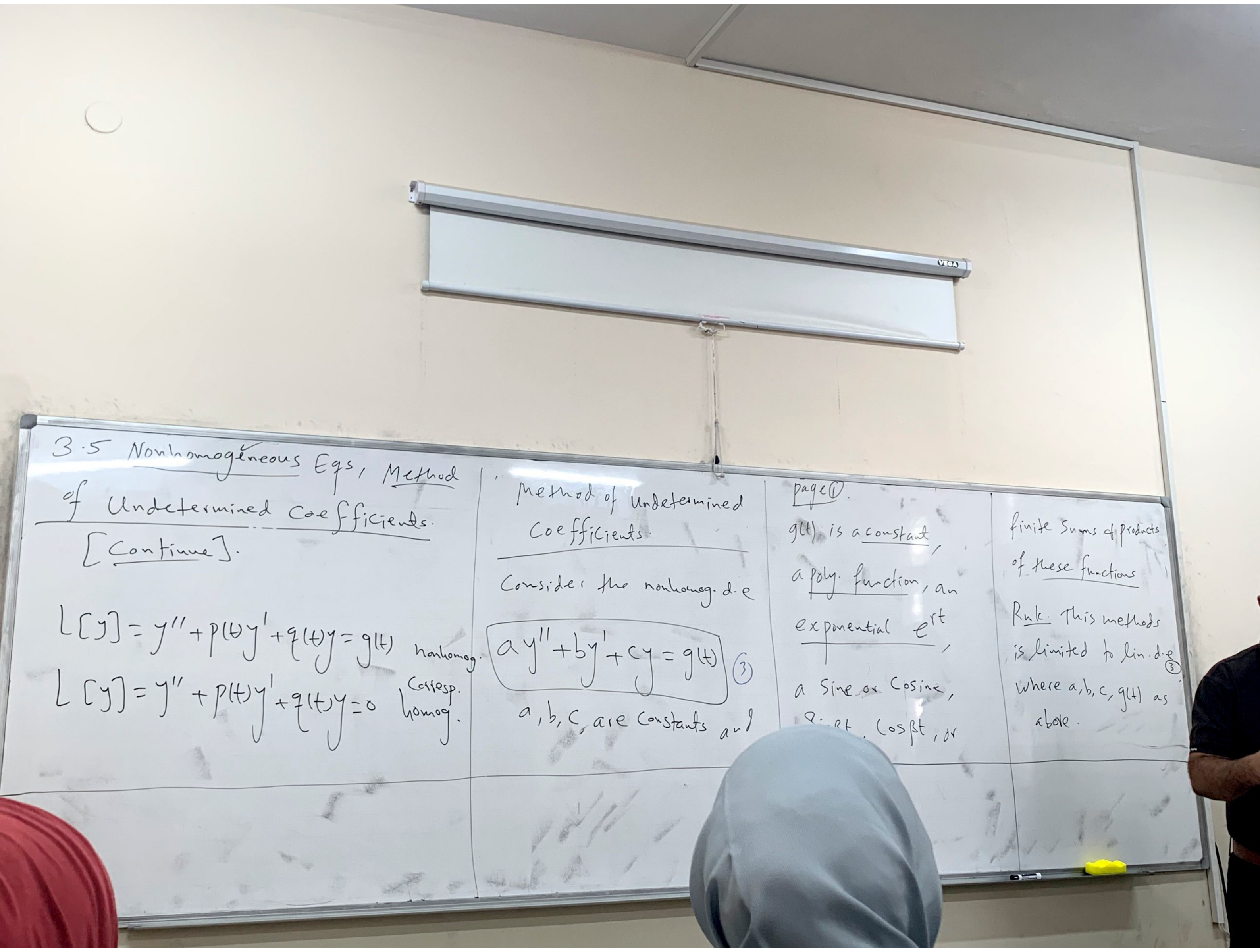
$\frac{1}{4} Y_1 + \frac{3}{4} Y_2$ is also a sol. of $L[y] = g(t)$.

Pf. $L\left[\frac{1}{4} Y_1 + \frac{3}{4} Y_2\right]$

$= \frac{1}{4} L[Y_1] + \frac{3}{4} L[Y_2]$

$= \frac{1}{4} g(t) + \frac{3}{4} g(t)$

$= g(t)$.



3.5 Nonhomogeneous Eqs, Method of Undetermined Coefficients.

[Continue].

$L[y] = y'' + p(t)y' + q(t)y = g(t)$ nonhomog.

$L[y] = y'' + p(t)y' + q(t)y = 0$ Corresp. homog.

Method of Undetermined Coefficients.

Consider the nonhomog. d.e

$$\boxed{ay'' + by' + cy = g(t)} \quad ③$$

a, b, c, are constants and

page ①.

g(t) is a constant,

a poly. function, an

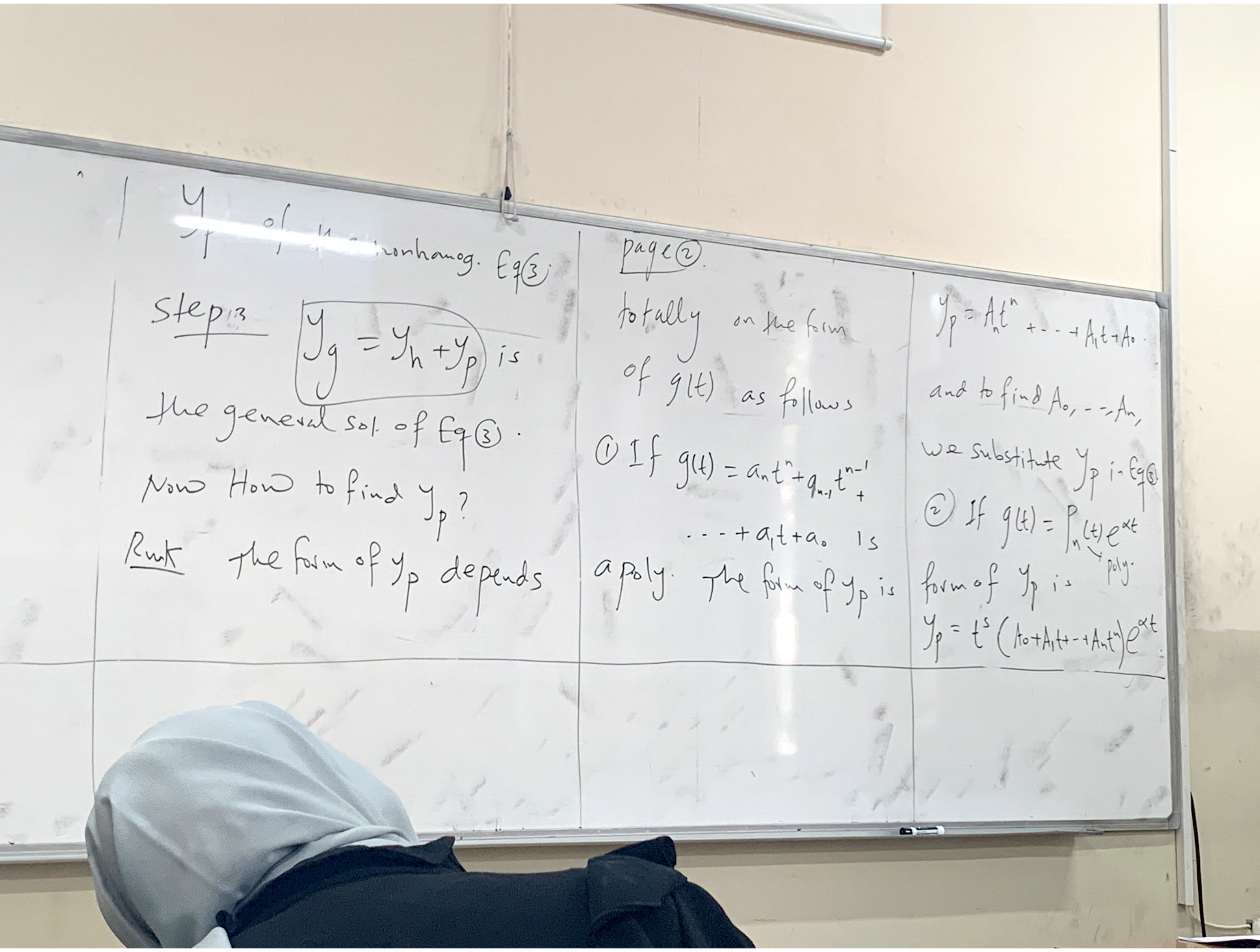
exponential e^{rt},

a Sine or Cosine,

S[illegible] βt, $\cos\beta t$, or

finite Sums of products of these functions

Rmk. This methods is limited to lin. d.e ③ where a, b, c, g(t) as above.



y_p of the nonhomog. Eq③:

Step 3 $\boxed{y_g = y_h + y_p}$ is the general sol. of Eq③.

Now How to find y_p?

Rmk The form of y_p depends

page②

totally on the form of $g(t)$ as follows

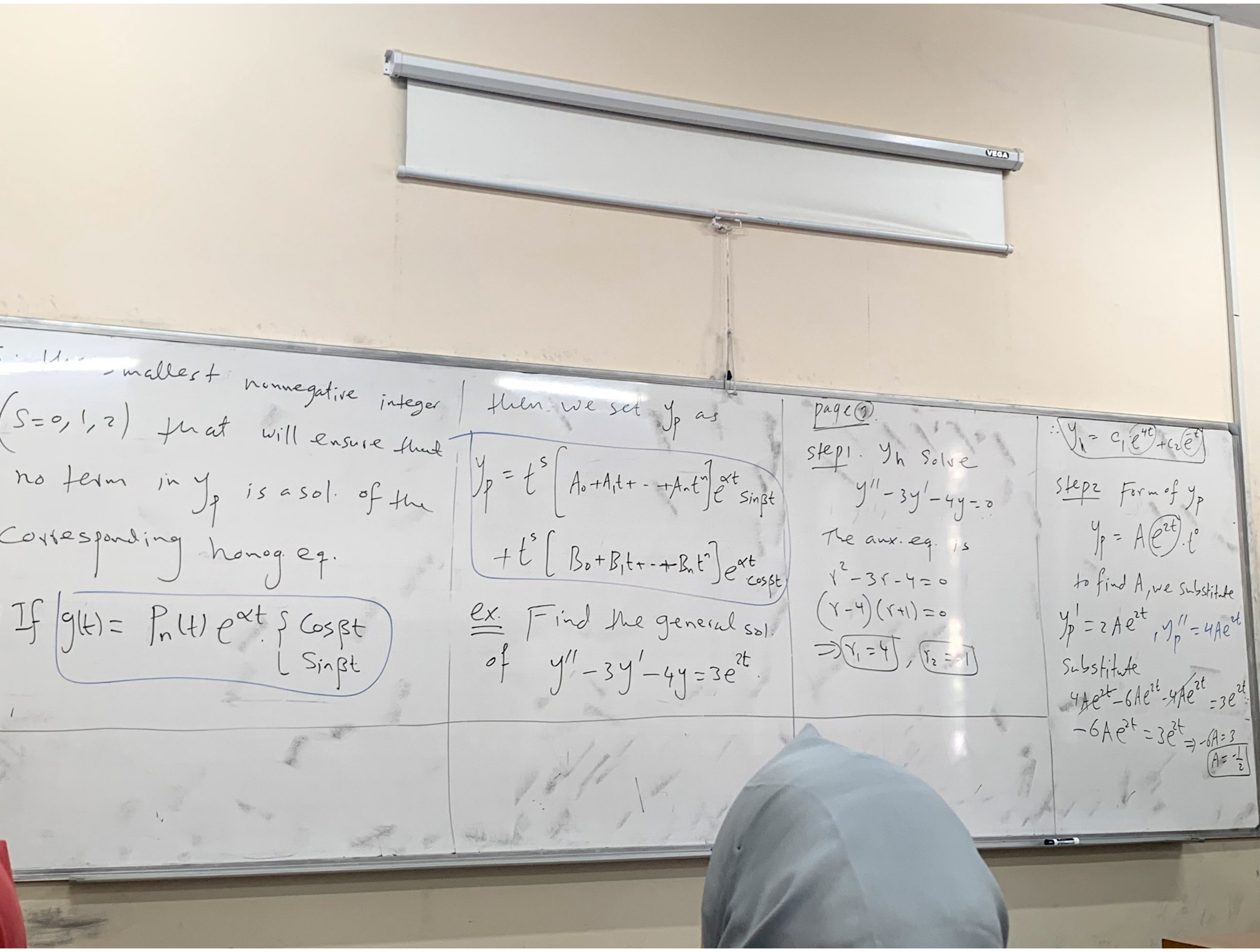
① If $g(t) = a_n t^n + q_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ is a poly. The form of y_p is

$$y_p = A_n t^n + \cdots + A_1 t + A_0.$$

and to find $A_0, \ldots, A_n$, we substitute y_p in Eq③

② If $g(t) = \underbrace{P_n(t)}_{\text{poly.}} e^{\alpha t}$ form of y_p is

$$y_p = t^s (A_0 + A_1 t + \cdots + A_n t^n) e^{\alpha t}$$



∴ the smallest nonnegative integer ($s = 0, 1, 2$) that will ensure that no term in y_p is a sol. of the corresponding homog. eq.

If $\boxed{g(t) = P_n(t)\, e^{\alpha t} \cdot \begin{cases} \cos\beta t \\ \sin\beta t \end{cases}}$

then we set y_p as

$$\boxed{\begin{aligned} y_p &= t^s\left[A_0 + A_1 t + \cdots + A_n t^n\right] e^{\alpha t} \sin\beta t \\ &+ t^s\left[B_0 + B_1 t + \cdots + B_n t^n\right] e^{\alpha t} \cos\beta t \end{aligned}}$$

ex. Find the general sol. of $y'' - 3y' - 4y = 3e^{2t}$.

page ③

step1. y_h Solve

$$y'' - 3y' - 4y = 0$$

The aux. eq. is

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0$$

$$\Rightarrow \boxed{r_1 = 4},\ \boxed{r_2 = -1}$$

$$\therefore \boxed{y_h = c_1 e^{4t} + c_2 e^{-t}}$$

step2 Form of y_p

$$y_p = A e^{2t} \cdot t^0$$

to find A, we substitute

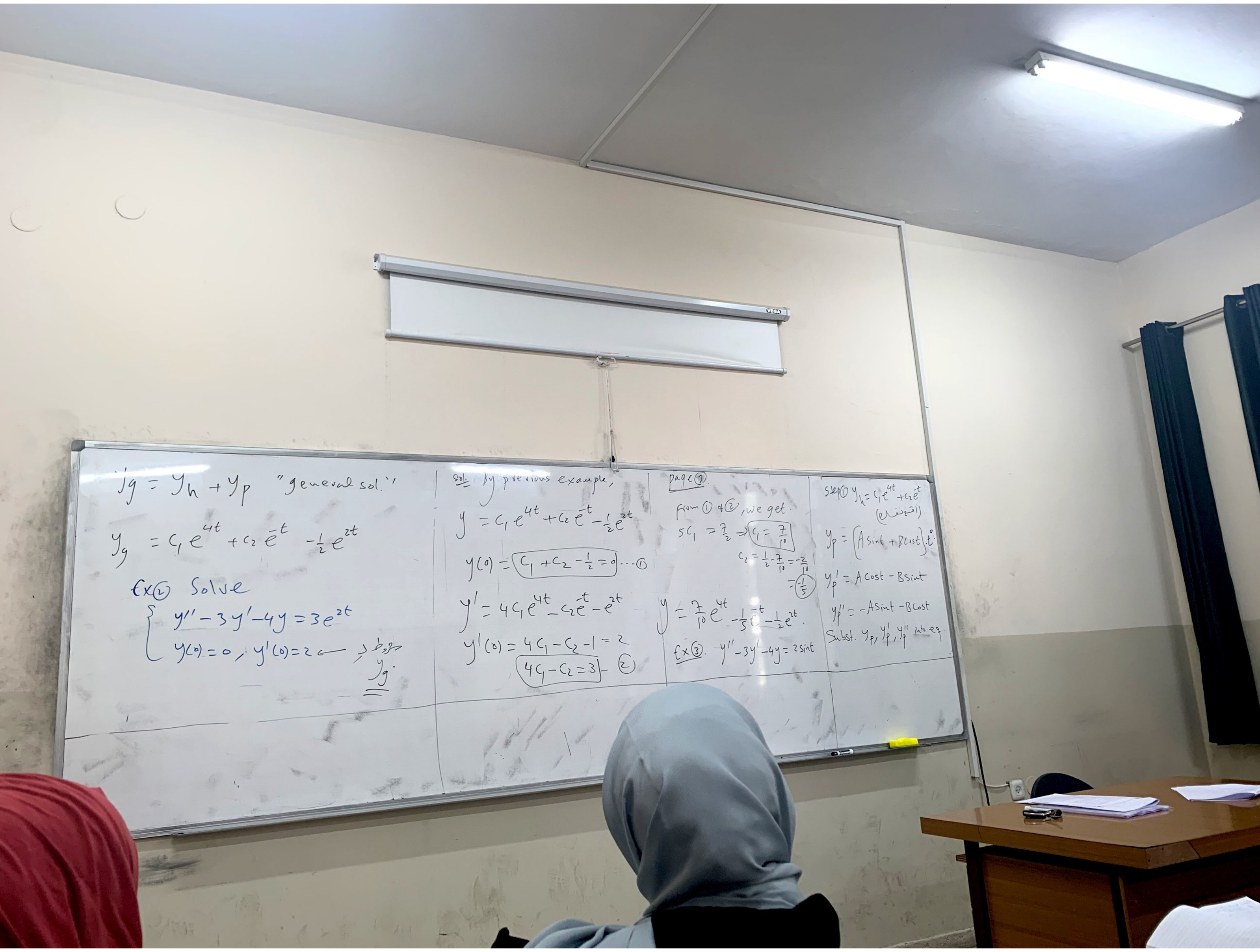
$$y_p' = 2Ae^{2t}, \quad y_p'' = 4Ae^{2t}$$

Substitute

$$\cancel{4Ae^{2t}} - 6Ae^{2t} - \cancel{4Ae^{2t}} = 3e^{2t}$$

$$-6Ae^{2t} = 3e^{2t} \Rightarrow -6A = 3$$

$$\boxed{A = -\frac{1}{2}}$$



$y_g = y_h + y_p$ "general sol."

$y_g = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t}$

Ex ② Solve

$$\begin{cases} y'' - 3y' - 4y = 3e^{2t} \\ y(0) = 0 , y'(0) = 2 \end{cases}$$

$\leftarrow$ [illegible] y_g.

Sol. By previous example,

$y = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t}$

$y(0) = \boxed{c_1 + c_2 - \frac{1}{2} = 0}$ ... ①

$y' = 4c_1 e^{4t} - c_2 e^{-t} - e^{2t}$

$y'(0) = 4c_1 - c_2 - 1 = 2$

$\boxed{4c_1 - c_2 = 3}$ – ②

page ⑦

From ① & ②, we get.

$5c_1 = \frac{7}{2} \Rightarrow \boxed{c_1 = \frac{7}{10}}$

$c_2 = \frac{1}{2} - \frac{7}{10} = \frac{-2}{10} = \frac{-1}{5}$

$y = \frac{7}{10} e^{4t} - \frac{1}{5} e^{-t} - \frac{1}{2} e^{2t}$.

Ex ③. $y'' - 3y' - 4y = 2\sin t$

Step ① $y_h = c_1 e^{4t} + c_2 e^{-t}$

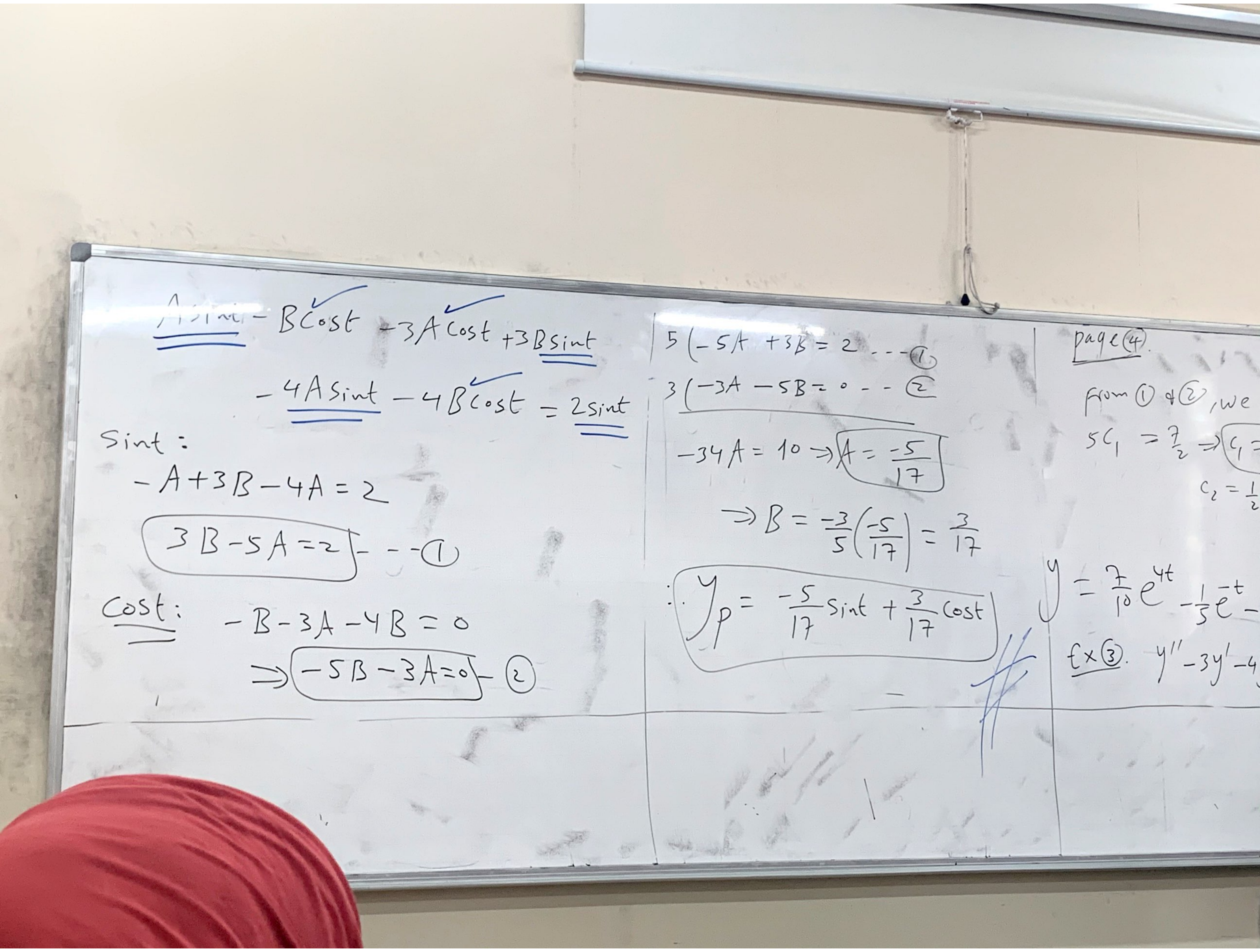
([illegible])

$y_p = \left[A\sin t + B\cos t \right] . t^0$

$y_p' = A\cos t - B\sin t$

$y_p'' = -A\sin t - B\cos t$

Subst. y_p, y_p', y_p'' into eq.



$A\sin t - B\cos t - 3A\cos t + 3B\sin t$

$-4A\sin t - 4B\cos t = 2\sin t$

$\underline{\sin t}$:

$-A+3B-4A=2$

$\boxed{3B-5A=2}$ - - - ①

$\underline{\cos t}$: $-B-3A-4B=0$

$\Rightarrow \boxed{-5B-3A=0}$ - ②

$5\,(-5A+3B=2$ - - - ①

$3\,(-3A-5B=0$ - - - ②

$-34A=10 \Rightarrow \boxed{A=\frac{-5}{17}}$

$\Rightarrow B=\frac{-3}{5}\left(\frac{-5}{17}\right)=\frac{3}{17}$

$\therefore \boxed{y_p=\frac{-5}{17}\sin t+\frac{3}{17}\cos t}$

$\underline{\text{page ④}}$.

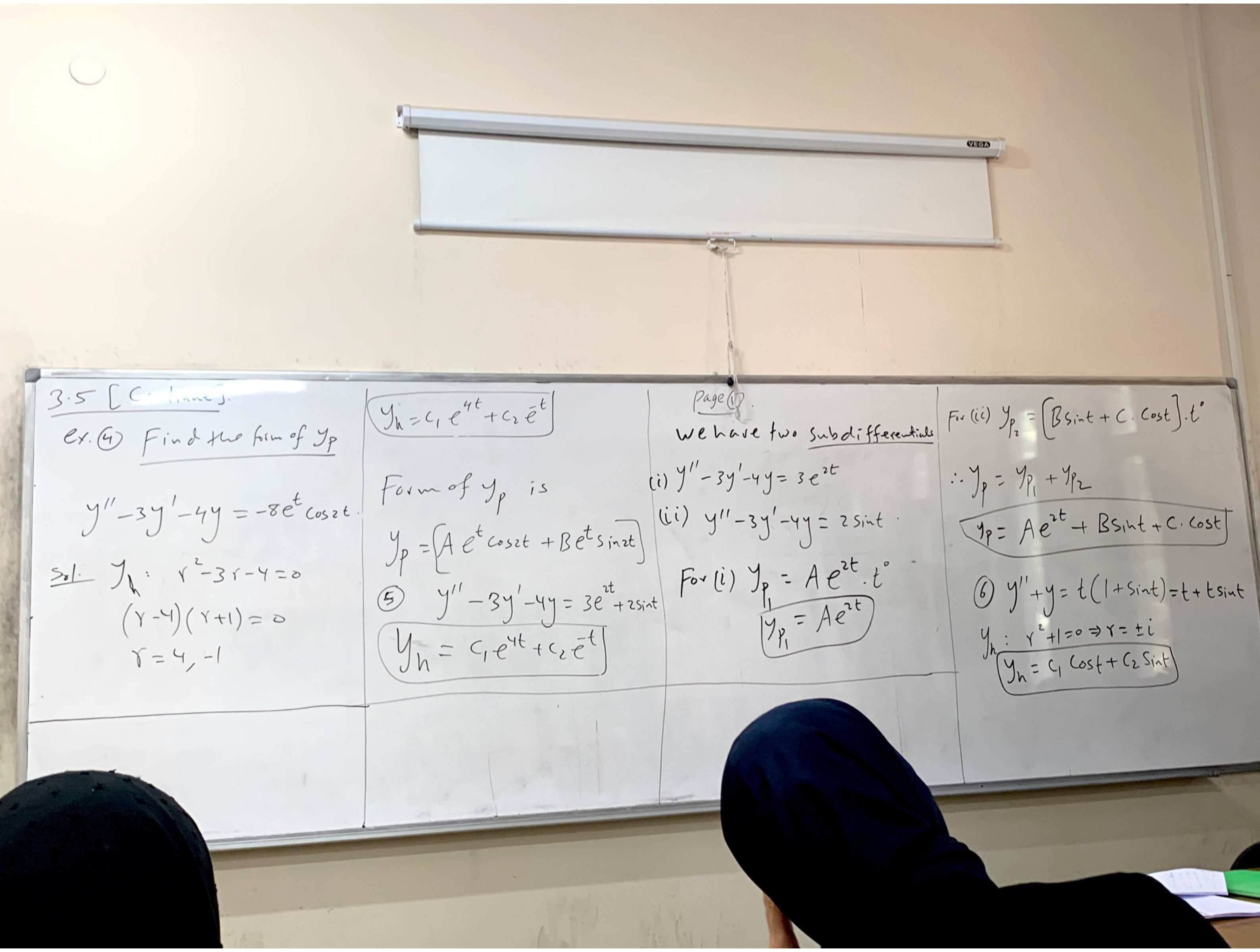
From ① & ②, we

$5c_1 = \frac{7}{2} \Rightarrow c_1 =$

$c_2 = \frac{1}{2}$

$y = \frac{7}{10}e^{4t} - \frac{1}{5}e^{-t} -$

$\underline{\text{Ex ③}}$. $y''-3y'-4$



3.5 [Continue].

ex. ④ Find the form of y_p

$$y''-3y'-4y=-8e^{t}\cos 2t.$$

Sol. y_h: $r^2-3r-4=0$

$(r-4)(r+1)=0$

$r=4,-1$

$$\boxed{y_h=c_1e^{4t}+c_2e^{-t}}$$

Form of y_p is

$$y_p=\left[Ae^{t}\cos 2t+Be^{t}\sin 2t\right]$$

⑤ $y''-3y'-4y=3e^{2t}+2\sin t$

$$\boxed{y_h=c_1e^{4t}+c_2e^{-t}}$$

Page ⑪.

we have two subdifferentials

(i) $y''-3y'-4y=3e^{2t}$

(ii) $y''-3y'-4y=2\sin t$.

For (i) $y_{p_1}=Ae^{2t}\cdot t^{0}$

$$\boxed{y_{p_1}=Ae^{2t}}$$

For (ii) $y_{p_2}=\left[B\sin t+C.\cos t\right].t^{0}$

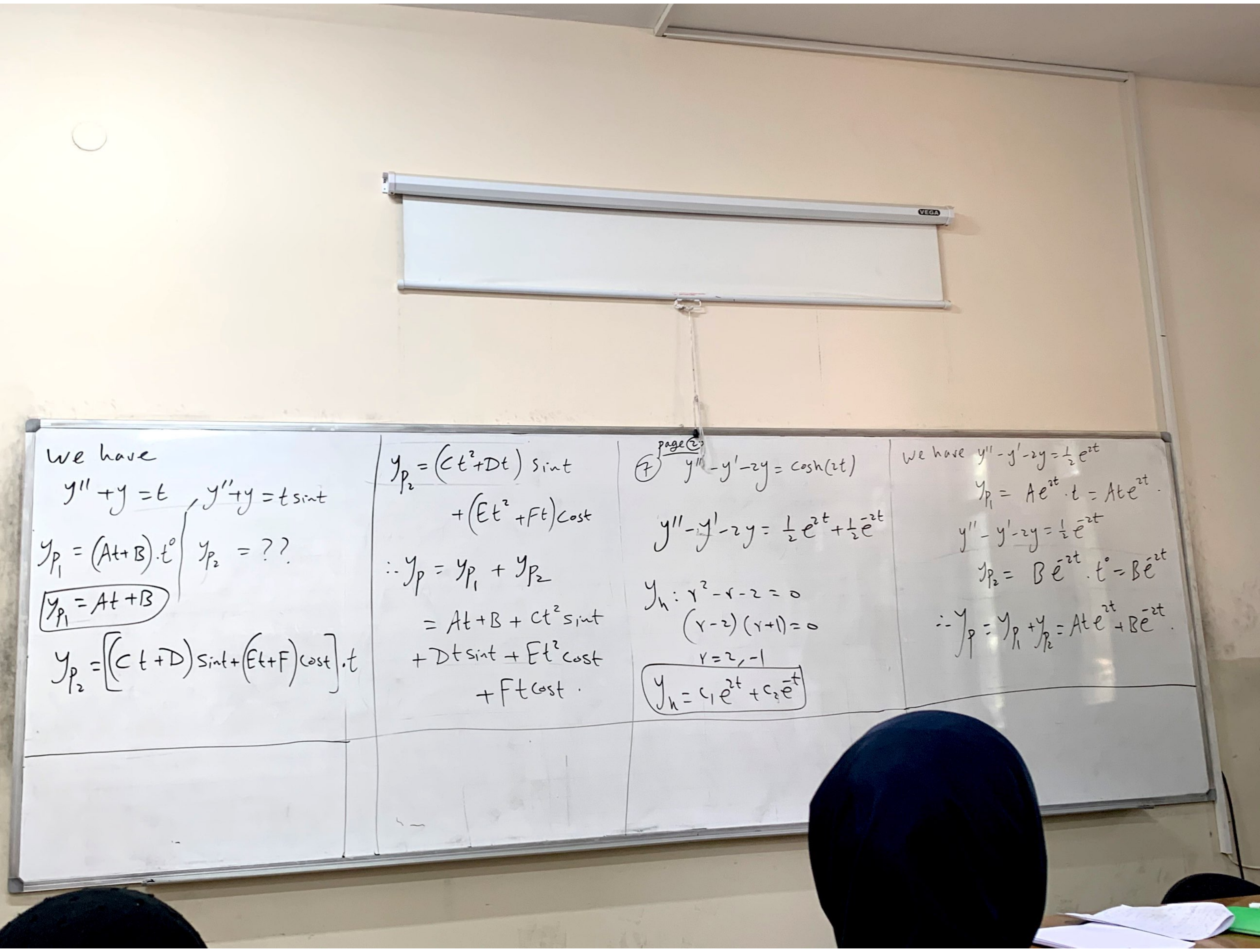
$\therefore y_p=y_{p_1}+y_{p_2}$

$$\boxed{y_p=Ae^{2t}+B\sin t+C\cdot\cos t}$$

⑥ $y''+y=t(1+\sin t)=t+t\sin t$

y_h: $r^2+1=0\Rightarrow r=\pm i$

$$\boxed{y_h=c_1\cos t+c_2\sin t}$$



We have

$y'' + y = t$, $y'' + y = t \sin t$

$y_{p_1} = (At + B) \cdot t^0$ | $y_{p_2} = ??$

$\boxed{y_{p_1} = At + B}$

$y_{p_2} = \left[(Ct + D) \sin t + (Et + F) \cos t \right] \cdot t$

$y_{p_2} = (Ct^2 + Dt) \sin t + (Et^2 + Ft) \cos t$

$\therefore y_p = y_{p_1} + y_{p_2}$

$= At + B + Ct^2 \sin t + Dt \sin t + Et^2 \cos t + Ft \cos t .$

page ②

⑦ $y'' - y' - 2y = \cosh(2t)$

$y'' - y' - 2y = \frac{1}{2} e^{2t} + \frac{1}{2} e^{-2t}$

$y_h : r^2 - r - 2 = 0$

$(r - 2)(r + 1) = 0$

$r = 2, -1$

$\boxed{y_h = c_1 e^{2t} + c_2 e^{-t}}$

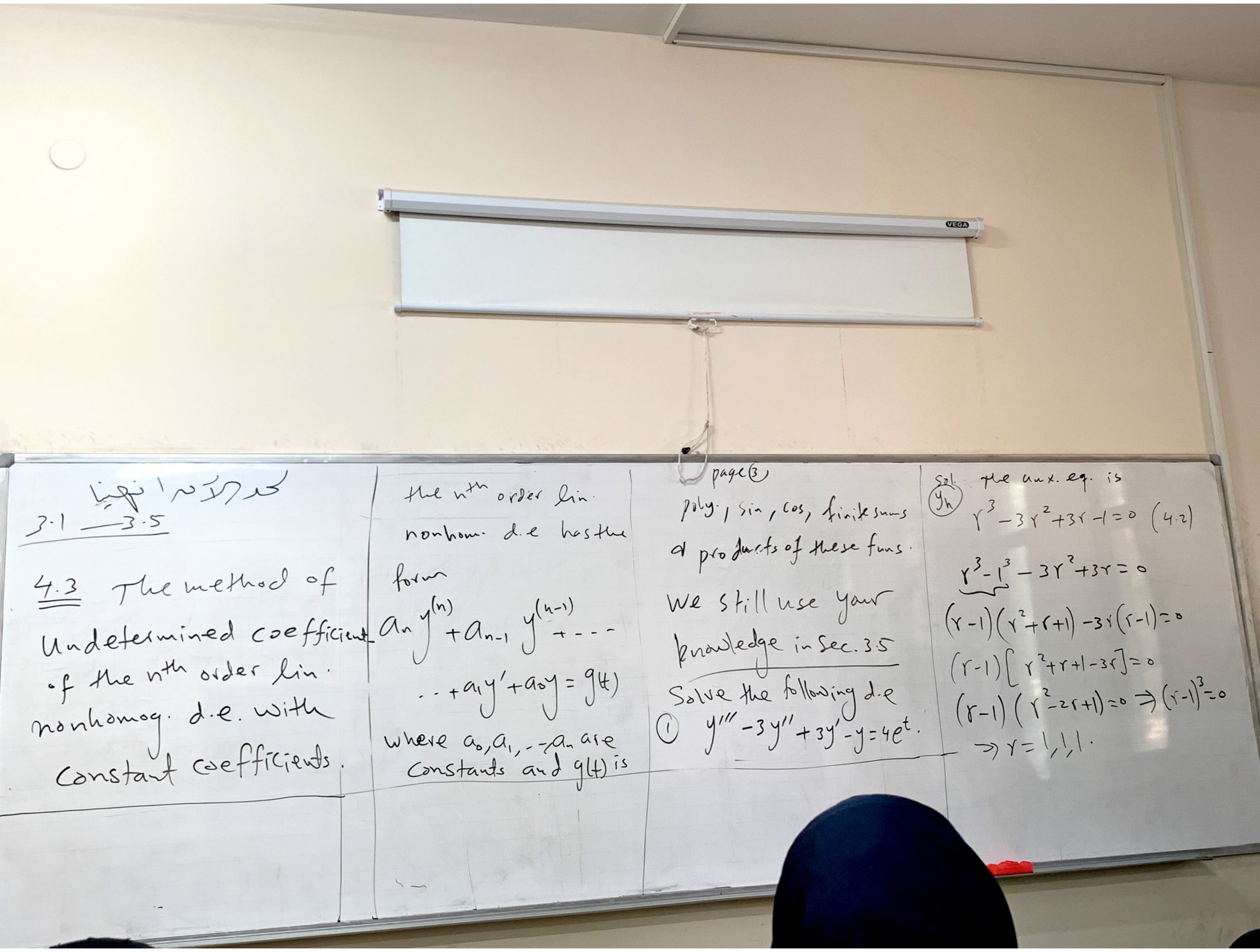
We have $y'' - y' - 2y = \frac{1}{2} e^{2t}$

$y_{p_1} = A e^{2t} \cdot t = A t e^{2t} .$

$y'' - y' - 2y = \frac{1}{2} e^{-2t}$

$y_{p_2} = B e^{-2t} \cdot t^0 = B e^{-2t}$

$\therefore y_p = y_{p_1} + y_{p_2} = A t e^{2t} + B e^{-2t} .$



لحد الآن انهينا

3.1 — 3.5

4.3 The method of Undetermined coefficient of the nth order lin. nonhomog. d.e. with Constant coefficients.

the nth order lin. nonhom. d.e has the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t)$$

where $a_0, a_1, \cdots, a_n$ are Constants and $g(t)$ is

page ③

poly., sin, cos, finite sums & products of these funs.

We still use your knowledge in Sec. 3.5

Solve the following d.e

① $y''' - 3y'' + 3y' - y = 4e^t$.

Sol. the aux. eq. is

$\boxed{y_h}$ $r^3 - 3r^2 + 3r - 1 = 0$ (4.2)

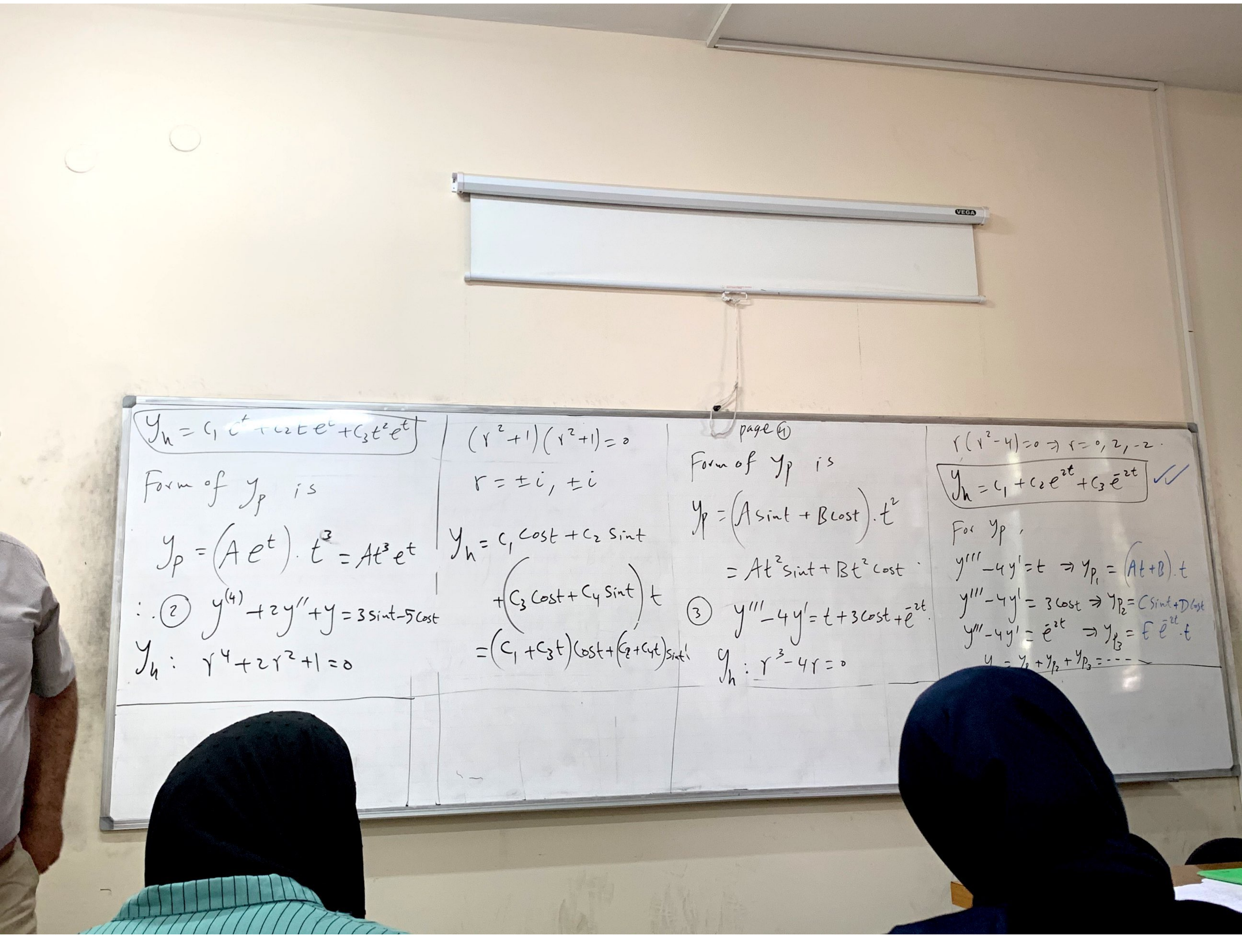
$\underbrace{r^3 - 1^3} - 3r^2 + 3r = 0$

$(r-1)(r^2+r+1) - 3r(r-1) = 0$

$(r-1)[r^2+r+1-3r] = 0$

$(r-1)(r^2-2r+1) = 0 \Rightarrow (r-1)^3 = 0$

$\Rightarrow r = 1, 1, 1.$



$$\boxed{y_h = c_1 e^t + c_2 t e^t + c_3 t^2 e^t}$$

Form of y_p is

$$y_p = (Ae^t) \cdot t^3 = At^3 e^t$$

∴ ② $y^{(4)} + 2y'' + y = 3\sin t - 5\cos t$

y_h : $r^4 + 2r^2 + 1 = 0$

$$(r^2+1)(r^2+1) = 0$$

$$r = \pm i, \pm i$$

$$y_h = c_1 \cos t + c_2 \sin t + (c_3 \cos t + c_4 \sin t)\, t$$

$$= (c_1 + c_3 t)\cos t + (c_2 + c_4 t)\sin t$$

page ④

Form of y_p is

$$y_p = (A\sin t + B\cos t) \cdot t^2$$

$$= At^2 \sin t + Bt^2 \cos t$$

③ $y''' - 4y' = t + 3\cos t + e^{-2t}$.

y_h : $r^3 - 4r = 0$

$r(r^2-4) = 0 \Rightarrow r = 0, 2, -2$.

$$\boxed{y_h = c_1 + c_2 e^{2t} + c_3 e^{-2t}} \checkmark\checkmark$$

For y_p,

$y''' - 4y' = t \Rightarrow y_{p_1} = (At+B) \cdot t$

$y''' - 4y' = 3\cos t \Rightarrow y_{p_2} = C\sin t + D\cos t$

$y''' - 4y' = e^{-2t} \Rightarrow y_{p_3} = E e^{-2t} \cdot t$

$y = y_{p_1} + y_{p_2} + y_{p_3} = \cdots$

④ $y^{(5)} + 4y^{(3)} = \underbrace{\cos^4 t - \sin^4 t}_{\cos(2t)}$

$y^{(5)} + 4y^{(3)} = \cos 2t$.

Sol. $r^5 + 4r^3 = 0$

$r^3(r^2+4) = 0$

$r = 0, 0, 0, \pm 2i$

$y_h = (C_1 + C_2 t$ … $ot)$

$+ (C_4 C$ … $2t)$

$\boxed{y_h = C_1 + C_2 t \quad + C_4 C}$

$y_p = (A$

page ④

⑤ $y^{(4)} + y = 0$

$y_h = ??$ $r^4 + 1 = 0$

$^4 + 2r^2 + 1 = 2r^2$

$1)^2 = 2r^2$

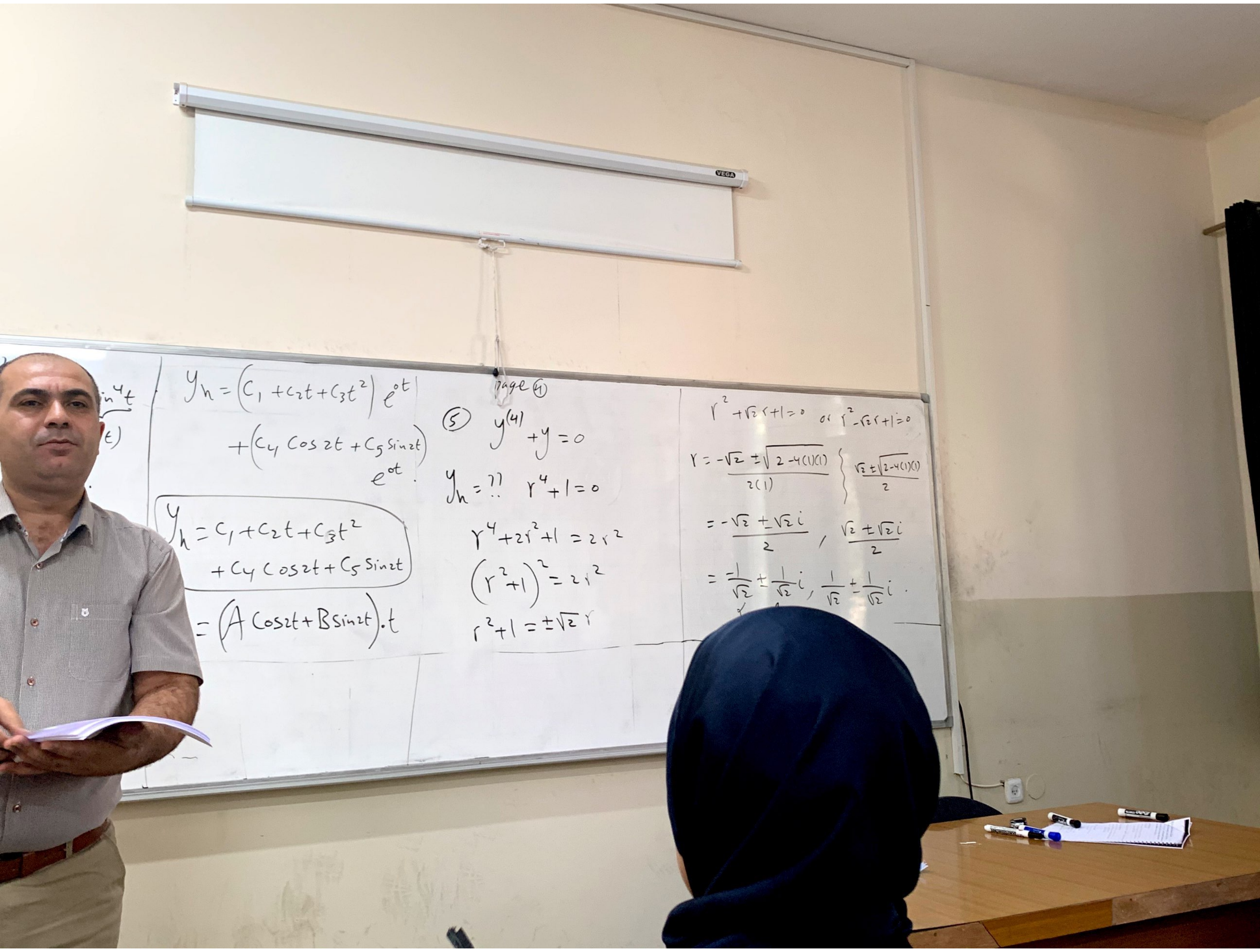
$= \pm\sqrt{2}\, r$

$r^2 + \sqrt{2}\, r + 1 = 0$ or $r^2 - \sqrt{2}\, r + 1 = 0$

$r = \frac{-\sqrt{2} \pm \sqrt{2 - 4(1)(1)}}{2(1)}$ $\Bigg\}$ $\frac{\sqrt{2} \pm \sqrt{2-4(1)(1)}}{2}$

$= \frac{-\sqrt{2} \pm \sqrt{2}\, i}{2}$, $\frac{\sqrt{2} \pm \sqrt{2}\, i}{2}$

$= \underset{\alpha}{\frac{-1}{\sqrt{2}}} \pm \underset{\beta}{\frac{1}{\sqrt{2}}} i$, $\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i$.



$$y_h = (c_1 + c_2 t + c_3 t^2)\, e^{0t} + (c_4 \cos 2t + c_5 \sin 2t)\, e^{0t}.$$

$$\boxed{y_h = c_1 + c_2 t + c_3 t^2 + c_4 \cos 2t + c_5 \sin 2t}$$

$$= (A \cos 2t + B \sin 2t) . t$$

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⑤ $y^{(4)} + y = 0$

$y_h = ??$ $r^4 + 1 = 0$

$$r^4 + 2r^2 + 1 = 2r^2$$

$$(r^2+1)^2 = 2r^2$$

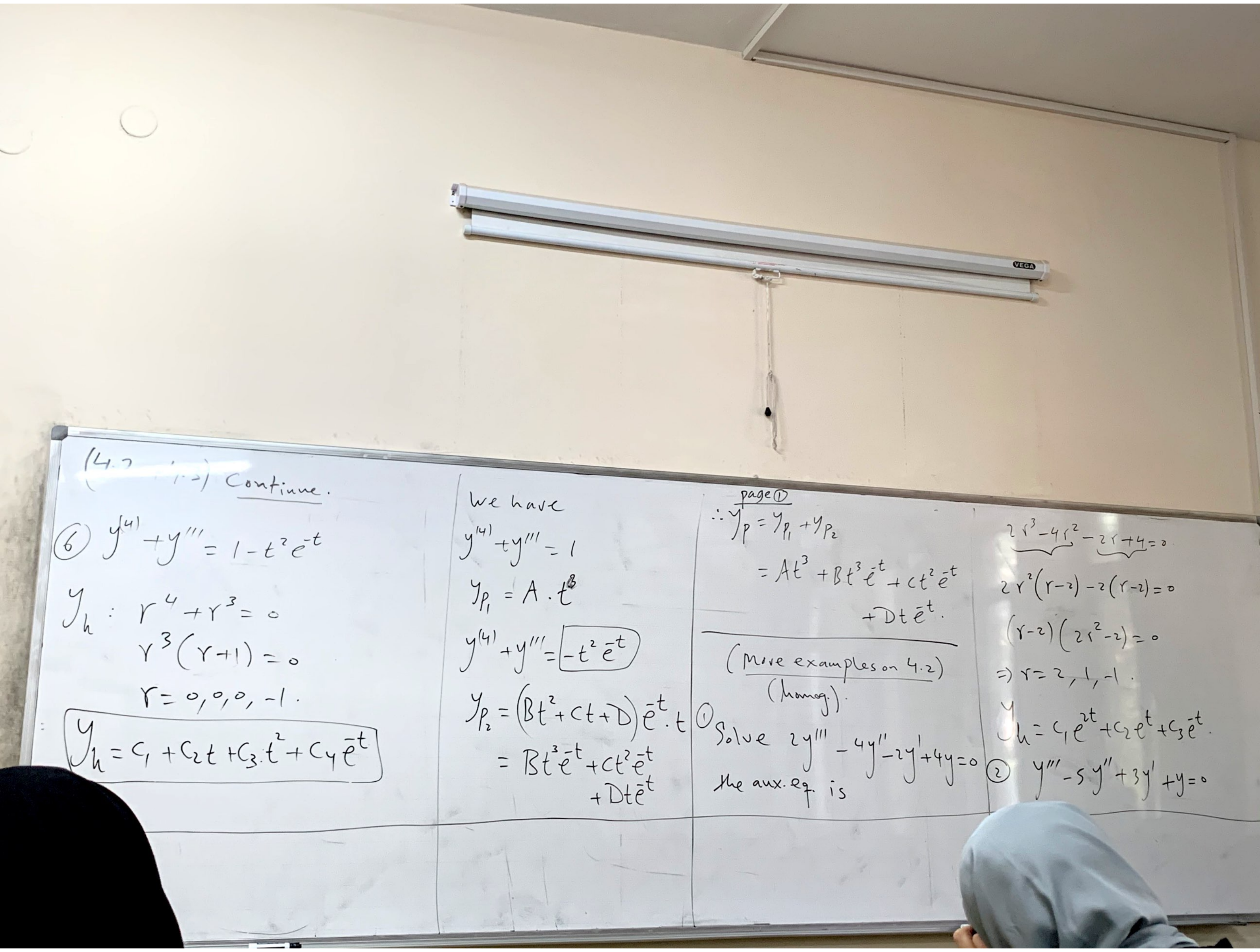
$$r^2 + 1 = \pm\sqrt{2}\, r$$

$r^2 + \sqrt{2} r + 1 = 0$ or $r^2 - \sqrt{2} r + 1 = 0$

$$r = \frac{-\sqrt{2} \pm \sqrt{2 - 4(1)(1)}}{2(1)} \Bigg\} \frac{\sqrt{2} \pm \sqrt{2-4(1)(1)}}{2}$$

$$= \frac{-\sqrt{2} \pm \sqrt{2} i}{2} , \frac{\sqrt{2} \pm \sqrt{2} i}{2}$$

$$= \frac{-1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i , \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}} i .$$



(4. [illegible]) Continue.

⑥ $y^{(4)} + y''' = 1 - t^2 e^{-t}$

y_h : $r^4 + r^3 = 0$

$r^3(r+1) = 0$

$r = 0, 0, 0, -1.$

$$\boxed{y_h = c_1 + c_2 t + c_3 . t^2 + c_4 e^{-t}}$$

We have

$y^{(4)} + y''' = 1$

$y_{p_1} = A . t^3$

$y^{(4)} + y''' = \boxed{-t^2 e^{-t}}$

$y_{p_2} = (Bt^2 + Ct + D) e^{-t} . t$

$= Bt^3 e^{-t} + Ct^2 e^{-t} + Dt e^{-t}$

page ①

$\therefore y_p = y_{p_1} + y_{p_2}$

$= At^3 + Bt^3 e^{-t} + Ct^2 e^{-t} + Dt e^{-t}.$

(More examples on 4.2)

(homog.)

① Solve $2y''' - 4y'' - 2y' + 4y = 0$

the aux. eq. is

$\underbrace{2r^3 - 4r^2} - \underbrace{2r + 4} = 0$

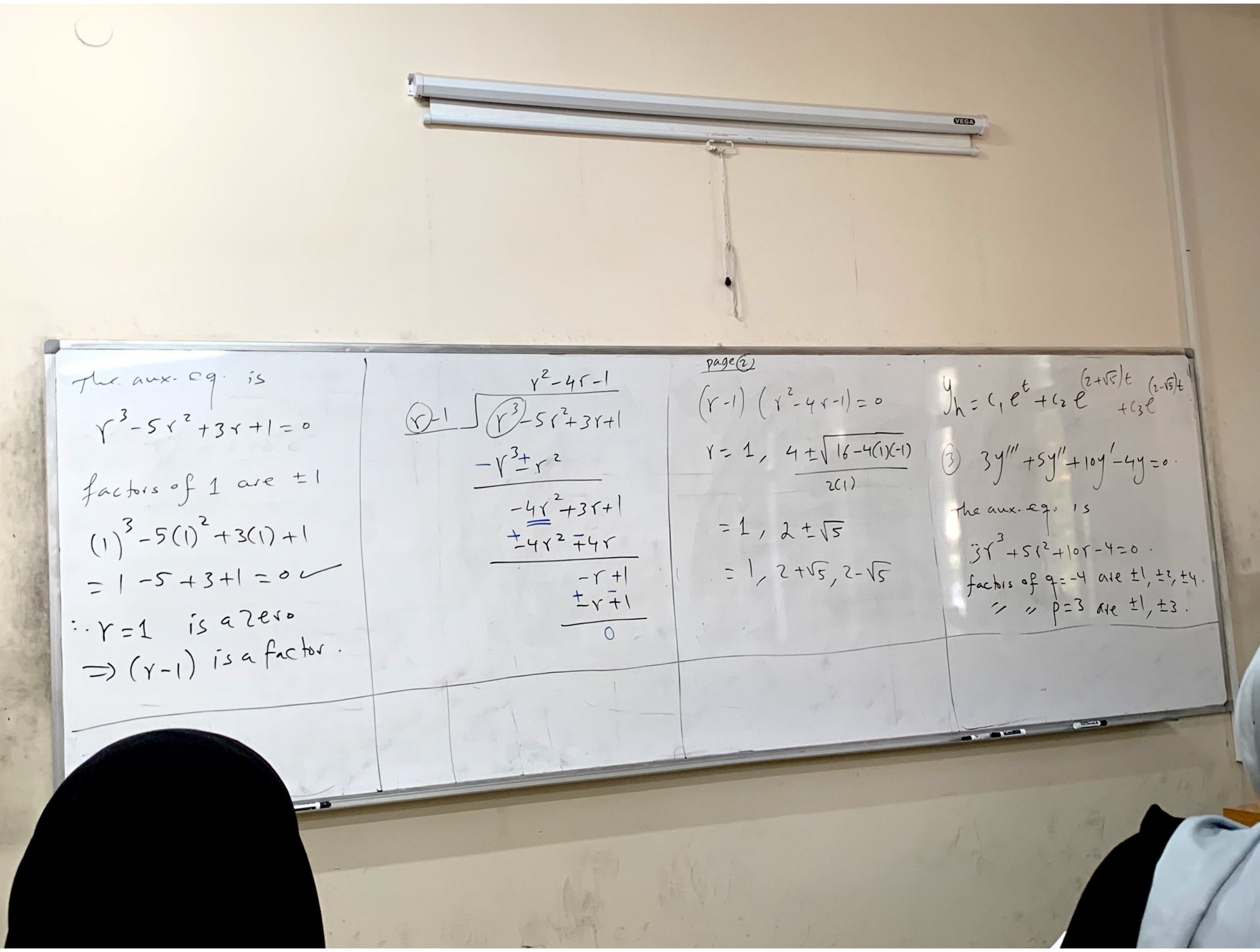
$2r^2(r-2) - 2(r-2) = 0$

$(r-2)(2r^2-2) = 0$

$\Rightarrow r = 2, 1, -1.$

$y_h = c_1 e^{2t} + c_2 e^{t} + c_3 e^{-t}.$

② $y''' - 5y'' + 3y' + y = 0$



The aux. eq. is

$r^3 - 5r^2 + 3r + 1 = 0$

factors of 1 are ±1

$(1)^3 - 5(1)^2 + 3(1) + 1$

$= 1 - 5 + 3 + 1 = 0$ ✓

$\therefore r = 1$ is a zero

$\Rightarrow (r-1)$ is a factor.

$$\begin{array}{r|l} & r^2 - 4r - 1 \\ \hline r - 1 & r^3 - 5r^2 + 3r + 1 \\ & -r^3 \pm r^2 \\ \hline & -4r^2 + 3r + 1 \\ & \pm 4r^2 \mp 4r \\ \hline & -r + 1 \\ & \pm r \mp 1 \\ \hline & 0 \end{array}$$

page②

$(r-1)(r^2 - 4r - 1) = 0$

$r = 1, \ \dfrac{4 \pm \sqrt{16 - 4(1)(-1)}}{2(1)}$

$= 1, \ 2 \pm \sqrt{5}$

$= 1, \ 2 + \sqrt{5}, \ 2 - \sqrt{5}$

$y_h = c_1 e^t + c_2 e^{(2+\sqrt{5})t} + c_3 e^{(2-\sqrt{5})t}$

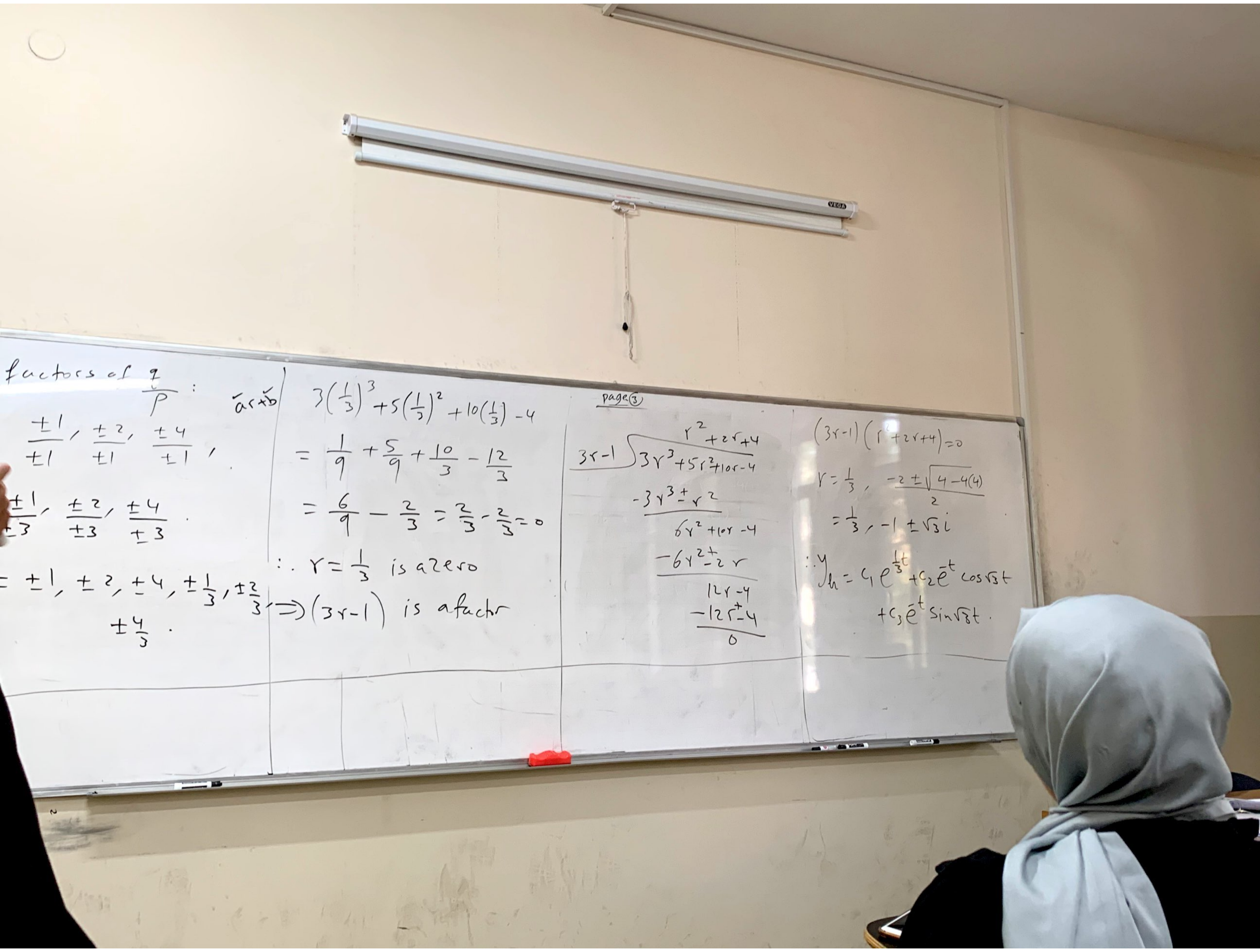
③ $3y''' + 5y'' + 10y' - 4y = 0.$

The aux. eq. is

$3r^3 + 5r^2 + 10r - 4 = 0.$

factors of $q = -4$ are ±1, ±2, ±4.

" " $p = 3$ are ±1, ±3.



factors of $\frac{q}{p}$: $\overset{\checkmark}{a}r+\overset{\checkmark}{b}$

$\frac{\pm 1}{\pm 1}, \frac{\pm 2}{\pm 1}, \frac{\pm 4}{\pm 1},$

$\frac{\pm 1}{\pm 3}, \frac{\pm 2}{\pm 3}, \frac{\pm 4}{\pm 3}.$

$= \pm 1, \pm 2, \pm 4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}.$

$3\left(\frac{1}{3}\right)^3 + 5\left(\frac{1}{3}\right)^2 + 10\left(\frac{1}{3}\right) - 4$

$= \frac{1}{9} + \frac{5}{9} + \frac{10}{3} - \frac{12}{3}$

$= \frac{6}{9} - \frac{2}{3} = \frac{2}{3} - \frac{2}{3} = 0$

$\therefore\ r = \frac{1}{3}$ is a zero

$\Rightarrow (3r-1)$ is a factor

page ③

$$
\begin{array}{r}
r^2 + 2r + 4 \\
3r-1 \overline{)\, 3r^3 + 5r^2 + 10r - 4} \\
-3r^3 \overset{+}{-} r^2 \qquad\quad \\
\hline
6r^2 + 10r - 4 \\
-6r^2 \overset{+}{-} 2r \qquad \\
\hline
12r - 4 \\
-12r \overset{+}{-} 4 \\
\hline
0
\end{array}
$$

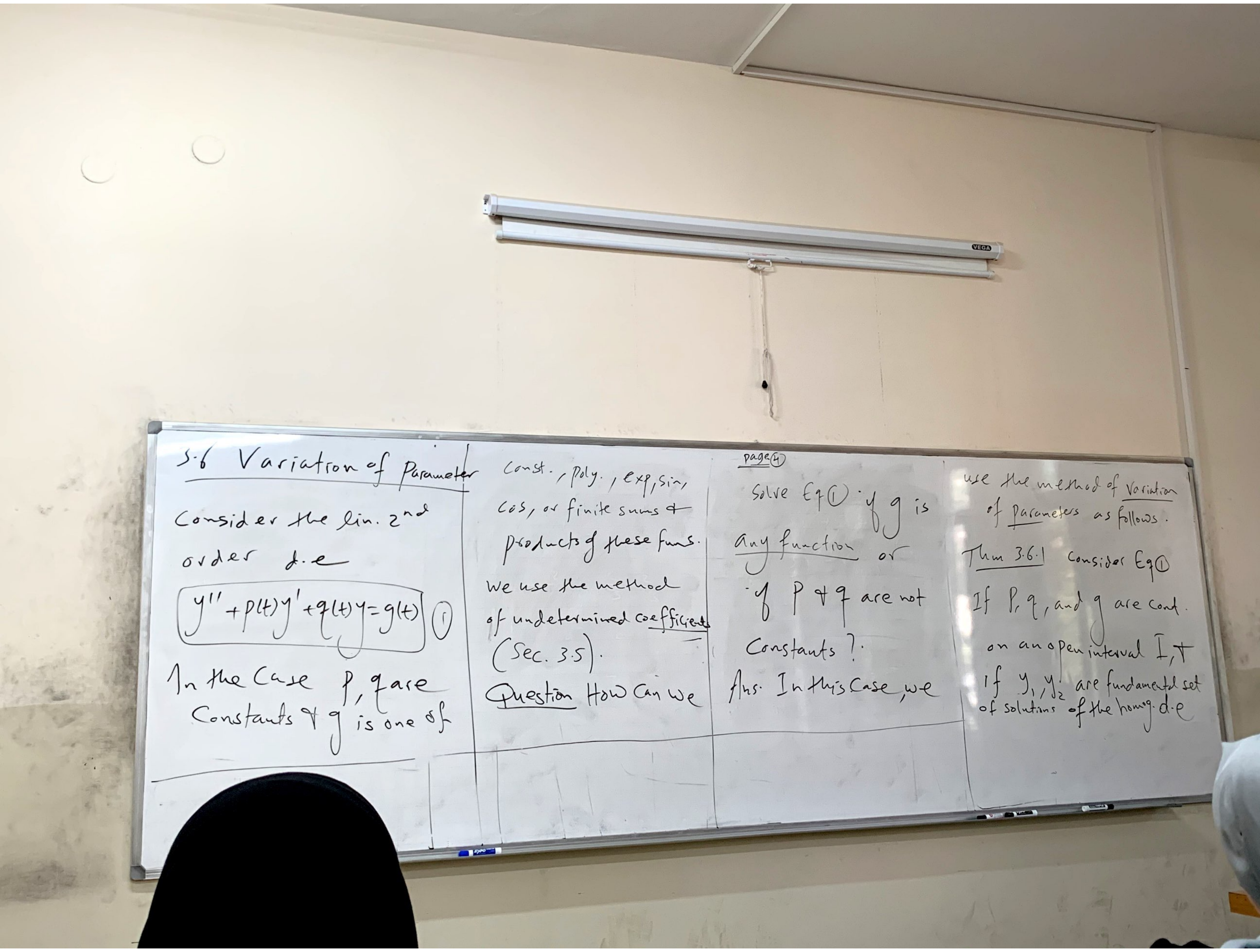
$(3r-1)(r^2+2r+4)=0$

$r = \frac{1}{3}, \ \frac{-2 \pm \sqrt{4-4(4)}}{2}$

$= \frac{1}{3}, -1 \pm \sqrt{3}i$

$\therefore y_h = c_1 e^{\frac{1}{3}t} + c_2 e^{-t} \cos\sqrt{3}t$

$+ c_3 e^{-t} \sin\sqrt{3}t.$



3.6 Variation of Parameter

Consider the lin. 2^{nd} order d.e

$$\boxed{y'' + p(t)y' + q(t)y = g(t)} \quad ①$$

In the Case p, q are Constants & g is one of const., poly., exp, sin, cos, or finite sums & products of these funs.

We use the method of undetermined coefficients (Sec. 3.5).

Question How can we

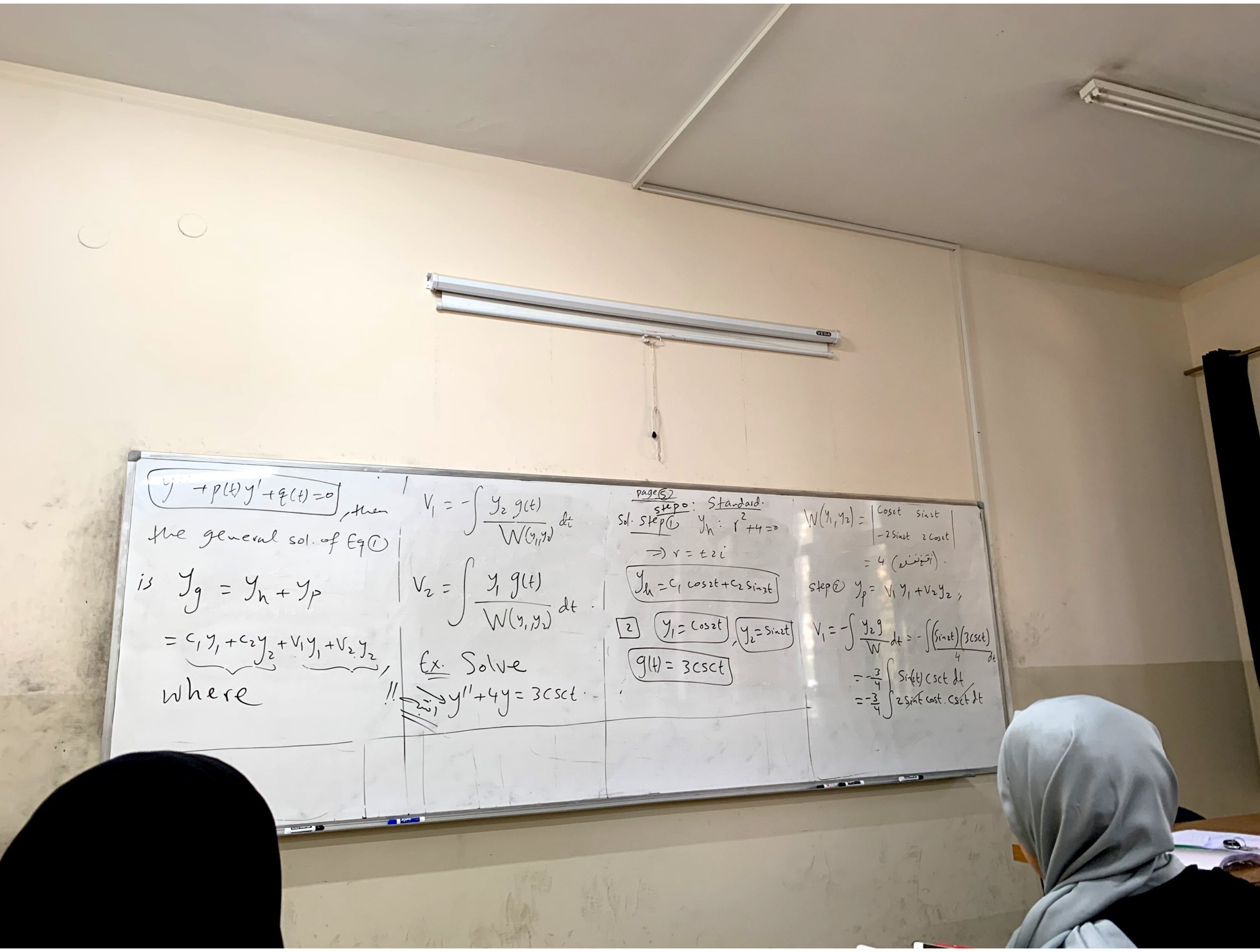
page ④

solve Eq ① if g is any function or if p & q are not Constants?.

Ans. In this case, we use the method of Variation of Parameters as follows.

Thm 3.6.1 Consider Eq ①

If p, q, and g are cont. on an open interval I, & if y_1, y_2 are fundamental set of solutions of the homog. d.e



$\boxed{y'' + p(t)y' + q(t) = 0}$, then

the general sol. of Eq ①

is $y_g = y_h + y_p$

$= \underbrace{c_1 y_1 + c_2 y_2} + \underbrace{V_1 y_1 + V_2 y_2}$,

where

$$V_1 = -\int \frac{y_2\, g(t)}{W(y_1, y_2)}\, dt$$

$$V_2 = \int \frac{y_1\, g(t)}{W(y_1, y_2)}\, dt .$$

Ex. Solve

!! [illegible] $\rightarrow y'' + 4y = 3\csc t$.

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step 0: Standard.

Sol. step ① y_h: $r^2 + 4 = 0$

$\Rightarrow r = \pm 2i$

$\boxed{y_h = c_1 \cos 2t + c_2 \sin 2t}$

$\boxed{2}$ $\boxed{y_1 = \cos 2t}$, $\boxed{y_2 = \sin 2t}$

$\boxed{g(t) = 3\csc t}$

$W(y_1, y_2) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}$

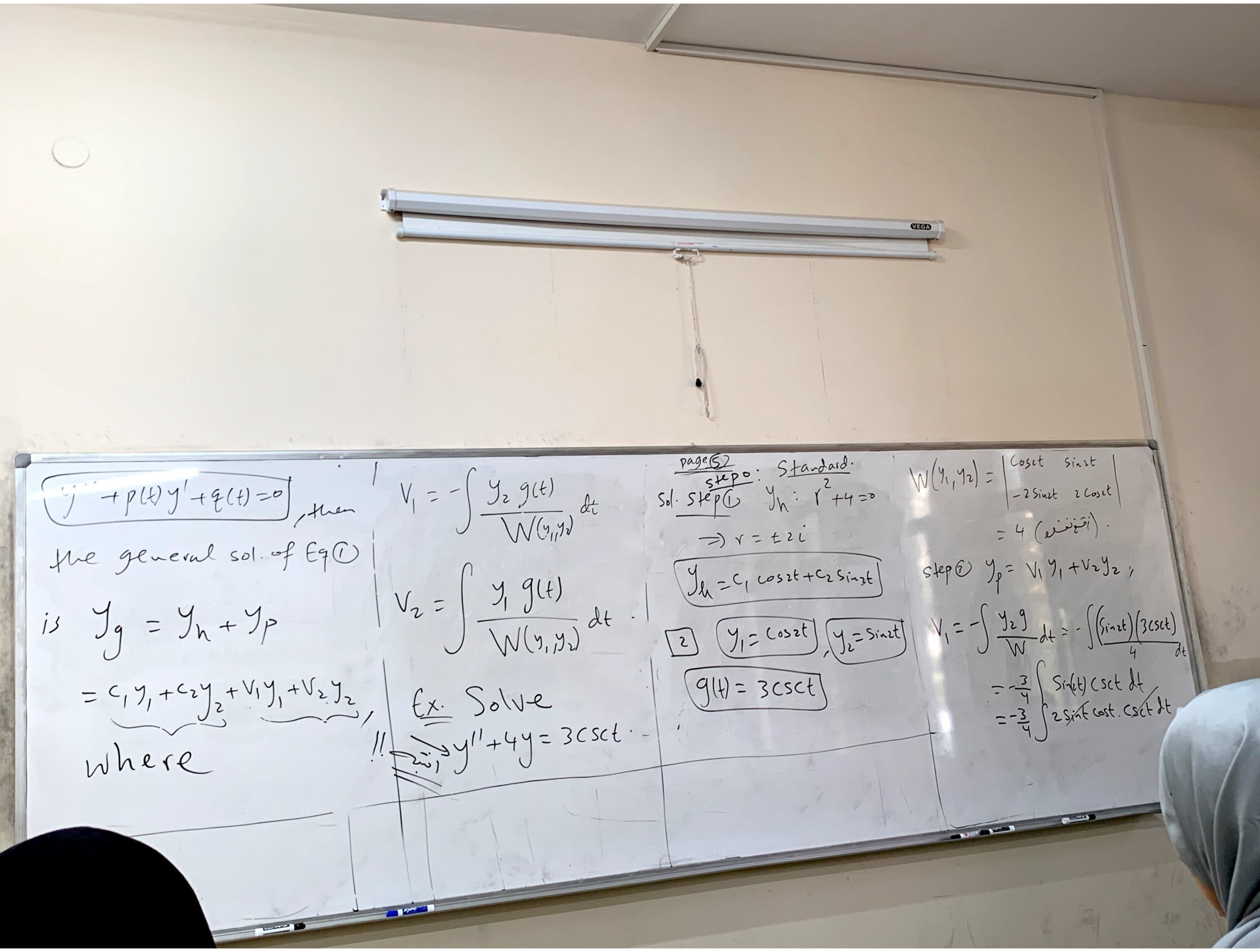
$= 4$ (اتحققوا).

step ② $y_p = V_1 y_1 + V_2 y_2$,

$V_1 = -\int \frac{y_2 g}{W} dt = -\int \frac{(\sin 2t)(3\csc t)}{4} dt$

$= -\frac{3}{4} \int \sin(2t) \csc t \, dt$

$= -\frac{3}{4} \int 2 \cancel{\sin t} \cos t \cdot \cancel{\csc t}\, dt$



$\boxed{y'' + p(t)y' + q(t) = 0}$, then

the general sol. of Eq ①

is $y_g = y_h + y_p$

$= \underbrace{c_1 y_1 + c_2 y_2} + \underbrace{V_1 y_1 + V_2 y_2}$,

where

$$V_1 = -\int \frac{y_2\, g(t)}{W(y_1, y_2)}\, dt$$

$$V_2 = \int \frac{y_1\, g(t)}{W(y_1, y_2)}\, dt .$$

Ex. Solve

!! [illegible] $\rightarrow y'' + 4y = 3\csc t$.

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step 0 : Standard.

Sol. step ① y_h : $r^2 + 4 = 0$

$\Rightarrow r = \pm 2i$

$\boxed{y_h = c_1 \cos 2t + c_2 \sin 2t}$

$\boxed{2}$ $\boxed{y_1 = \cos 2t}$, $\boxed{y_2 = \sin 2t}$

$\boxed{g(t) = 3\csc t}$

$$W(y_1, y_2) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix}$$

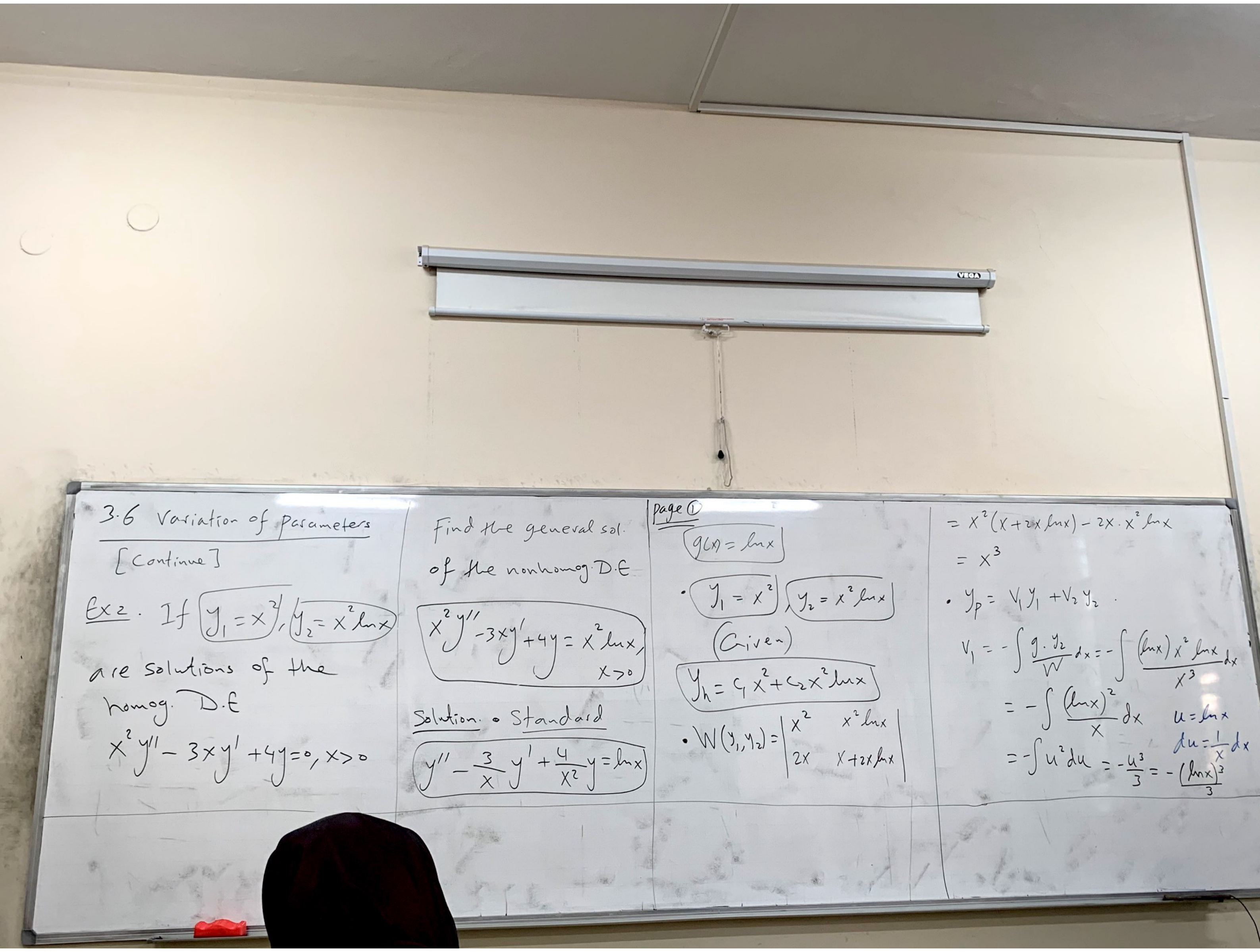
$= 4$ (اتحققي) .

step ② $y_p = V_1 y_1 + V_2 y_2$;

$$V_1 = -\int \frac{y_2\, g}{W}\, dt = -\int \frac{(\sin 2t)(3\csc t)}{4}\, dt$$

$$= -\frac{3}{4}\int \sin(2t)\csc t\, dt$$

$$= -\frac{3}{4}\int 2\cancel{\sin t}\cos t . \cancel{\csc t}\, dt$$



3.6 Variation of parameters

[Continue]

Ex2. If $\boxed{y_1 = x^2}$, $\boxed{y_2 = x^2 \ln x}$

are solutions of the homog. D.E

$x^2 y'' - 3xy' + 4y = 0, x > 0$

Find the general sol. of the nonhomog. D.E

$\boxed{x^2 y'' - 3xy' + 4y = x^2 \ln x, \quad x > 0}$

Solution. • Standard

$\boxed{y'' - \frac{3}{x} y' + \frac{4}{x^2} y = \ln x}$

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$\boxed{g(x) = \ln x}$

• $\boxed{y_1 = x^2}$, $\boxed{y_2 = x^2 \ln x}$ (Given)

$\boxed{y_h = c_1 x^2 + c_2 x^2 \ln x}$

• $W(y_1, y_2) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & x + 2x \ln x \end{vmatrix}$

$= x^2(x + 2x \ln x) - 2x \cdot x^2 \ln x$

$= x^3$

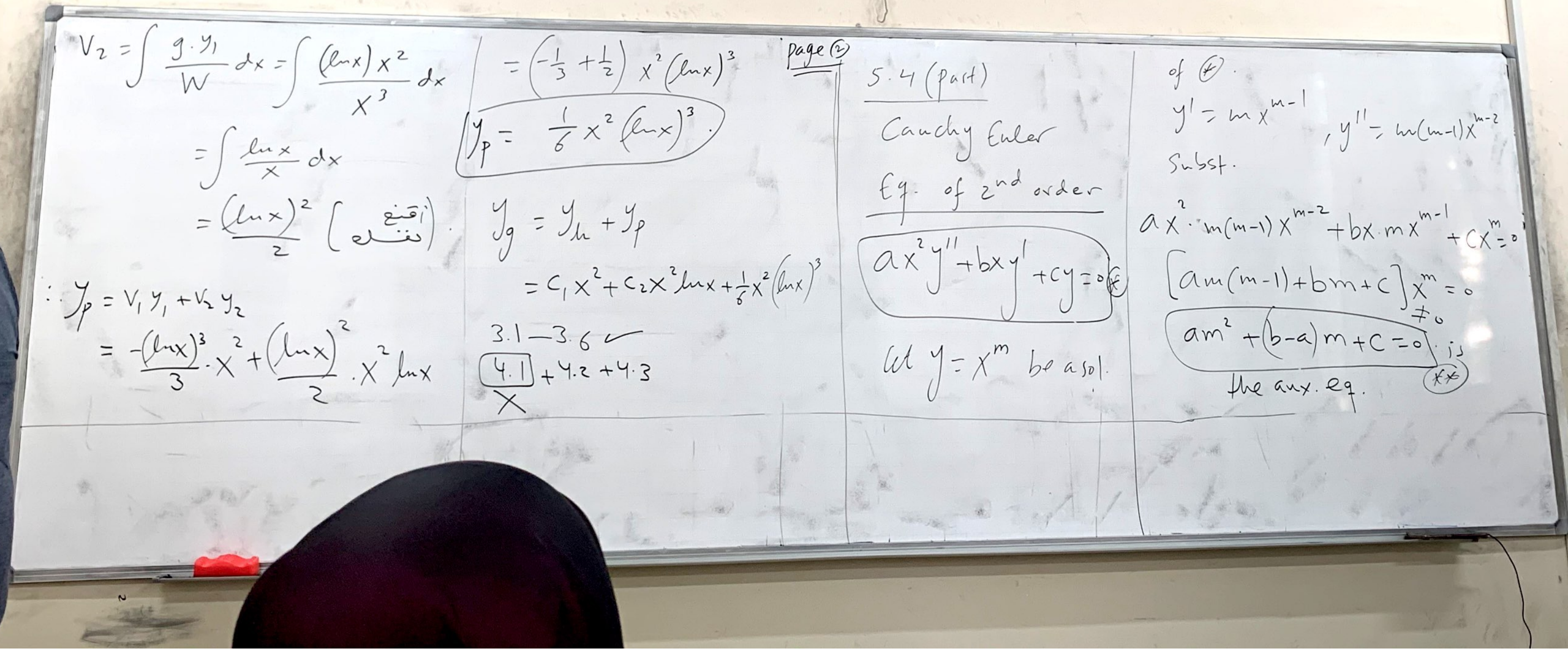
• $y_p = v_1 y_1 + v_2 y_2$.

$v_1 = -\int \frac{g \cdot y_2}{W} dx = -\int \frac{(\ln x)\, x^2 \ln x}{x^3} dx$

$= -\int \frac{(\ln x)^2}{x} dx$ $\quad u = \ln x$

$\quad du = \frac{1}{x} dx$

$= -\int u^2 du = -\frac{u^3}{3} = -\frac{(\ln x)^3}{3}$



$$V_2 = \int \frac{g \cdot y_1}{W}\,dx = \int \frac{(\ln x)\,x^2}{x^3}\,dx$$

$$= \int \frac{\ln x}{x}\,dx$$

$$= \frac{(\ln x)^2}{2}$$ (أقنع نفسك).

$$\therefore y_p = V_1 y_1 + V_2 y_2$$

$$= -\frac{(\ln x)^3}{3} \cdot x^2 + \frac{(\ln x)^2}{2} \cdot x^2 \ln x$$

$$= \left(-\frac{1}{3} + \frac{1}{2}\right) x^2 (\ln x)^3$$

$$\boxed{y_p = \frac{1}{6} x^2 (\ln x)^3.}$$

$$y_g = y_h + y_p$$

$$= c_1 x^2 + c_2 x^2 \ln x + \frac{1}{6} x^2 (\ln x)^3$$

3.1 — 3.6 ✓

$\boxed{4.1}$ + 4.2 + 4.3

×

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5.4 (part)

Cauchy Euler Eq. of 2nd order

$$\boxed{a x^2 y'' + b x y' + c y = 0} \quad \circledast$$

let $y = x^m$ be a sol. of $\circledast$.

$$y' = m x^{m-1}, \quad y'' = m(m-1) x^{m-2}$$

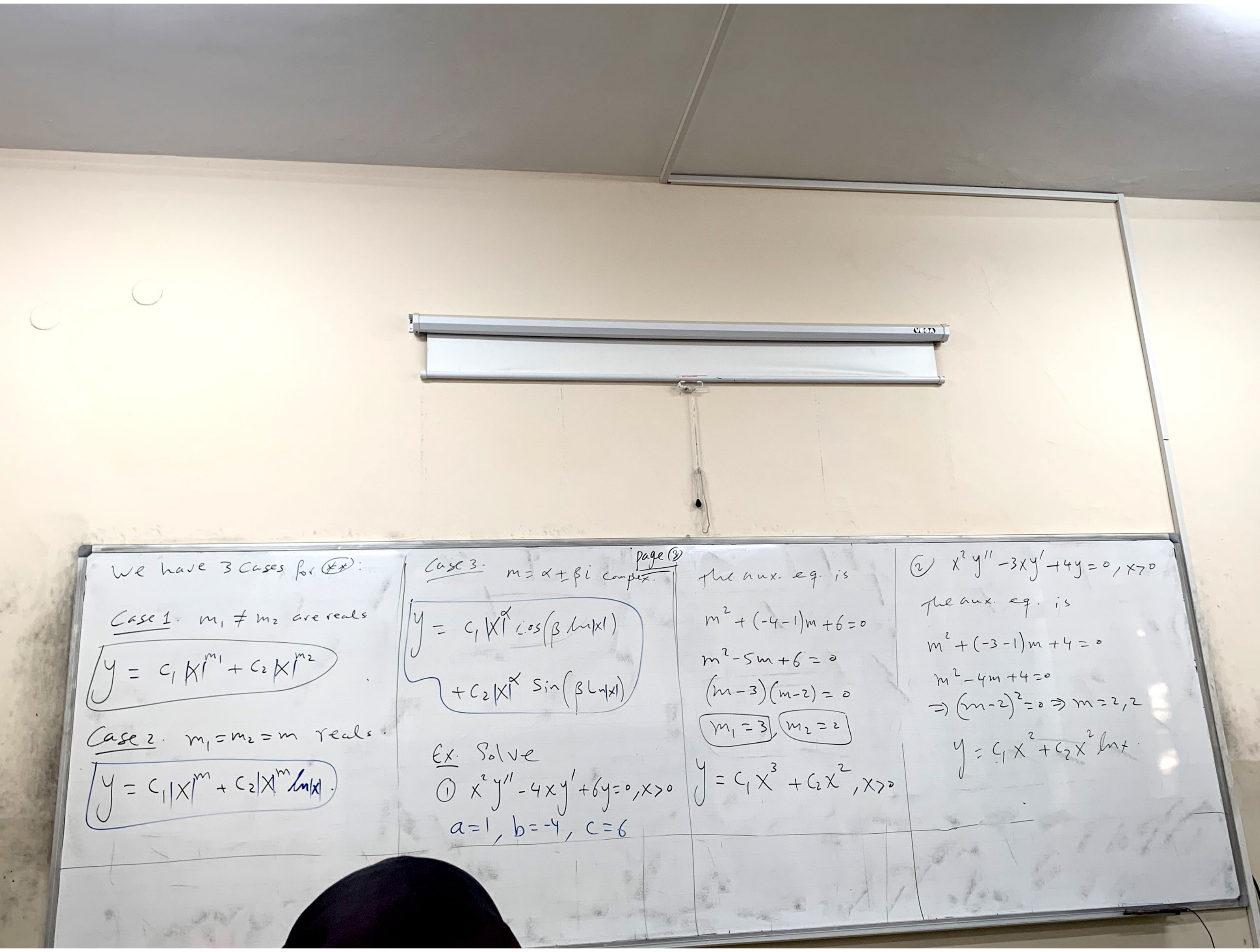
Subst.

$$a x^2 \cdot m(m-1) x^{m-2} + b x \cdot m x^{m-1} + c x^m = 0$$

$$[a m(m-1) + b m + c]\, \underset{\neq 0}{x^m} = 0$$

$$\boxed{a m^2 + (b-a) m + c = 0} \quad (**)$$

is the aux. eq.



We have 3 cases for $(**)$:

Case 1. $m_1 \neq m_2$ are reals

$$\boxed{y = c_1 |x|^{m_1} + c_2 |x|^{m_2}}.$$

Case 2. $m_1 = m_2 = m$ reals.

$$\boxed{y = c_1 |x|^{m} + c_2 |x|^{m} \ln|x|}.$$

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Case 3. $m = \alpha \pm \beta i$ complex.

$$\boxed{y = c_1 |x|^{\alpha} \cos(\beta \ln|x|) + c_2 |x|^{\alpha} \sin(\beta \ln|x|)}$$

Ex. Solve

① $x^2 y'' - 4xy' + 6y = 0, x > 0$

$a = 1, b = -4, c = 6$

The aux. eq. is

$m^2 + (-4-1)m + 6 = 0$

$m^2 - 5m + 6 = 0$

$(m-3)(m-2) = 0$

$\boxed{m_1 = 3}, \boxed{m_2 = 2}$

$y = c_1 x^3 + c_2 x^2, x > 0$

② $x^2 y'' - 3xy' + 4y = 0, x > 0$

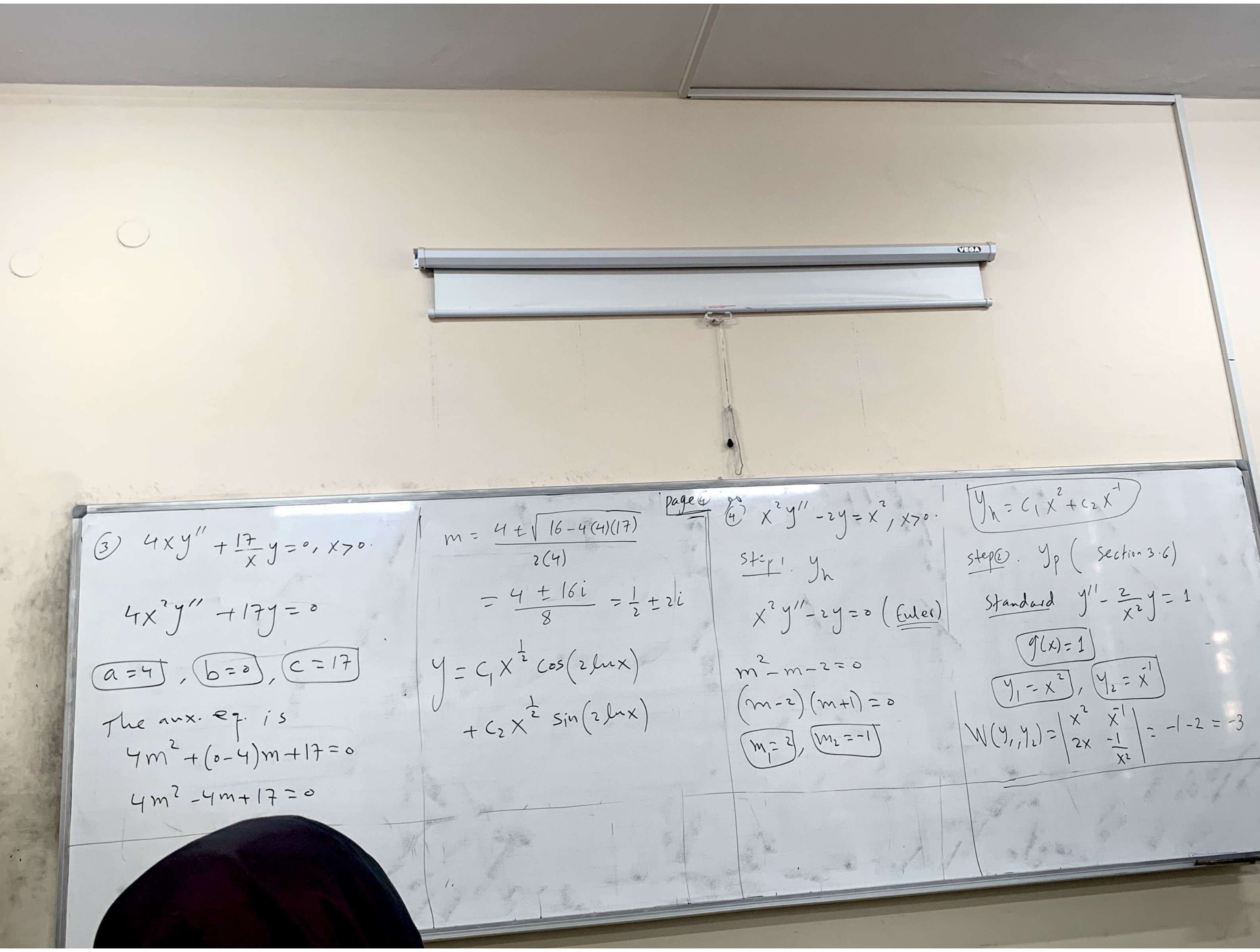
The aux. eq. is

$m^2 + (-3-1)m + 4 = 0$

$m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

$y = c_1 x^2 + c_2 x^2 \ln x$



③ $4xy'' + \frac{17}{x}y = 0, \quad x > 0.$

$$4x^2y'' + 17y = 0$$

$\boxed{a=4}$, $\boxed{b=0}$, $\boxed{c=17}$

The aux. eq. is

$$4m^2 + (0-4)m + 17 = 0$$

$$4m^2 - 4m + 17 = 0$$

$$m = \frac{4 \pm \sqrt{16 - 4(4)(17)}}{2(4)}$$

$$= \frac{4 \pm 16i}{8} = \frac{1}{2} \pm 2i$$

$$y = c_1 x^{\frac{1}{2}} \cos(2\ln x) + c_2 x^{\frac{1}{2}} \sin(2\ln x)$$

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④ $x^2y'' - 2y = x^2, \quad x > 0.$

Step 1. y_h

$$x^2y'' - 2y = 0 \quad (\text{Euler})$$

$$m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$\boxed{m_1 = 2}$, $\boxed{m_2 = -1}$

$$\boxed{y_h = c_1x^2 + c_2x^{-1}}$$

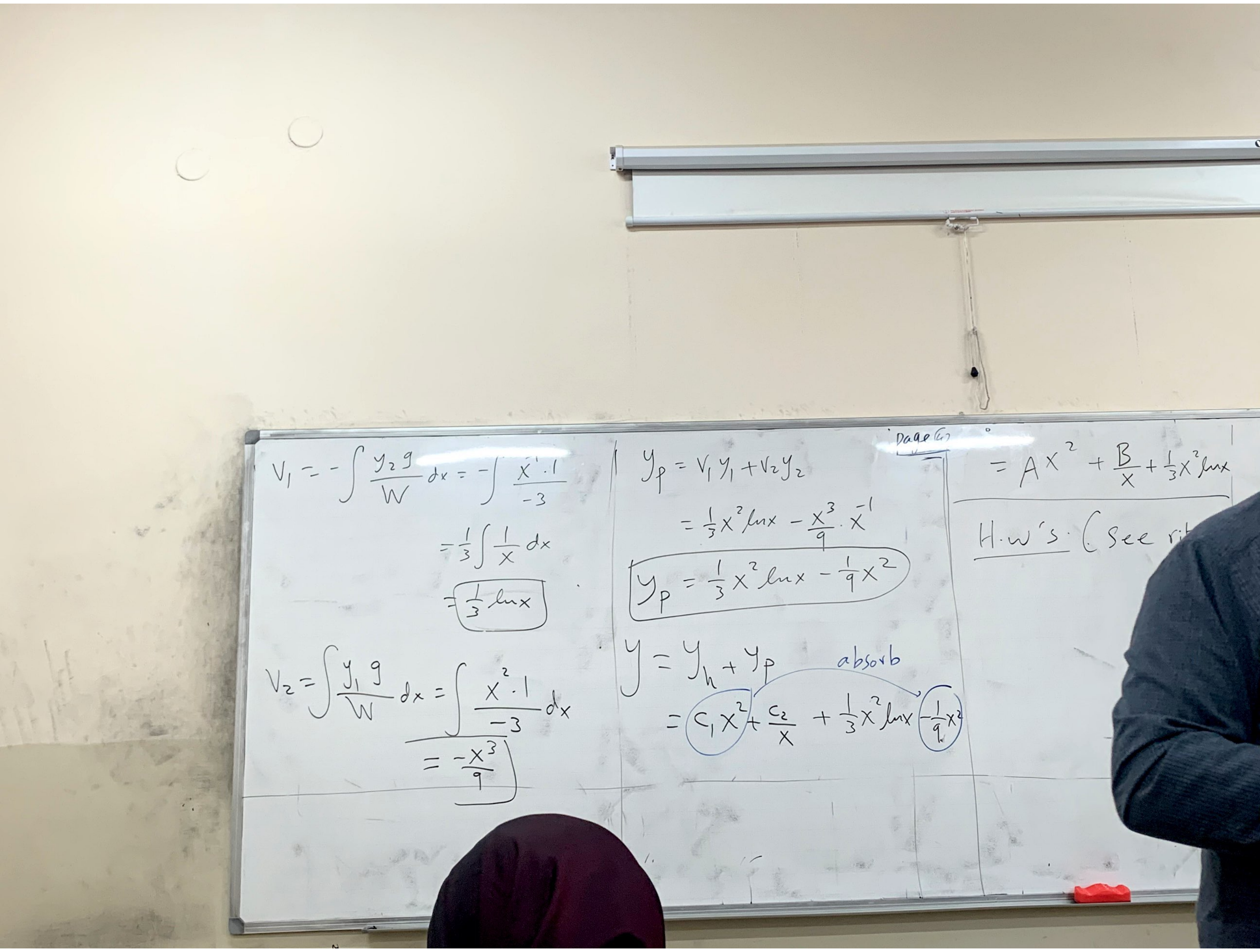
Step ②. y_p (Section 3.6)

Standard $y'' - \frac{2}{x^2}y = 1$

$$\boxed{g(x) = 1}$$

$\boxed{y_1 = x^2}$, $\boxed{y_2 = x^{-1}}$

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^{-1} \\ 2x & -\frac{1}{x^2} \end{vmatrix} = -1 - 2 = -3$$



$$V_1 = -\int \frac{y_2 g}{W} dx = -\int \frac{x^{-1} \cdot 1}{-3}$$

$$= \frac{1}{3}\int \frac{1}{x} dx$$

$$= \boxed{\frac{1}{3} \ln x}$$

$$V_2 = \int \frac{y_1 g}{W} dx = \int \frac{x^2 \cdot 1}{-3} dx$$

$$= \boxed{\frac{-x^3}{9}}$$

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$$y_p = v_1 y_1 + v_2 y_2$$

$$= \frac{1}{3}x^2 \ln x - \frac{x^3}{9} \cdot x^{-1}$$

$$\boxed{y_p = \frac{1}{3}x^2 \ln x - \frac{1}{9}x^2}$$

$$y = y_h + y_p$$

absorb

$$= c_1 x^2 + \frac{c_2}{x} + \frac{1}{3}x^2 \ln x - \frac{1}{9}x^2$$

$$= A x^2 + \frac{B}{x} + \frac{1}{3}x^2 \ln x$$

H.w's. (See ri