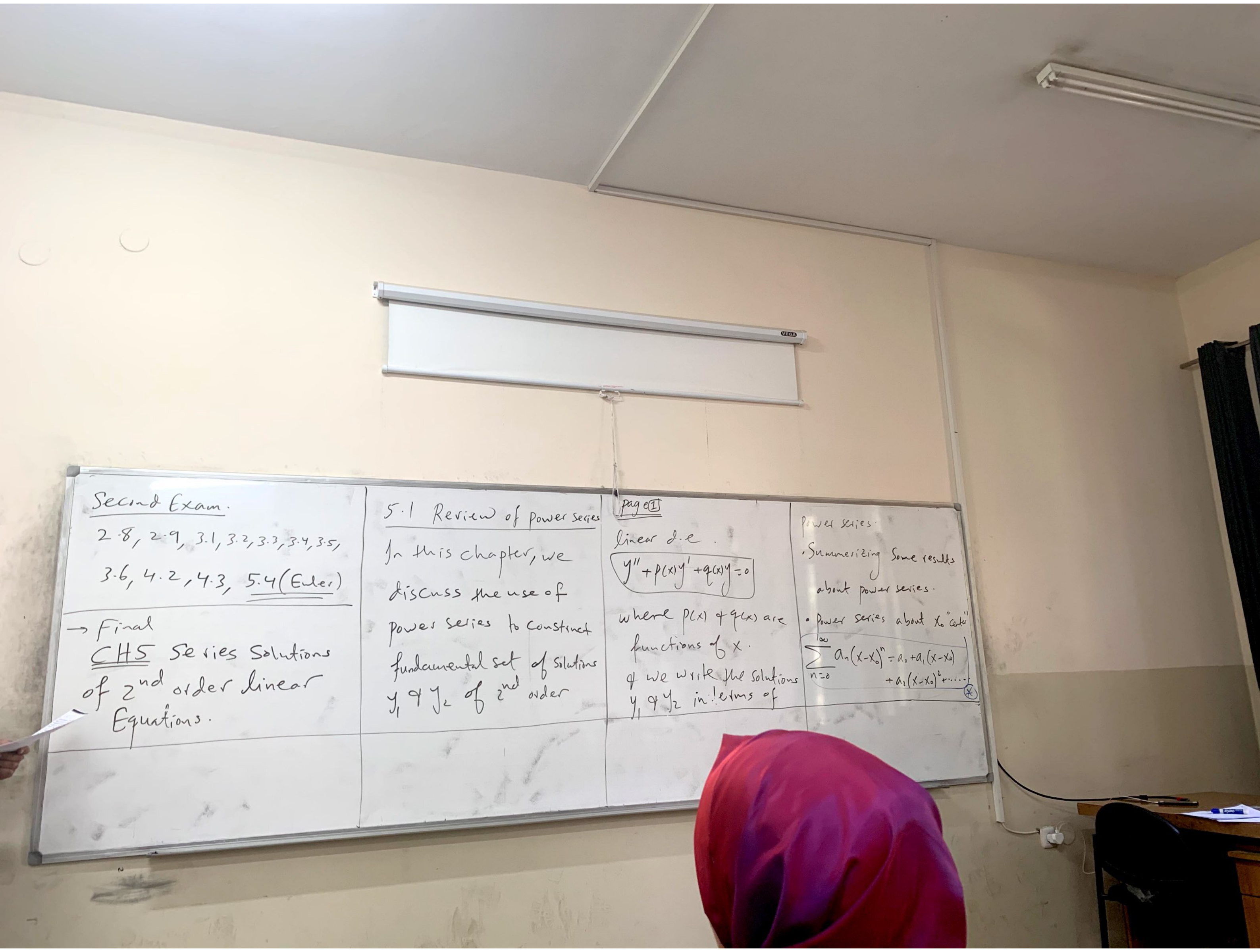

Second Exam.
2.8, 2.9, 3.1, 3.2, 3.3, 3.4, 3.5,
3.6, 4.2, 4.3, 5.4 (Euler)
→ Final
CH5 Series Solutions of 2nd order linear Equations.
5.1 Review of power series
In this chapter, we discuss the use of power series to construct fundamental set of solutions y1 & y2 of 2nd order
page (1)
linear d.e.
y'' + p(x)y' + q(x)y = 0
where p(x) & q(x) are functions of x.
& we write the solutions y1 & y2 in terms of
Power series.
· Summerizing some results about power series.
· Power series about x0 "center"
∑ n=0 to ∞ an(x-x0)^n = a0 + a1(x-x0) + a2(x-x0)^2 + ...
(*)

the series (*) conv. at x

if $\lim_{k\to\infty} \sum_{n=0}^{k} a_n (x-x_0)^n$

exists for all x.

and it is conv. absolutely

at x if $\sum_{n=0}^{\infty} |a_n(x-x_0)^n|$ conv.

To test the convergence for (*), we use Ratio Test

$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right|$

$= \lim_{n\to\infty} \left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right|$

$= |x-x_0| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-x_0| \cdot L$

page (2)

(*) conv. if $|x-x_0| \cdot L < 1$

if $|x-x_0| \cdot L = 1 \rightsquigarrow$ test fails.

- The radius of convergence is a $\rho > 0$ such that (*) conv. abs. for $|x-x_0| < \rho$.

the interval $(x_0-\rho, x_0+\rho)$ is called the interval of convergence.

Ex. Determine the radius & the interval of convergence of the power series

$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot 2^n}$.

Sol. $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|$

$= \lim_{n\to\infty}\left|\frac{(x+1)^{n+1}}{2^{n+1}(n+1)}\cdot\frac{n\cdot 2^n}{(x+1)^n}\right|$

$= \frac{|x+1|}{2}\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^{1}$

$= \frac{|x+1|}{2} < 1$

$\Rightarrow |x+1| < 2 \Rightarrow -2 < x+1 < 2$

$-3 < x < 1$.

$\boxed{x=-3}$, $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ conv. by A.T.

$x=1$, $\sum_{n=1}^{\infty}\frac{1}{n}$ div. by p test.

Interval $[-3,1)$ $-3 \quad -1 \quad 1$

Center -1, Radius $= 2$.

page ③

Differentiation & Integration

$y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$

$y' = \sum_{n=1}^{\infty} n a_n(x-x_0)^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$

$\vdots$

$\int y\,dx = \sum_{n=0}^{\infty} a_n\frac{(x-x_0)^{n+1}}{n+1} + C$.

· Taylor series for f at $x=a$.

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Shifting of Index of Summation

ex. Write the series $S = \sum_{n=2}^{\infty} n(n+2) a_n (x-1)^{n-2}$

page (4)

as a series involve $(x-1)^n$

$$S = \sum_{n=0}^{\infty} (n+2)(n+4) a_{n+2} (x-1)^n$$

Ex. Write as a single sum

(1) $\sum_{n=1}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 2\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} + 2a_n\right] x^n$$

(2) $x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$

$$= \sum_{n=1}^{\infty} n a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} (n-1) a_{n-1} x^n + a_0 x^0 + a_1 x^1 + \sum_{n=2}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + \sum_{n=2}^{\infty} \left[(n-1) a_{n-1} + a_n \right] x^n .$$

5.2 Series Solutions Near ordinary points, Part I.

Df. Consider

$$(1) \quad P(x) y'' + Q(x) y' + R(x) y = 0$$

page (4)

A point x_0 such that

$$P(x_0) \neq 0 \text{ in Eq (1)},$$

is called an ordinary point. If $P(x_0) = 0$, then x_0 is called singular point.

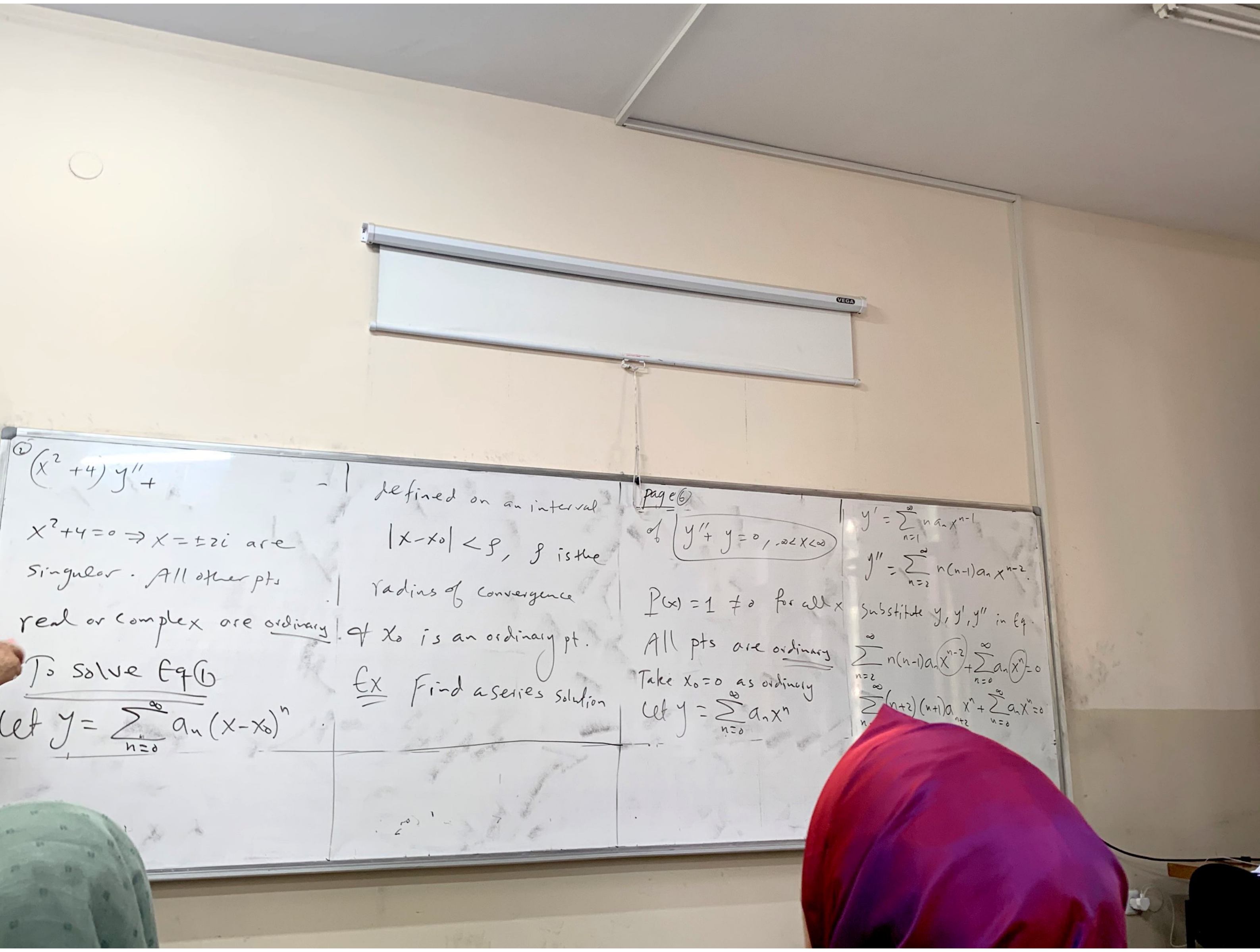
Ex. Find ordinary & singular pts of

$$(x^2 - x) y'' + x y' - 2x^2 y = 0$$

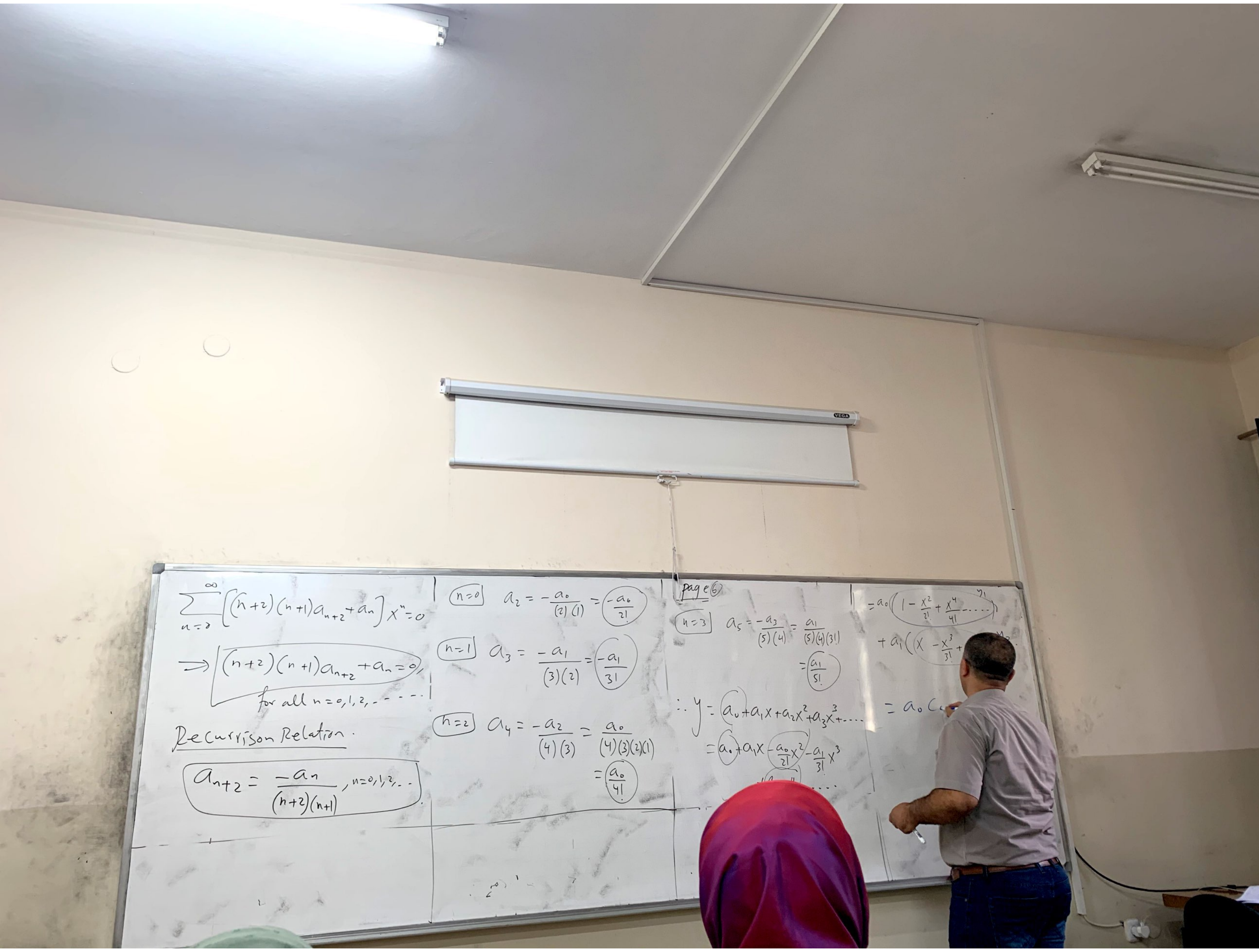
$P(x) = x^2 - x = 0 \Rightarrow x = 0, x = 1$ are singular pts

All other pts (real or comp are ordinary pts.


All other pts
are ordinary
defined on an interval
|x-x0| < ρ, ρ is the
radius of convergence
if x0 is an ordinary pt.
Ex Find a series solution
page 6
of y'' + y = 0, -∞<x<∞
P(x) = 1 ≠ 0 for all x
All pts are
Take x0 = 0 as
Let y = Σ a
Substitute y, y', y'' in Eq.
VEGA



① (x² + 4) y'' +
x² + 4 = 0 ⇒ x = ±2i are singular. All other pts real or complex are ordinary
To solve Eq ①
Let y = Σ_{n=0}^∞ a_n (x − x₀)^n
defined on an interval |x − x₀| < ρ, ρ is the radius of convergence
If x₀ is an ordinary pt.
Ex Find a series solution
page ⑥
of y'' + y = 0, −∞ < x < ∞
P(x) = 1 ≠ 0 for all x
All pts are ordinary
Take x₀ = 0 as ordinary
Let y = Σ_{n=0}^∞ a_n x^n
y' = Σ_{n=1}^∞ n a_n x^{n−1}
y'' = Σ_{n=2}^∞ n(n−1) a_n x^{n−2}
Substitute y, y', y'' in Eq
Σ_{n=2}^∞ n(n−1) a_n x^{n−2} + Σ_{n=0}^∞ a_n x^n = 0
Σ (n+2)(n+1) a_{n+2} x^n + Σ_{n=0}^∞ a_n x^n = 0


∑ [(n+2)(n+1)a_{n+2} + a_n] x^n = 0
⇒ (n+2)(n+1)a_{n+2} + a_n = 0, for all n = 0, 1, 2, ...
Recursion Relation
a_{n+2} = −a_n / ((n+2)(n+1)), n = 0, 1, 2, ..
n=0 a_2 = −a_0/((2)(1)) = −a_0/2!
n=1 a_3 = −a_1/((3)(2)) = −a_1/3!
n=2 a_4 = −a_2/((4)(3)) = a_0/((4)(3)(2)(1)) = a_0/4!
page 6
n=3 a_5 = −a_3/((5)(4)) = a_1/((5)(4)(3!)) = a_1/5!
∴ y = a_0 + a_1x + a_2x^2 + a_3x^3 + ...
= a_0 + a_1x − (a_0/2!)x^2 − (a_1/3!)x^3
= a_0(1 − x^2/2! + x^4/4! − ....)
+ a_1(x − x^3/3! + ...
= a_0 C

5.2 Continue.

Ex 2 Find two linearly indep. power series of the Airy's Eq.

$$y'' - xy = 0, \quad -\infty < x < \infty$$

about the ordinary point $x_0 = 0$.

Give the first nonzero terms for each series Solution.

Sol. Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

be the Solution.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

page ①

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x^1 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$(2)(1) a_2 x^0 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - a_{n-1} \right] x^n = 0$$

$$2a_2 = 0, \quad (n+2)(n+1) a_{n+2} - a_{n-1} = 0,$$

$$\boxed{a_2 = 0}, \quad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \quad \text{for } n = 1, 2, \ldots$$

$n=1$, $a_3 = \frac{a_0}{(3)(2)} = \frac{a_0}{6}$

$n=2$, $a_4 = \frac{a_1}{(4)(3)} = \frac{a_1}{12}$

$n=3$, $a_5 = \frac{a_2}{(5)(4)} = 0$

$n=4$, $a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{6(6)(5)} = \frac{a_0}{180}$

$n=5$, $a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(12)(42)}$

$\vdots$

$$y = a_0 + a_1 x + a_2 x^2 + \dots$$

$$= a_0 + a_1 x + 0x^2 + \frac{a_0}{6}x^3 + \frac{a_1}{12}x^4 + 0x^5 + \frac{a_0}{180}x^6 + \frac{a_1}{(12)(42)}x^7 + \dots$$

page (2)

$$= a_0\left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \dots\right) + a_1\left(x + \frac{1}{12}x^4 + \frac{1}{(12)(42)}x^7 + \dots\right)$$

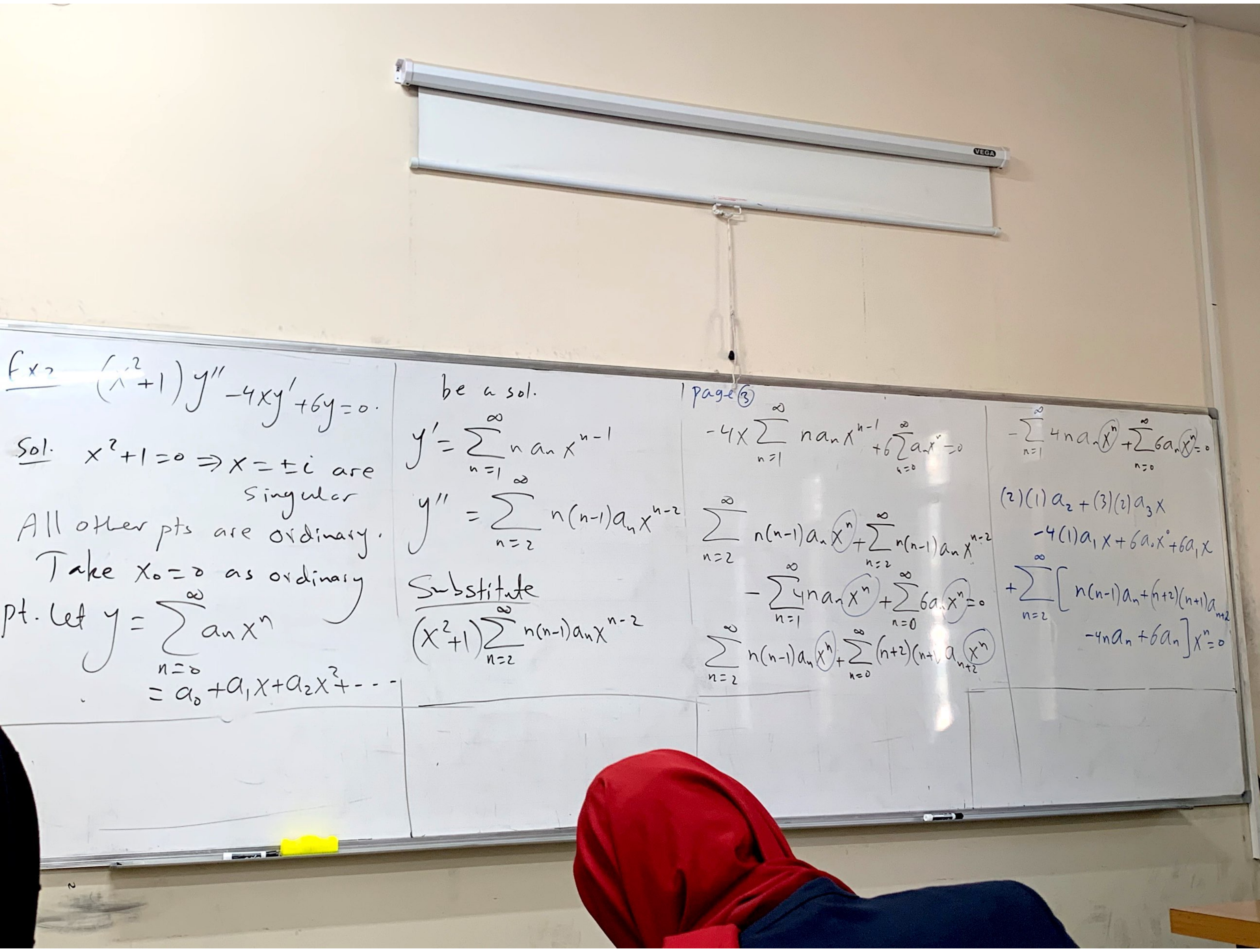
$$y_1 = 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \dots$$

$$y_2 = x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \dots$$

lin. Independence

$$W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\therefore \{y_1, y_2\}$ form a fundamental set of solution.


Ex: $(x^2+1)y'' - 4xy' + 6y = 0$.
Sol. $x^2+1=0 \Rightarrow x = \pm i$ are singular
All other pts are ordinary.
Take $x_0 = 0$ as ordinary pt. Let $y = \sum_{n=0}^{\infty} a_n x^n$
$= a_0 + a_1 x + a_2 x^2 + \cdots$
be a sol.
$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$
$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$
Substitute
$(x^2+1)\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$
page (3)
$-4x\sum_{n=1}^{\infty} n a_n x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$
$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$
$-\sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$
$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$
$-\sum_{n=1}^{\infty} 4n a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$
$(2)(1)a_2 + (3)(2)a_3 x$
$-4(1)a_1 x + 6a_0 x^0 + 6a_1 x$
$+\sum_{n=2}^{\infty} [n(n-1)a_n + (n+2)(n+1)a_{n+2}$
$-4n a_n + 6a_n] x^n = 0$

$$\Rightarrow 2a_2 + 6a_0 + (6a_3 + 2a_1)x + \sum_{n=2}^{\infty}\Big[(\underbrace{n^2 - n - 4n + 6})a_n + (n+2)(n+1)a_{n+2}\Big]x^n = $$

$$\Rightarrow 2a_2 + 6a_0 = 0, \quad 6a_3 + 2a_1 = 0$$

$$(n-2)(n-3)a_n + (n+2)(n+1)a_{n+2} = 0, \quad \text{for } n = 2, 3, 4, \ldots$$

$$\boxed{a_2 = -3a_0}$$

$$\boxed{a_3 = -\frac{1}{3}a_1}$$

$$a_{n+2} = \frac{-(n-2)(n-3)a_n}{(n+2)(n+1)}, \quad n = 2, 3, 4, \ldots$$

$\boxed{n=2}$, $a_4 = 0$

$n=3$, $a_5 = 0$

page (4)

$$n=4, \quad a_6 = \frac{-(2)(1)a_4}{(6)(5)} = 0$$

$\vdots$

$a_n = 0$, for $n = 4, 5, \ldots$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + 0$$
$$= a_0 + a_1x - 3a_0x^2 - \frac{1}{3}a_1x^3$$
$$= a_0(1 - 3x^2) + a_1\left(x - \frac{1}{3}x^3\right).$$

$y_1 = 1 - 3x^2, \quad y_2 = x - \frac{1}{3}x^3$

$y_1' = -6x, \quad y_2' = 1 - x^2$

$$W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

lin. independence.

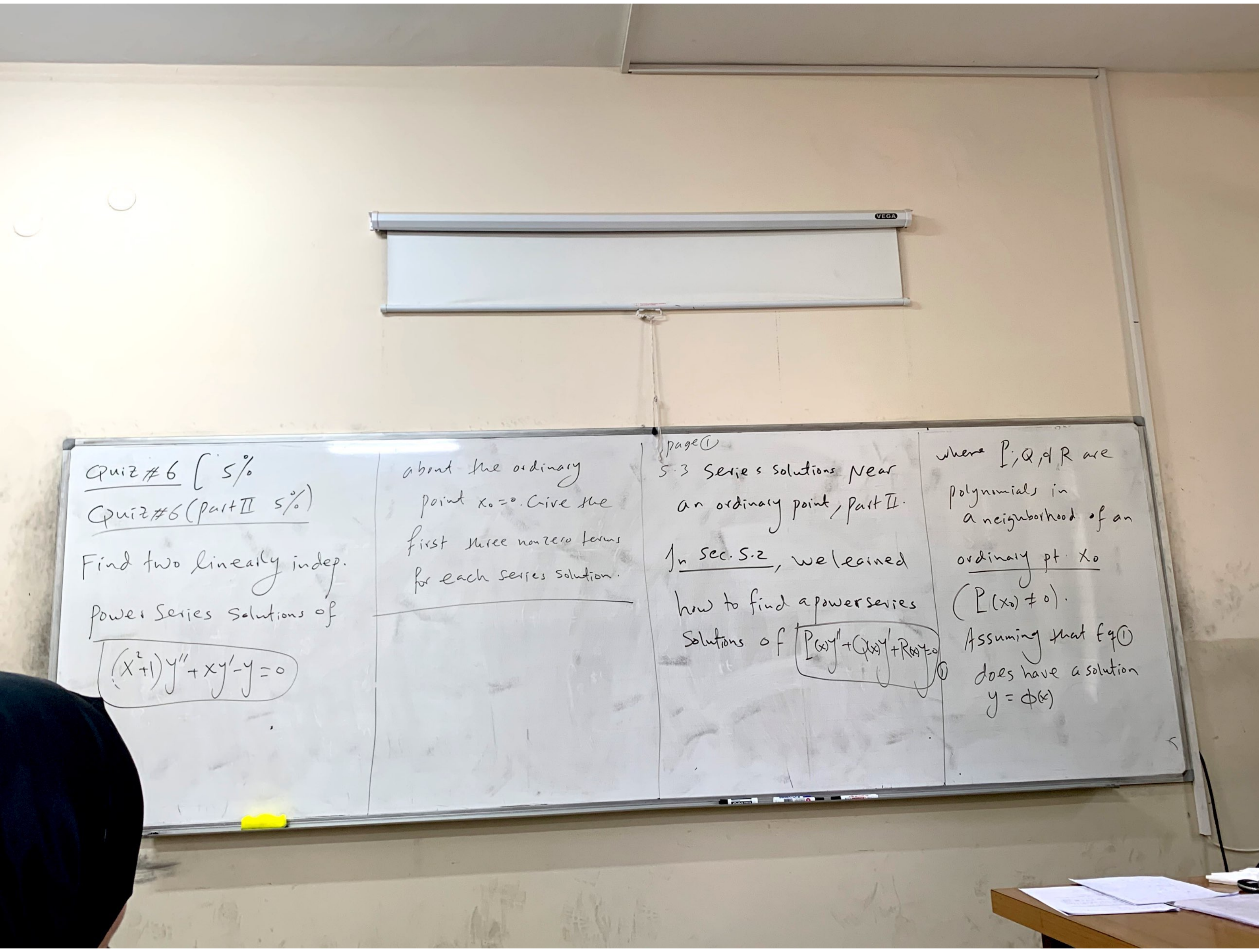
Quiz #6 [5%

Quiz#6 (part II 5%)

Find two linearly indep. power series solutions of

$$(x^2+1)y''+xy'-y=0$$

about the ordinary point $x_0=0$. Give the first three nonzero terms for each series solution.


Quiz#6 [5%
Quiz#6 (part II 5%)
Find two linearly indep.
power series solutions of
$(x^2+1)y'' + xy' - y = 0$
about the ordinary
point $x_0 = 0$. Give the
first three nonzero terms
for each series solution.
page①
5.3 Series Solutions Near
an ordinary point, part II.
In Sec. 5.2, we learned
how to find a power series
Solutions of $P(x)y'' + Q(x)y' + R(x)y = 0$ ①
where P, Q, & R are
polynomials in
a neighborhood of an
ordinary pt. x_0
$(P(x_0) \neq 0)$.
Assuming that Eq①
does have a solution
$y = \phi(x)$

and that ϕ has a Taylor Series. That is,

$$y = \phi(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

$$a_n = \frac{\phi^{(n)}(x_0)}{n!} = \frac{y^{(n)}(x_0)}{n!}$$

$a_0 = y(x_0)$, $a_1 = y'(x_0)$

$$a_2 = \frac{y''(x_0)}{2!}$$

$$a_3 = \frac{y'''(x_0)}{3!}, \ldots$$

Ex. Spse that $y = \sum_{n=0}^{\infty} a_n x^n$ is a sol. of the IVP

$$\begin{cases} y'' + e^x y = 0 \\ y(0) = 1, \quad y'(0) = 1 \end{cases}$$

page (1)

Find the first fifth nonzero terms.

Sol. $y = a_0 + a_1 x + a_2 x^2 + \ldots$

$$a_n = \frac{y^{(n)}(0)}{n!}$$

$a_0 = y(0) = 1$

$a_1 = y'(0) = 1$

$$a_2 = \frac{y''(0)}{2!}$$

Now, $y'' = -e^x y$

$y''(0) = -e^0 y(0) = -1$

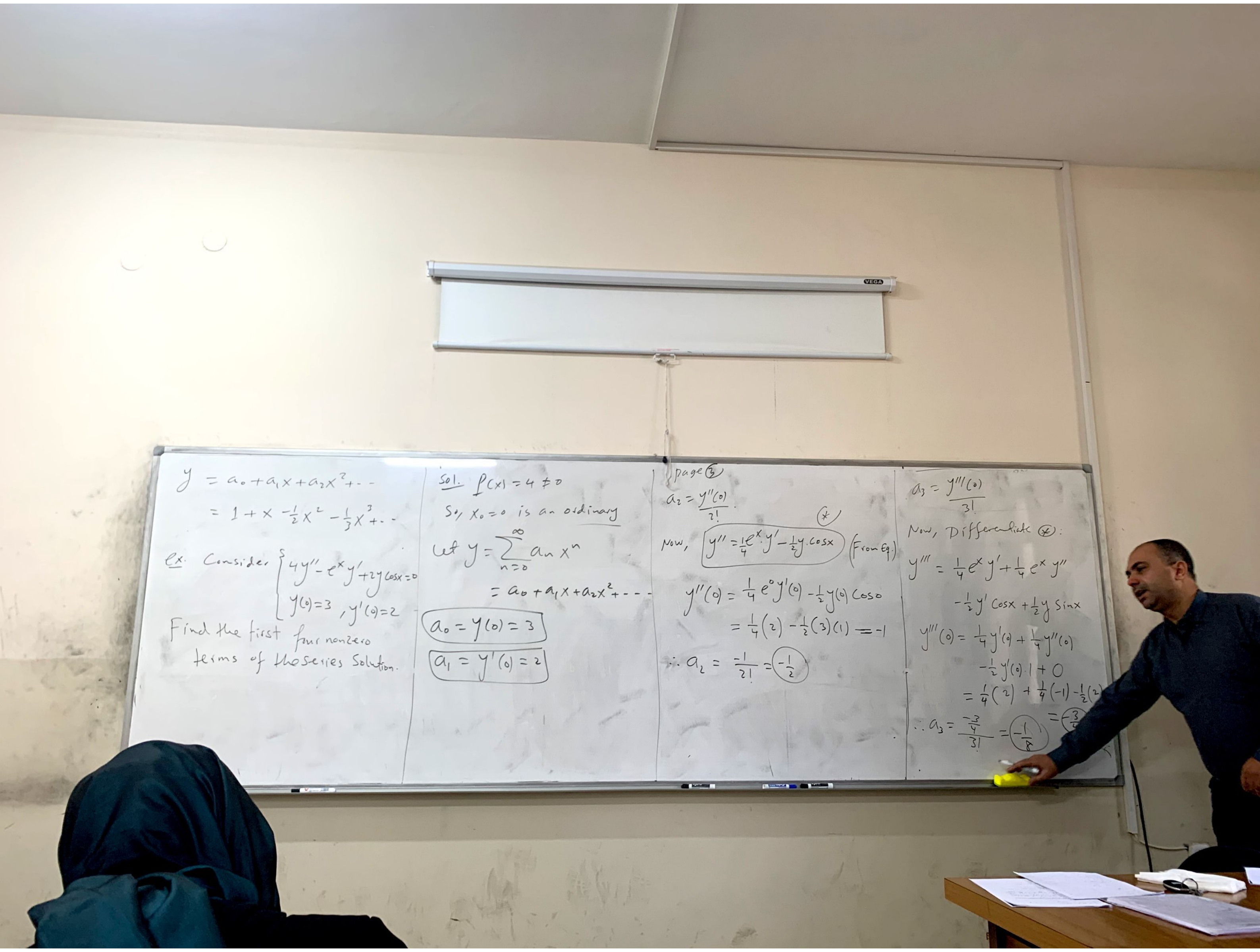
$$a_2 = \frac{-1}{2!} = -\frac{1}{2}$$

$$a_3 = \frac{y'''(0)}{3!}$$

$y''' = (-e^x y)' = -e^x y - e^x y'$

$y'''(0) = -y(0) - y'(0) = -1 - 1 = -2$

$$a_3 = \frac{-2}{3!} = -\frac{1}{3}$$



y = a0 + a1x + a2x^2 + ...
= 1 + x − 1/2 x^2 − 1/3 x^3 + ...
ex. Consider { 4y'' − e^x y' + 2y cosx = 0 ; y(0) = 3 , y'(0) = 2
Find the first four nonzero terms of the series Solution.
Sol. P(x) = 4 ≠ 0
So, x0 = 0 is an ordinary
Let y = Σ_{n=0}^{∞} an x^n
= a0 + a1x + a2x^2 + ...
a0 = y(0) = 3
a1 = y'(0) = 2
page (3)
a2 = y''(0)/2!
Now, y'' = 1/4 e^x · y' − 1/2 y · cosx (*) (From Eq.)
y''(0) = 1/4 e^0 y'(0) − 1/2 y(0) cos0
= 1/4 (2) − 1/2 (3)(1) = −1
∴ a2 = −1/2! = −1/2
a3 = y'''(0)/3!
Now, Differentiate (*):
y''' = 1/4 e^x y' + 1/4 e^x y'' − 1/2 y' cosx + 1/2 y sinx
y'''(0) = 1/4 y'(0) + 1/4 y''(0) − 1/2 y'(0) · 1 + 0
= 1/4 (2) + 1/4 (−1) − 1/2 (2)
= −3/4
∴ a3 = (−3/4)/3! = −1/8

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= 1 + x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \cdots$$

Ex. Consider $\begin{cases} 4y'' - e^x y' + 2y\cos x = 0 \\ y(0) = 3, \ y'(0) = 2 \end{cases}$

Find the first four nonzero terms of the series Solution.

Sol. $P(x) = 4 \neq 0$

So, $x_0 = 0$ is an ordinary

Let $y = \sum_{n=0}^{\infty} a_n x^n$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$a_0 = y(0) = 3$$

$$a_1 = y'(0) = 2$$

page (3)

$$a_2 = \frac{y''(0)}{2!}$$

Now, $y'' = \frac{1}{4}e^x \cdot y' - \frac{1}{2}y \cdot \cos x$ (*) (From Eq.)

$$y''(0) = \frac{1}{4}e^0 y'(0) - \frac{1}{2}y(0)\cos 0$$
$$= \frac{1}{4}(2) - \frac{1}{2}(3)(1) = -1$$

$$\therefore a_2 = \frac{-1}{2!} = -\frac{1}{2}$$

$$a_3 = \frac{y'''(0)}{3!}$$

Now, Differentiate (*):

$$y''' = \frac{1}{4}e^x y' + \frac{1}{4}e^x y'' - \frac{1}{2}y'\cos x + \frac{1}{2}y \sin x$$

$$y'''(0) = \frac{1}{4}y'(0) + \frac{1}{4}y''(0) - \frac{1}{2}y'(0)\cdot 1 + 0$$
$$= \frac{1}{4}(2) + \frac{1}{4}(-1) - \frac{1}{2}(2) = -\frac{3}{4}$$

$$\therefore a_3 = \frac{-\frac{3}{4}}{3!} = -\frac{1}{8}$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$= 3 + 2x - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^3 + \cdots$$

Ex. (3). Find a power series solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ for the eq. $y'' - (x^3+3x+2)y' + (3\cos 2x)y = 0$

Sol. Let $y = a_0 + a_1 x + a_2 x^2 + \cdots$

$a_0 = y(0)$; $a_1 = y'(0)$

$$a_2 = \frac{y''(0)}{2!}$$

Now, $y'' = (x^3+3x+2)y' - 3(\cos 2x)y$ (*)

$$y''(0) = 2y'(0) - 3y(0)$$
$$= 2a_1 - 3a_0$$

page (3)

$$a_2 = \frac{y''(0)}{2!} = \frac{2a_1 - 3a_0}{2}$$
$$\boxed{a_2 = a_1 - \tfrac{3}{2}a_0}$$

$$a_3 = \frac{y'''(0)}{3!}$$

Diff. (*)

$$y''' = (3x^2+3)y' + (x^3+3x+2)y'' + 6\sin 2x \cdot y - (3\cos 2x)y'$$

$$y'''(0) = 3y'(0) + 2y''(0) - 3y'(0)$$
$$= 2(2a_1 - 3a_0)$$
$$y'''(0) = 4a_1 - 6a_0$$
$$a_3 = \frac{4a_1 - 6a_0}{3!}$$
$$\boxed{a_3 = \tfrac{2}{3}a_1 - a_0}$$

$$y = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$= a_0 + a_1 x + \left(a_1 - \tfrac{3}{2} a_0\right) x^2 + \left(\tfrac{2}{3} a_1 - a_0\right) x^3 + \cdots$$
$$= a_0\left(1 - \tfrac{3}{2} x^2 - x^3 + \cdots\right) + a_1\left(x + x^2 + \tfrac{2}{3} x^3 + \cdots\right)$$
$$y_1 = 1 - \tfrac{3}{2} x^2 - x^3 + \cdots$$
$$y_2 = x + x^2 + \tfrac{2}{3} x^3 + \cdots$$

Thm. 5.3.1 If x_0 is an ordinary point of the d.e

$$P(x) y'' + Q(x) y' + R(x) y = 0 \qquad (\eta)$$

then the general sol. of Eq (η) is

$\eta \quad \int \quad \Delta \quad \nabla \quad \sim \quad \nu \quad \ell \quad \mu$

$\kappa \quad \lambda \quad \in \quad \varepsilon \quad \psi \quad \phi, \Phi$

page (5)

$$y = a_0 y_1 + a_1 y_2 = \sum_{n=0}^{\infty} a_n (x - x_0)$$

where a_0, a_1 are constants, and y_1 & y_2 are lin. indep. that are analytic at x_0. The solutions y_1 & y_2 form a fundamental set of solutions.

Further, the radius of convergence of each y_1 & y_2 is at least as large as the minimum of radii of convergence of the series $p = \frac{Q(x)}{P(x)}$, $q(x) = \frac{R(x)}{P(x)}$, Ex. ① What is the radius of convergence of the Taylor series for $f(x) = \frac{1}{x^2-2x-3}$ about $x_0 = 1$ (ordinary)

Sol. Singular points

$x^2-2x-3=0 \Rightarrow (x-3)(x+1)=0$

$x=3$, $x=-1$

$\rho_1 = \text{dist.}(1, 3) = 2$ (1: ord., 3: sing.)

$\rho_2 = \text{dist.}(1, -1) = 2$ (1: ord., −1: sing.)

$\rho \le \min\{\rho_1, \rho_2\} = 2$

radius $= 2$

page (5)

Interval of convergence

$|x-x_0| < \rho$

$|x-1| < 2$

$-2 < x-1 < 2$

$-1 < x < 3$

>> continue.

ex. Determine a lower bound for the radius of convergence of

(1) $(x^2-2x-3)y'' + xy' + 4y = 0$ about $x_0 = 4$.

Sol. $y'' + \frac{x}{x^2-2x-3}y' + \frac{4}{x^2-2x-3}y = 0$

$p(x) = \frac{x}{x^2-2x-3}$, $q(x) = \frac{4}{x^2-2x-3}$

Singular pts are

$x^2-2x-3 = 0$

$(x-3)(x+1) = 0$

$\boxed{x=3}$ $\boxed{x=-1}$

page (1) (ord. ✓ sing. ✓)

$\rho_1 = \text{dist.}(4, 3) = 1$

$\rho_2 = \text{dist.}(4, -1) = 5$

radius of convergence

$\rho = \min\{\rho_1, \rho_2\} = 1$.

the series conv. for at least on $|x-x_0| < \rho$

$\Rightarrow |x-4| < 1$

$-1 < x-4 < 1$

$\boxed{3 < x < 5}$

(2) $y'' + \frac{2x}{1+x^2}y' + \frac{4x^2}{1+x^2}y = 0$ about $x=1$.

Singular pts are

$x^2+1=0 \Rightarrow x=\pm i$

$x=i$, $x=-i$

$\rho_1 = dist(1, i)$ $\quad (x+iy) \sim (x,y)$.

$= dist((1,0),(0,1))$

$= \sqrt{(1-0)^2+(0-1)^2} = \sqrt{2}$.

$\rho_2 = dist.(1, -i)$

$= dist.((1,0),(0,-1))$

$= \sqrt{(1-0)^2+(0+1)^2} = \sqrt{2}$

$\therefore \rho = \min\{\sqrt{2}, \sqrt{2}\} = \sqrt{2}$.

the series conv. for at least on $|x-1| < \sqrt{2}$.

page (2)

(3) (*) $y'' + (\sin x)y' + (1+x^2)y =$

about $x_0 = 0$.

$p(x) = \sin x$ is analytic on $(-\infty, \infty)$

$q(x) = 1+x^2$ " " " $(-\infty, \infty)$

$\rho_1 = \infty$, $\rho_2 = \infty$

$\Longrightarrow \rho = \infty$ or is infinite

5.4 Euler, Regular Singular point.

Df. Consider the d.e

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad (1)$$

eq (1) Can be rewritten as.

$\rho = \infty$ or infinite.. $\longrightarrow$ theres no singularity continuous في every wher... interval $(-\infty, \infty)$.

5.4: Euler, Singular regular or irregular singular point :-

Def :- Consider the d.e $\boxed{P(x)y'' + Q(x)y' + R(x)y = 0}$ ①

eq ① can be rewitten as:

$$\boxed{y'' + p(x)y' + q(x)y = 0} \longrightarrow ②$$

$x = x_0$ is singular regular of eq ① if $\lim_{x \to x_0} (x - x_0)^1 p(x)$ and $\lim_{x \to x_0} (x - x_0)^2 q(x)$ are finite.

اذا وصل ∞ ← irregular بدون ما نشوف التاني ،

~~Asing.~~ Assuming point in eq ① that is not regular singular point is called irregular singular point.

ex : Determine the singular point of the d.e Determine whether they are regular or irregular :

① $2x(x-2)^2 y'' + 3xy' + (x-2)y = 0$

CH6 The Laplace Transform

6.1 Definition of the Laplace transform.

Review (Calculus 2)

Improper Integral

$$\int_a^\infty f(x)\,dx = \lim_{A\to\infty}\int_a^A f(x)\,dx$$

Where $A > 0$

If this limits exists, then the integral is said to be converge. Otherwise, the integral diverges.

page ①

ex. $\int_0^\infty e^{ct}\,dt$, $c \neq 0$

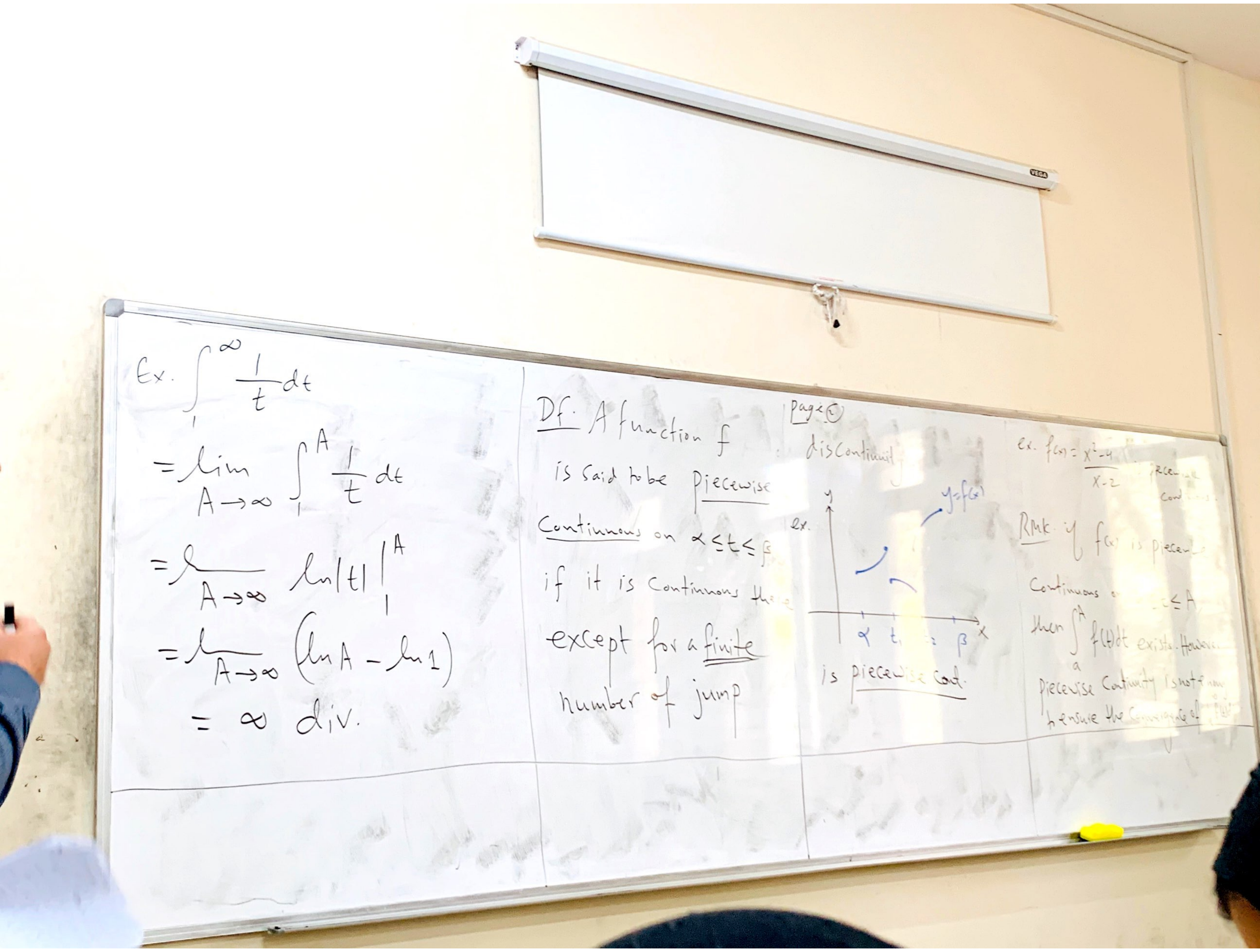
$$= \lim_{A\to\infty}\int_0^A e^{ct}\,dt$$

$$= \lim_{A\to\infty} \frac{e^{ct}}{c}\Big|_0^A$$

$$= \lim_{A\to\infty} \frac{e^{cA}-1}{c}$$

$$= \begin{cases} \text{diverge}, & \text{if } c>0 \\ -\frac{1}{c}, & \text{if } c<0 \\ \text{div.}, & \text{if } c=0 \end{cases}$$

$$= \begin{cases} \text{conv. if } c<0 \\ \text{div. if } c\geq 0 \end{cases}$$


Ex. $\int_1^\infty \frac{1}{t} dt$
$= \lim_{A\to\infty} \int_1^A \frac{1}{t} dt$
$= \lim_{A\to\infty} \ln|t| \Big|_1^A$
$= \lim_{A\to\infty} (\ln A - \ln 1)$
$= \infty$ div.
Df. A function f is said to be piecewise Continuous on $\alpha \le t \le \beta$ if it is Continuous there except for a finite number of jump
Page 2
discontinuity
ex.
y = f(x)
α t_1 β x
is piecewise Cont.
ex. $f(x) = \frac{x^2-4}{x-2}$
RMK. If f(x) is piecewise Continuous
then $\int_a^A f(t) dt$ exists. However piecewise Continuity is not
to ensure the

Df. An <u>integral transform</u> is a relation of the form

$$\int_{\alpha}^{\beta} f(t)\, \underbrace{K(s,t)}_{\text{kernel}}\, dt = F(s)$$

$f(t) = t^3$

$2 \xrightarrow{f=t^3} 8$

Df. (Laplace Transform) $\mathcal{L}.T$

The Laplace transform of $f(t)$ is defined as

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt = F(s)$$

provided this integral

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converges.

Thm (Existence of $\mathcal{L}.T$)

Suppose that

(i) f is <u>piecewise cont.</u> on $0 \le t \le A$, for $A > 0$

(ii) $|f(t)| \le K e^{at}$, $t \ge M$

$K, a, \& M \in \mathbb{R}$, $K, M > 0$.

then the $\mathcal{L}.T$ $\mathcal{L}\{f(t)\} = F(s)$ exists for all $s > a$

Examples.

① $\mathcal{L}\{1\} = \int_0^{\infty} 1 \cdot e^{-st}\, dt$

$= \lim_{A \to \infty} \int_0^A e^{-st}\, dt$

$= \lim_{A \to \infty} \left. \frac{e^{-st}}{-s} \right|_0^A , \quad s > 0$

$$= \lim_{A\to\infty} \frac{e^{-sA} - 1}{-s} = \frac{0-1}{-s} = \frac{1}{s}$$

$$\therefore \mathcal{L}\{1\} = \frac{1}{s}, \quad s>0$$

In general, $\boxed{\mathcal{L}\{k\} = \frac{k}{s}, \quad s>0}$ (1)

ex. $\mathcal{L}\{2019\} = \frac{2019}{s}, \quad s>0$

ex. $\mathcal{L}\{t^2\} = \int_0^\infty t^2 e^{-st}\,dt$

$$= \lim_{A\to\infty} \int_0^A t^2 e^{-st}\,dt$$

اشتقاق		تكامل
t^2	$+$	e^{-st}
$2t$	$(-)$	$\frac{e^{-st}}{-s}$
2	$(+)$	$\frac{e^{-st}}{s^2}$
0		$\frac{e^{-st}}{-s^3}$

Page (4)

$$\mathcal{L}\{t^2\} = \lim_{A\to\infty} -\frac{t^2}{s e^{st}} - \frac{2t}{s^2 e^{st}} - \frac{2}{s^3 e^{st}} \Big|_0^A$$

تعويض

$$= \lim_{A\to\infty} \left[\left(\frac{-A^2}{s e^{sA}} - \frac{2A}{s^2 e^{sA}} - \frac{2}{s^3 e^{sA}} \right) + \frac{2}{s^3} \right]$$

$$= \frac{2}{s^3}, \quad s>0$$

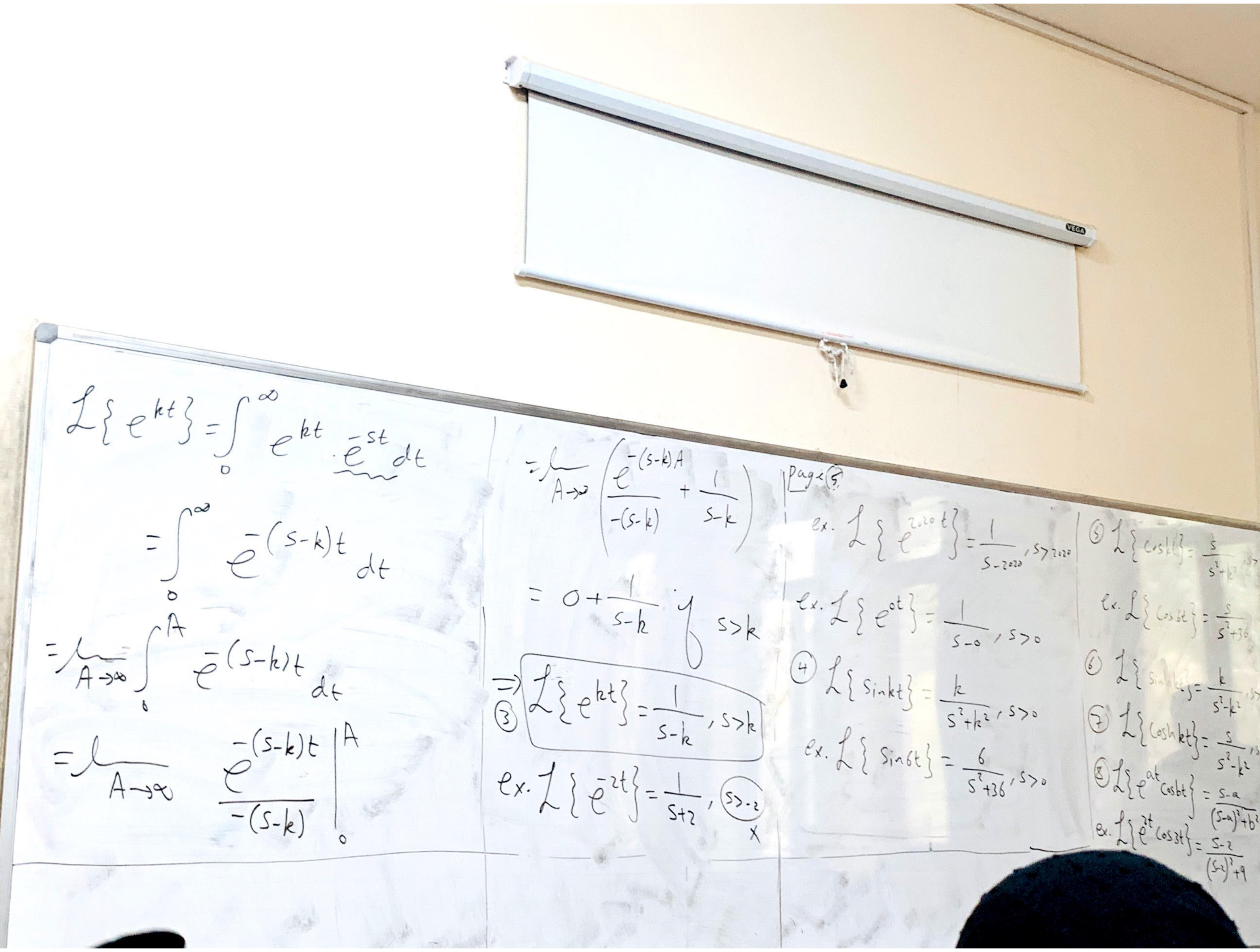
$$\boxed{\mathcal{L}\{t^2\} = \frac{2}{s^3}, \quad s>0}$$

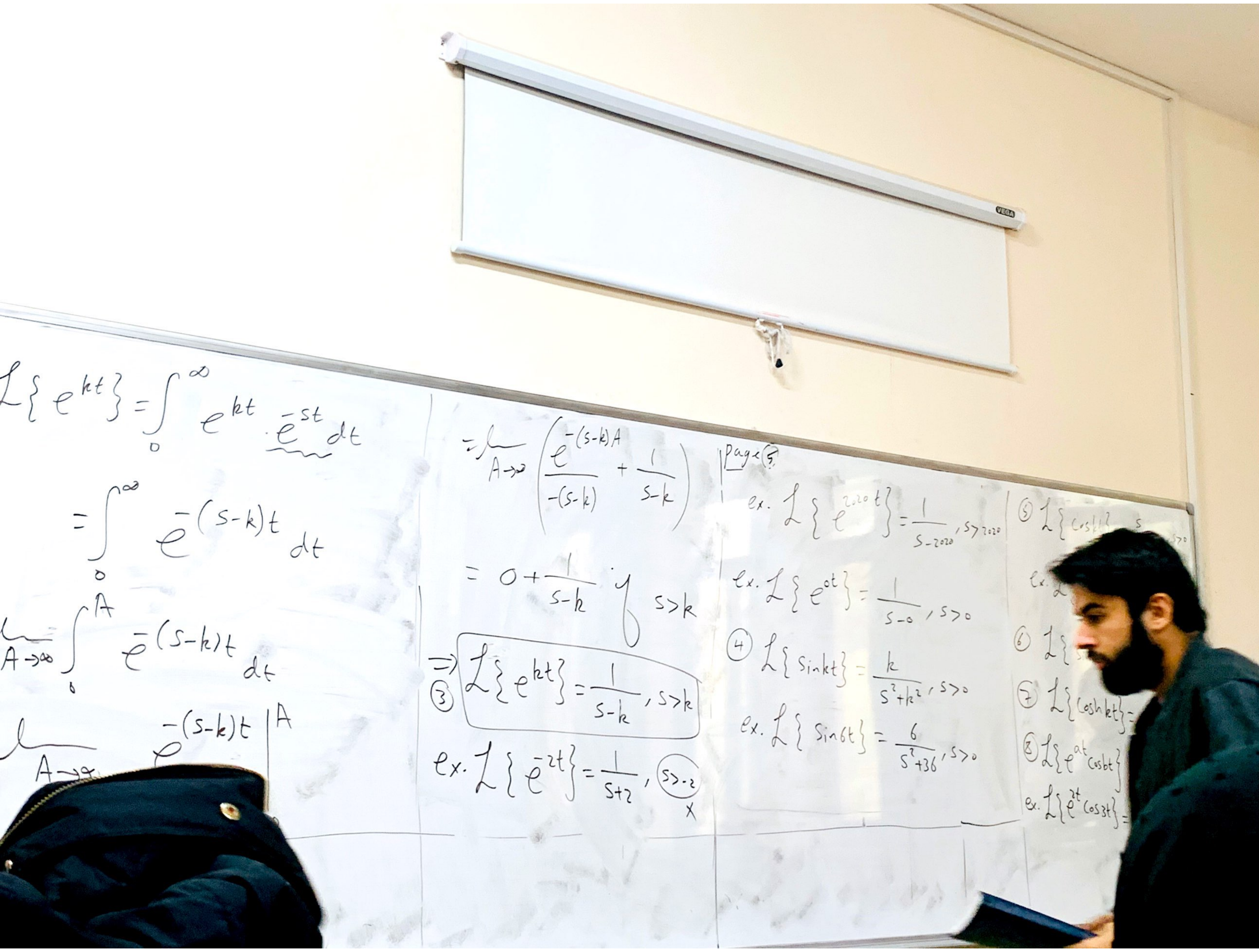
In general, (2) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

ex. $\mathcal{L}\{t^7\} = \frac{7!}{s^8}$

$$\mathcal{L}\{t^0\} = \frac{0!}{s^{0+1}} = \frac{1}{s}$$

Ex (3). $\mathcal{L}\{e^{kt}\} = ??$


$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{kt} e^{-st} dt$
$= \int_0^\infty e^{-(s-k)t} dt$
$= \lim_{A\to\infty} \int_0^A e^{-(s-k)t} dt$
$= \lim_{A\to\infty} \frac{e^{-(s-k)t}}{-(s-k)} \Big|_0^A$
$= \lim_{A\to\infty} \left(\frac{e^{-(s-k)A}}{-(s-k)} + \frac{1}{s-k} \right)$
$= 0 + \frac{1}{s-k}$ if $s>k$
$\Rightarrow$ ③ $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}, s>k$
ex. $\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}, s>-2$
page 5
ex. $\mathcal{L}\{e^{2020t}\} = \frac{1}{s-2020}, s>2020$
ex. $\mathcal{L}\{e^{0t}\} = \frac{1}{s-0}, s>0$
④ $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}, s>0$
ex. $\mathcal{L}\{\sin 6t\} = \frac{6}{s^2+36}, s>0$
⑤ $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$
ex. $\mathcal{L}\{\cos 6t\} = \frac{s}{s^2+36}$
⑥ $\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$
⑦ $\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$
⑧ $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$
ex. $\mathcal{L}\{e^{2t}\cos 3t\} = \frac{s-2}{(s-2)^2+9}$


L{e^kt} = ∫_0^∞ e^kt e^-st dt
= ∫_0^∞ e^-(s-k)t dt
lim A→∞ ∫_0^A e^-(s-k)t dt
= lim A→∞ (e^-(s-k)A / -(s-k) + 1/(s-k))
= 0 + 1/(s-k) , s>k
⇒ ③ L{e^kt} = 1/(s-k) , s>k
ex. L{e^-2t} = 1/(s+2) , s>-2
Page 5
ex. L{e^2020t} = 1/(s-2020) , s>2020
ex. L{e^0t} = 1/(s-0) , s>0
④ L{sin kt} = k/(s²+k²) , s>0
ex. L{sin 6t} = 6/(s²+36) , s>0
⑦ L{cosh kt} =
⑧ L{e^at cos bt}
ex. L{e^2t cos 3t} =

6.1 [continue]

① $\mathcal{L}\{k\} = \frac{k}{s}, \quad s>0$

② $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s>0$

③ $\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}, \quad s>k$

④ $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s>0$

⑤ $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s>0.$

⑥ $\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}, \quad s>|k|.$

⑦ $\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}, \quad s>|k|.$

⑧ $\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}, \quad s>a$

page ①

⑨ $\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}, \quad s>a$

⑩ $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$

ex. $\mathcal{L}\{t^2 e^{3t}\} = (-1)^2 \frac{d^2}{ds^2}\left(\mathcal{L}\{e^{3t}\}\right)$

$= \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)$

let $F(s) = \frac{1}{s-3}$

$F'(s) = \frac{-1}{(s-3)^2}$

$F''(s) = \frac{2}{(s-3)^3}$

$\therefore \mathcal{L}\{t^2 e^{3t}\} = (-1)^2 \cdot \frac{2}{(s-3)^3}$

$= \frac{2}{(s-3)^3}$

(11) $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}, \ s>a$

$n = 0, 1, 2, \dots$

ex. $\mathcal{L}\{t^{14} e^{2t}\} = \frac{14!}{(s-2)^{15}}$.

(12) <u>linearity</u>

$\mathcal{L}\{\alpha f(t) \pm \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} \pm \beta \mathcal{L}\{g(t)\}$

ex. $\mathcal{L}\{\cos^2 t\}$

$= \mathcal{L}\left\{\frac{1+\cos 2t}{2}\right\}$

$= \frac{1}{2}\mathcal{L}\{1+\cos 2t\}$

$= \frac{1}{2}\left[\mathcal{L}\{1\} + \mathcal{L}\{\cos 2t\}\right]$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$

page (17)

ex. $\mathcal{L}\{\sin(2t + \frac{\pi}{3})\}$

$= \mathcal{L}\{\sin 2t \cos\frac{\pi}{3} + \cos 2t \sin\frac{\pi}{3}\}$

$= \frac{1}{2}\mathcal{L}\{\sin 2t\} + \frac{\sqrt{3}}{2}\mathcal{L}\{\cos 2t\}$

$= \frac{1}{2} \cdot \frac{2}{s^2+4} + \frac{\sqrt{3}}{2} \cdot \frac{s}{s^2+4}$

<u>6.2 Solutions of IVPs</u>

<u>Inverse Laplace</u>

$\mathcal{L}\{f(t)\} = F(s)$

$\Rightarrow \mathcal{L}^{-1}\{F(s)\} = f(t)$.

ex. $\mathcal{L}^{-1}\left\{\frac{6}{s}\right\} = 6$.

ex. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$

$= \frac{1}{3}\sin(3t)$.

ex. $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2!}\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}$

$= \frac{1}{2}t^2$.

ex. $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{(s-2)^4}\right\}$

$= \frac{1}{6}e^{2t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$

$= \frac{1}{6}e^{2t}\cdot t^3$

ex. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+6}\right\}$

$s^2+4s+6 = \underbrace{s^2+4s+4}-4+6$

$= (s+2)^2+2$.

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$

$= e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+(\sqrt{2})^2}\right\}$

page (4)

$= \frac{1}{\sqrt{2}}e^{-2t}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\}$

$= \frac{1}{\sqrt{2}}e^{-2t}\sin\sqrt{2}t$.

ex. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s-3}\right\}$

$$\boxed{\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}}$$

$\frac{1}{(s-3)(s+1)} = \frac{A(s+1)+B(s-3)}{(s-3)(s+1)}$

$$\boxed{1 = A(s+1)+B(s-3)}$$

$\boxed{s=-1} \Rightarrow 1 = -4B \Rightarrow \boxed{B = -\frac{1}{4}}$

$\boxed{s=3} \Rightarrow 1 = 4A \Rightarrow \boxed{A = \frac{1}{4}}$

$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2-2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{1}{4}}{s-3} - \frac{\frac{1}{4}}{s+1}\right\}$

$$= \frac{1}{4} \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{4} \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t} .$$

- $\mathcal{L}\{y(t)\} = Y(s)$
- $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$
- $\mathcal{L}\{y''\} = s^2 Y - sy(0) - y'(0)$

- $\mathcal{L}\{y'''(t)\} = s^3 Y - s^2 y(0) - sy'(0) - y''(0)$

$$\mathcal{L}\{y^{(4)}\} = s^4 Y - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0) .$$

$\vdots$

and so on .

Applications

Ex. Solve the IVP using $\mathcal{L}.T$

page (5)

$$\begin{cases} y'' + 2y' + \\ y(0) = 2 , \end{cases}$$

Sol. Take

$\mathcal{L}\{y''\} + 2\mathcal{L}$

$s^2 Y - sy(0) - y$

$+ Y = 4$

$\mathcal{L}\{y'''(t)\} = s^3 Y - s^2 y(0) - s y'(0) - y''(0)$

$\mathcal{L}\{y^{(4)}\} = s^4 Y - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0)$.

⋮

and so on.

Applications

Ex. Solve the IVP using $\mathcal{L}.T$

page ⑤

$$\begin{cases} y'' + 2y' + y = 4e^{-t} \\ y(0) = 2, \; y'(0) = -1 \end{cases}$$

Sol Take $\mathcal{L}.T.$

$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 4\mathcal{L}\{e^{-t}\}$

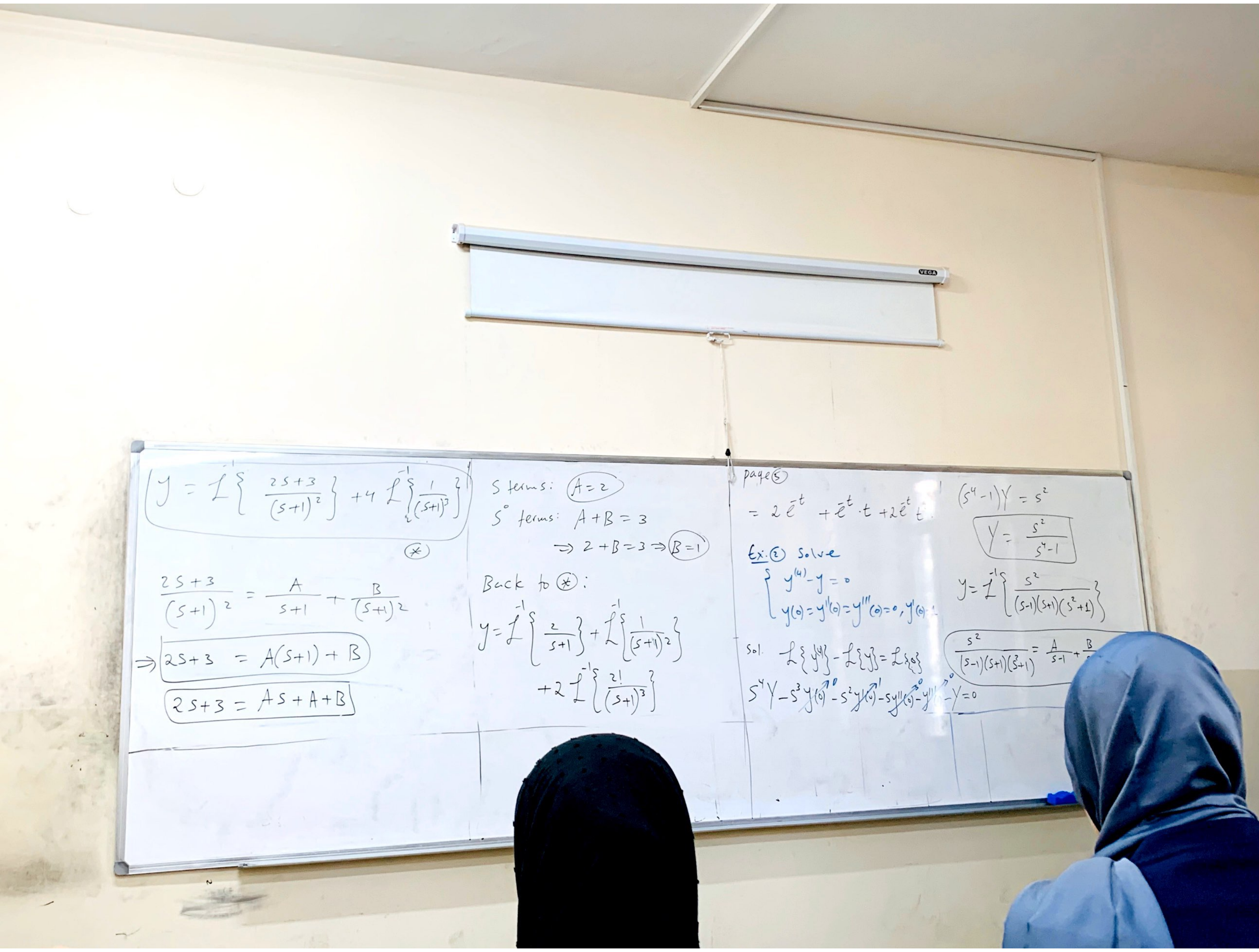
$s^2 Y - s y(0) - y'(0) + 2[sY - y(0)] + Y = 4 \frac{1}{s+1}$

$s^2 Y - 2s + 1 + 2sY - 4 + Y = \frac{4}{s+1}$

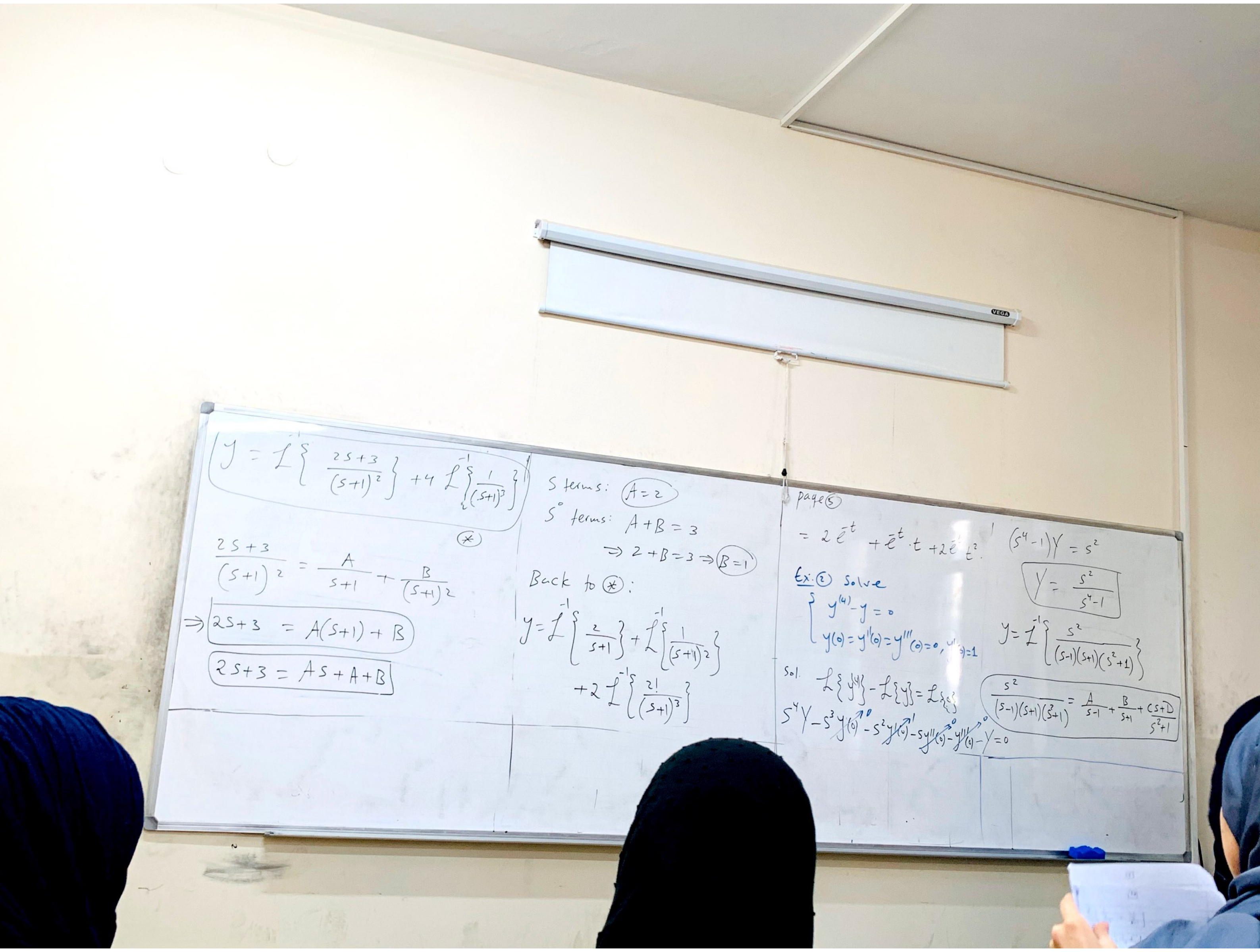
$(s^2 + 2s + 1) Y = 2s + 3 + \frac{4}{s+1}$

$$\boxed{Y = \frac{2s+3}{(s+1)^2} + \frac{4}{(s+1)^3}}$$

$\Rightarrow y = \mathcal{L}^{-1}\{Y\}$



$y = \mathcal{L}^{-1}\left\{\frac{2s+3}{(s+1)^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\}$ (*)
$\frac{2s+3}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$
$\Rightarrow 2s+3 = A(s+1) + B$
$2s+3 = As + A + B$
s terms: $A=2$
s^0 terms: $A+B=3$
$\Rightarrow 2+B=3 \Rightarrow B=1$
Back to (*):
$y = \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$
$+2\mathcal{L}^{-1}\left\{\frac{2!}{(s+1)^3}\right\}$
page (5)
$= 2e^{-t} + e^{-t}\cdot t + 2e^{-t}t$
Ex. (2) Solve
$y^{(4)} - y = 0$
$y(0) = y''(0) = y'''(0) = 0, y'(0)$
Sol. $\mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} = \mathcal{L}\{0\}$
$s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y = 0$
$(s^4-1)Y = s^2$
$Y = \frac{s^2}{s^4-1}$
$y = \mathcal{L}^{-1}\left\{\frac{s^2}{(s-1)(s+1)(s^2+1)}\right\}$
$\frac{s^2}{(s-1)(s+1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{}$


y = L^-1 { (2s+3)/(s+1)^2 } + 4 L^-1 { 1/(s+1)^3 }
(2s+3)/(s+1)^2 = A/(s+1) + B/(s+1)^2
2s+3 = A(s+1) + B
2s+3 = As + A + B
s terms: A = 2
s^0 terms: A + B = 3
⇒ 2 + B = 3 ⇒ B = 1
Back to (*):
y = L^-1 { 2/(s+1) } + L^-1 { 1/(s+1)^2 }
+ 2 L^-1 { 2!/(s+1)^3 }
page 5
= 2e^-t + e^-t · t + 2e^-t t^2
Ex. 2 Solve
y^(4) - y = 0
y(0) = y'(0) = y'''(0) = 0, y''(0) = 1
Sol.
L{y^(4)} - L{y} = L{0}
s^4 Y - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y = 0
(s^4 - 1) Y = s^2
Y = s^2/(s^4 - 1)
y = L^-1 { s^2/((s-1)(s+1)(s^2+1)) }
s^2/((s-1)(s+1)(s^2+1)) = A/(s-1) + B/(s+1) + (Cs+D)/(s^2+1)