6.3 Step Functions.

1st Translation theorem

If $\mathcal{L}\{f(t)\} = F(s)$, and $a \in \mathbb{R}$, then

$$\mathcal{L}\{e^{at} f(t)\} = \mathcal{L}\{f(t)\}\Big|_{s \to s-a} = F(s-a)$$

or

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$$

ex $\mathcal{L}\{e^{6t} \cdot t^2\}$

$$= \mathcal{L}\{t^2\}_{s \to s-6}$$

$$= \frac{2!}{s^3}\Big|_{s \to s-6} = \frac{2}{(s-6)^3}$$

page(1)

ex. $\mathcal{L}\{e^{-2t}\cos 4t\} = \mathcal{L}\{$

$$= \frac{s}{s^2 +}$$

$$= \frac{s+2}{(s+2)^2 +}$$

$\} = e^{at} \mathcal{L}^{-1}\{F(s)\}$

page (1)

ex. $\mathcal{L}\{e^{-2t}\cos 4t\} = \mathcal{L}\{\cos 4t\}_{s \to s+2}$

$$= \frac{s}{s^2+16}\Bigg|_{s \to s+2}$$

$$= \frac{s+2}{(s+2)^2+16}$$

$\to s-6$

$s-6 \quad = \frac{2}{(s-6)^3}$

ex. $\mathcal{L}^{-1}\left\{\frac{1}{(s-4)^3}\right\}$

$= e^{4t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$

$= \frac{1}{2!} e^{4t}\,\mathcal{L}^{-1}\left\{\frac{2!}{s^3}\right\}$

$= \frac{1}{2} e^{4t} \cdot t^2.$

ex $\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}$

$\frac{2s+5}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$ (*)

$\Rightarrow 2s+5 = A(s-3) + B$

$s=3 \Rightarrow 11 = B$

$s=0 \Rightarrow 5 = -3A + 11 \Rightarrow$

page

Back to (*):

$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}$

page (2)

Back to (*):

$$\ldots \frac{B}{(s-3)^2} \quad (*)$$

$$\mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= 2e^{3t} + 11e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 2e^{3t} + 11e^{3t}\cdot t$$

$$= (2+11t)e^{3t}.$$

Ex. Solve the d.e using $\mathcal{L}$.T.

$$\begin{cases} y'' - 6y' + 9y = t^2 e^{3t} \\ y(0) = 2, \quad y'(0) = 17 \end{cases}$$

Sol. Take $\mathcal{L}$ T:

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{t^2 e^{3t}\}$$

$$s^2 Y - s y(0) - y'(0) - 6[sY - y(0)] + 9Y = \frac{2}{(s-3)^3}$$

$$\Rightarrow s^2 Y - 2s - 17 - 6sY + 12 + 9Y = \frac{2}{(s-3)^3}$$

$$(s^2 - 6s + 9)Y = 2s + 5 + \frac{2}{(s-3)^3}$$

$$Y = \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}$$

$$y = \mathcal{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\} + \frac{2}{4!}\mathcal{L}^{-1}\left\{\frac{1 \cdot 4!}{(s-3)^5}\right\}$$

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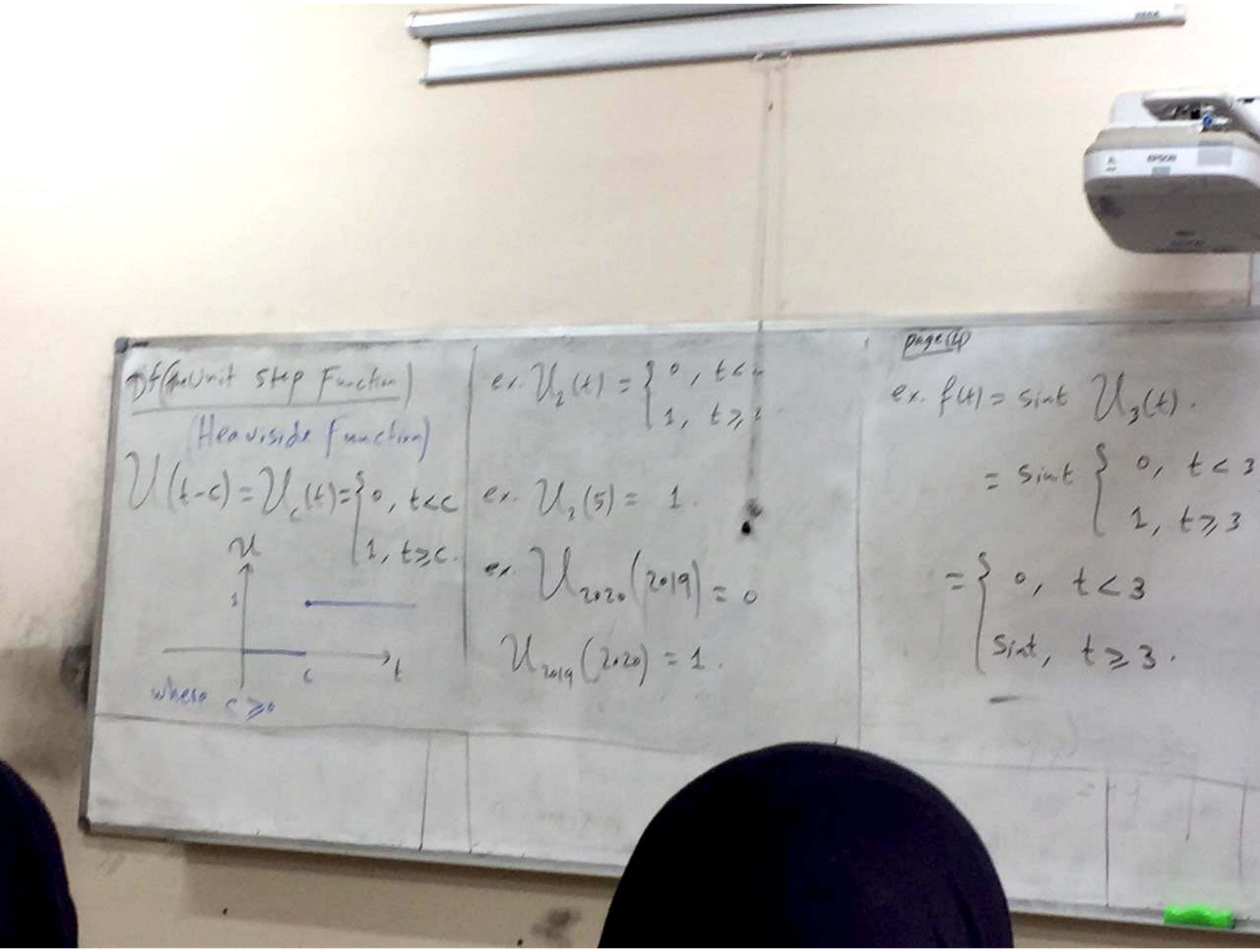
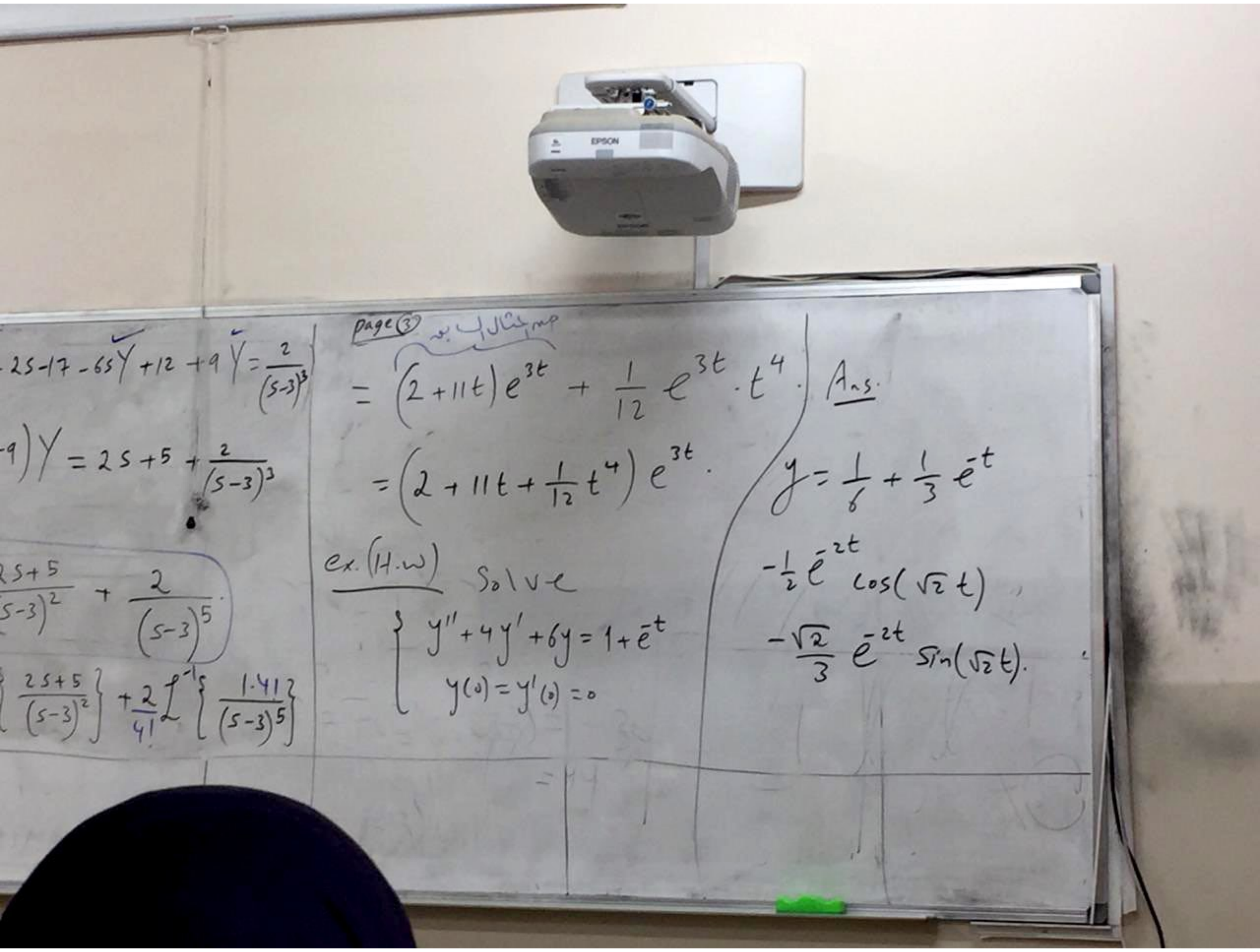
$$= (2 + 11t)e^{3t} + \frac{1}{12}$$

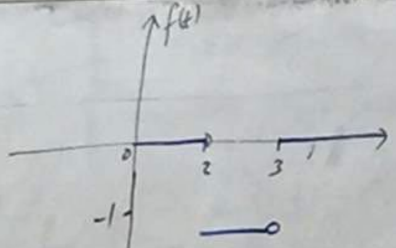
$$= \left(2 + 11t + \frac{1}{12}t^4\right)e$$

ex. (H.W) Solve

$$\begin{cases} y'' + 4y' + 6y = 1 + e^{-} \\ y(0) = y'(0) = 0 \end{cases}$$


Df (the Unit step Function)
(Heaviside Function)
U(t-c) = U_c(t) = { 0, t<c ; 1, t≥c.
u
1
c
t
where c ≥ 0
ex. U_2(t) = { 0, t<2 ; 1, t≥2
ex. U_2(5) = 1.
ex. U_2020(2019) = 0
U_2019(2020) = 1.
page(14)
ex. f(t) = sint U_3(t).
= sint { 0, t<3 ; 1, t≥3
= { 0, t<3 ; sint, t≥3.


page 3
Ans.
= (2 + 11t) e^{3t} + 1/12 e^{3t} · t^4
= (2 + 11t + 1/12 t^4) e^{3t}.
ex. (H.w) Solve
y'' + 4y' + 6y = 1 + e^{-t}
y(0) = y'(0) = 0
y = 1/6 + 1/3 e^{-t}
- 1/2 e^{-2t} cos(√2 t)
- √2/3 e^{-2t} sin(√2 t).
(s-9) Y = 2s + 5 + 2/(s-3)^3



Rmk. The unit step function can be used to write piece-wise function in a Compact form as follows.

ex. write $f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & a \le t < b \\ k(t), & t \ge b. \end{cases}$

Ans. (0 — a — b)

$$f(t) = g(t) + [h(t) - g(t)]\,\mathcal{U}_a(t) + [k(t) - h(t)]\,\mathcal{U}_b(t).$$

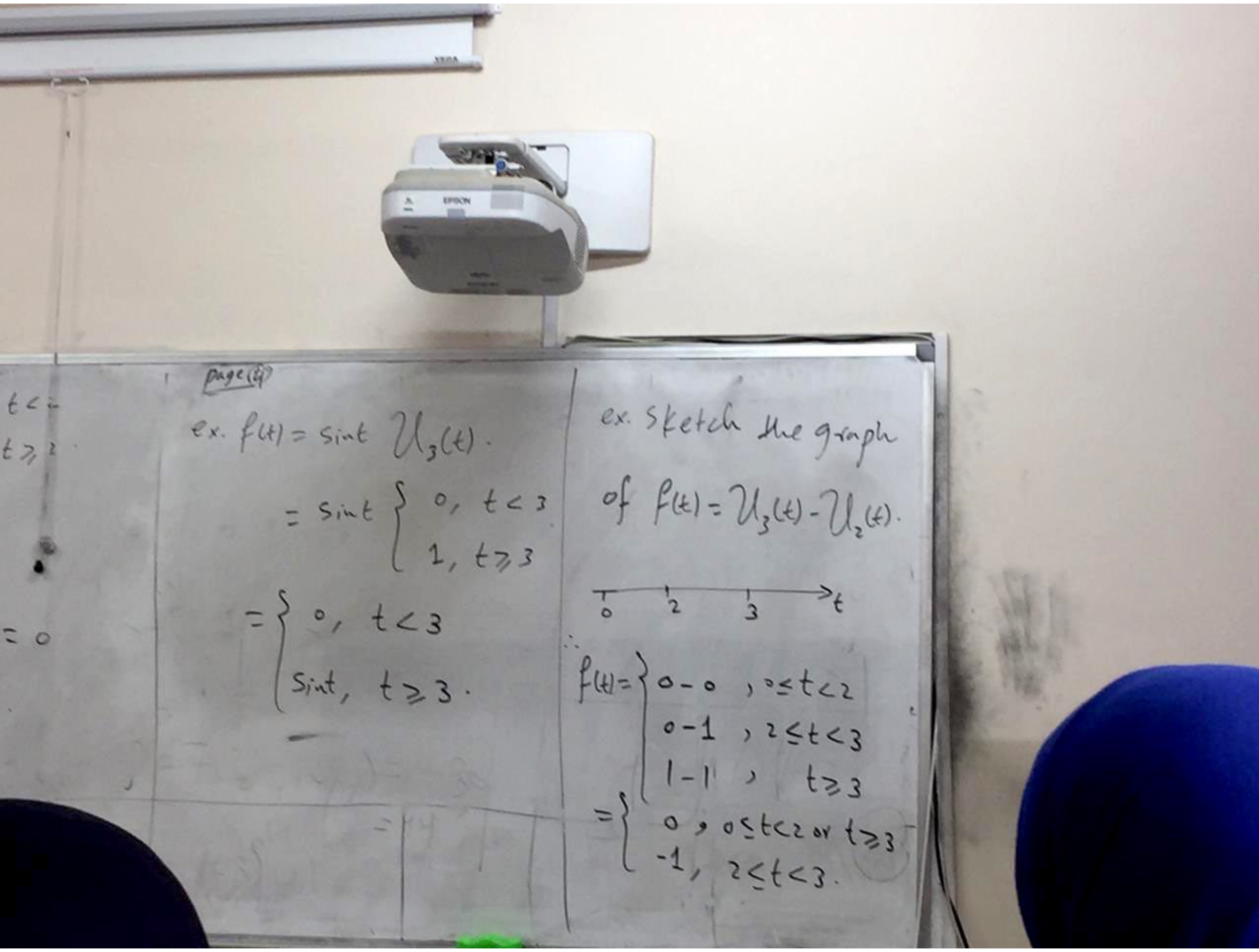
[Compact form].

ex. write

$f(t) = \begin{cases} \frac{t}{2} \\ 3, \end{cases}$

Sol. $f(t) = \frac{t}{2}$

ex. $f(t) = \begin{cases} \ldots \end{cases}$


page (9)
ex. f(t) = sin t U_3(t).
= sin t { 0, t < 3 ; 1, t ≥ 3
= { 0, t < 3 ; sin t, t ≥ 3.
ex. Sketch the graph
of f(t) = U_3(t) - U_2(t).
0 2 3 t
∴ f(t) = { 0 - 0 , 0 ≤ t < 2 ; 0 - 1 , 2 ≤ t < 3 ; 1 - 1 , t ≥ 3
= { 0 , 0 ≤ t < 2 or t ≥ 3 ; -1 , 2 ≤ t < 3.

$t < a$

$t \le b$

$\ge b.$

$_a(t)$

$).$

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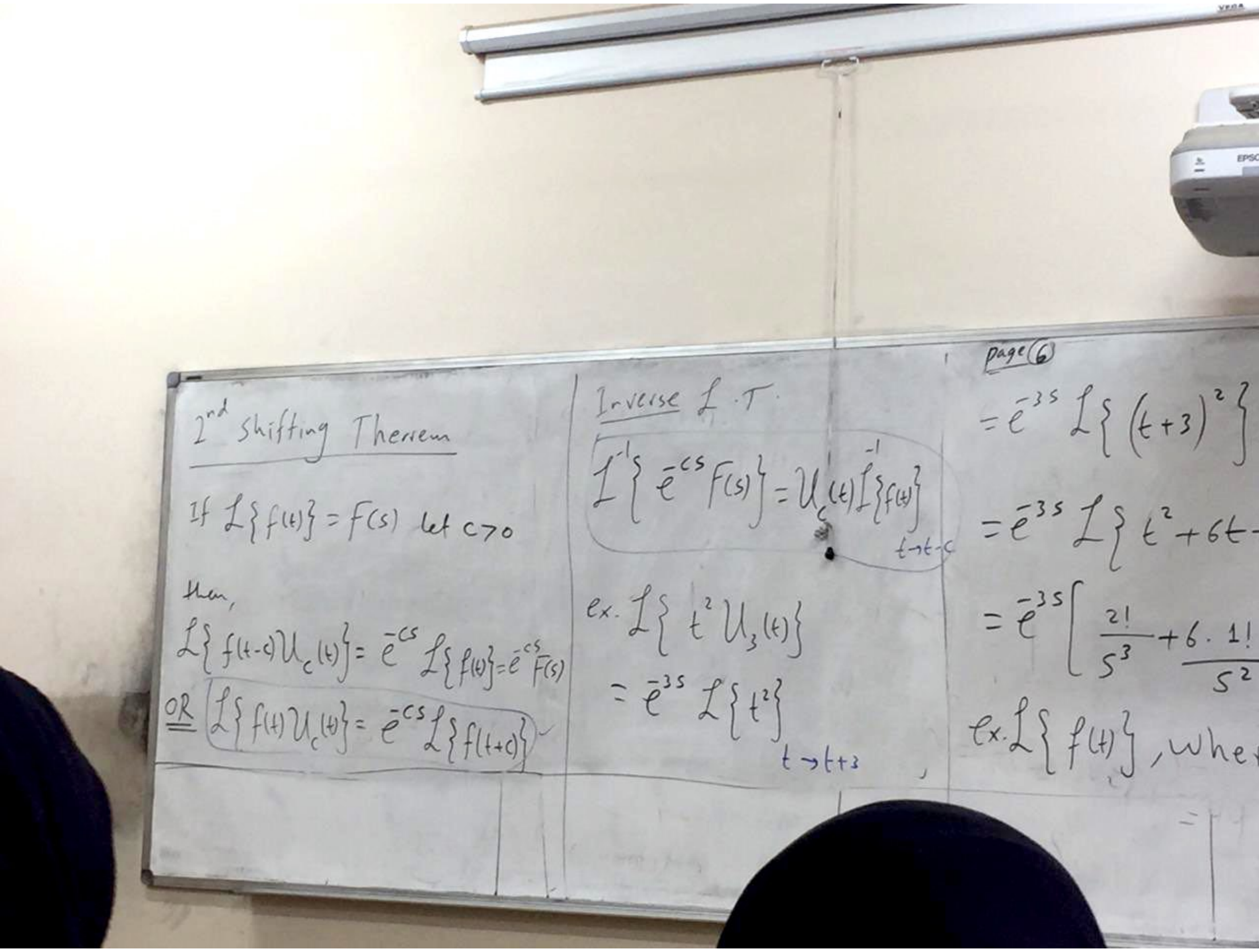
ex. Write in a compact form

$$f(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3, & t \ge 6. \end{cases}$$

Sol. $f(t) = \frac{t}{2} + \left(3 - \frac{t}{2}\right) \mathcal{U}_6(t).$

ex. $$f(t) = \begin{cases} 1, & 0 \le t < 3 \\ 5, & 3 \le t < 6 \\ t, & t \ge 6 \end{cases}$$

Sol. $f(t) = 1 + (5-1)\,\mathcal{U}_3(t) + (t-5)\,\mathcal{U}_6(t).$


2nd Shifting Theorem
If $\mathcal{L}\{f(t)\} = F(s)$ let $c>0$
then,
$\mathcal{L}\{f(t-c)\mathcal{U}_c(t)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$
OR $\mathcal{L}\{f(t)\mathcal{U}_c(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}$
Inverse $\mathcal{L}$.T.
$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = \mathcal{U}_c(t)\mathcal{L}^{-1}\{f(t)\}_{t\to t-c}$
ex. $\mathcal{L}\{t^2\mathcal{U}_3(t)\}$
$= e^{-3s}\mathcal{L}\{t^2\}_{t\to t+3}$
page (6)
$= e^{-3s}\mathcal{L}\{(t+3)^2\}$
$= e^{-3s}\mathcal{L}\{t^2+6t$
$= e^{-3s}\left[\frac{2!}{s^3}+\frac{6\cdot 1!}{s^2}\right.$
ex. $\mathcal{L}\{f(t)\}$, whe

...rse L.T.

$\{ e^{-cs}F(s)\} = U_c(t)\,\mathcal{L}^{-1}\{f(t)\}_{t\to t-c}$

$\mathcal{L}\{ t^2 U_3(t)\}$

$= e^{-3s}\,\mathcal{L}\{t^2\}_{t\to t+3}$

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$= e^{-3s}\,\mathcal{L}\{(t+3)^2\}$

$= e^{-3s}\,\mathcal{L}\{t^2+6t+9\}$

$= e^{-3s}\left[\frac{2!}{s^3}+\frac{6\cdot 1!}{s^2}+\frac{9}{s}\right]$

ex. $\mathcal{L}\{f(t)\}$, where

$$f(t)=\begin{cases} \sin t, & 0\le t<\pi/4 \\ \sin t+\cos(t-\frac{\pi}{4}), & t\ge \frac{\pi}{4} \end{cases}$$

Sol. (Compact form)

$$f(t)=\sin t+\cos(t-\tfrac{\pi}{4})\,U_{\frac{\pi}{4}}(t).$$

$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{\cos(t-\frac{\pi}{4})\,\mathcal{U}_{\frac{\pi}{4}}(t)\}$

$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\,\mathcal{L}\{\cos(t-\frac{\pi}{4})\}$ $\quad t \to t+\frac{\pi}{4}$

$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\,\mathcal{L}\{\cos t\}$

$= \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\,\frac{s}{s^2+1}$

Recall,
Inverse $\mathcal{L}.T$

$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = \mathcal{U}_c(t)\,\mathcal{L}^{-1}\{f(t)\}$ $\quad t \to t-c$

ex. $\mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^2+1}\right\}$

$= \mathcal{L}^{-1}\left\{e^{-4s} \cdot \frac{1}{s^2+1}\right\}$ $\quad \frac{1}{s^2+1} \to F(s)$

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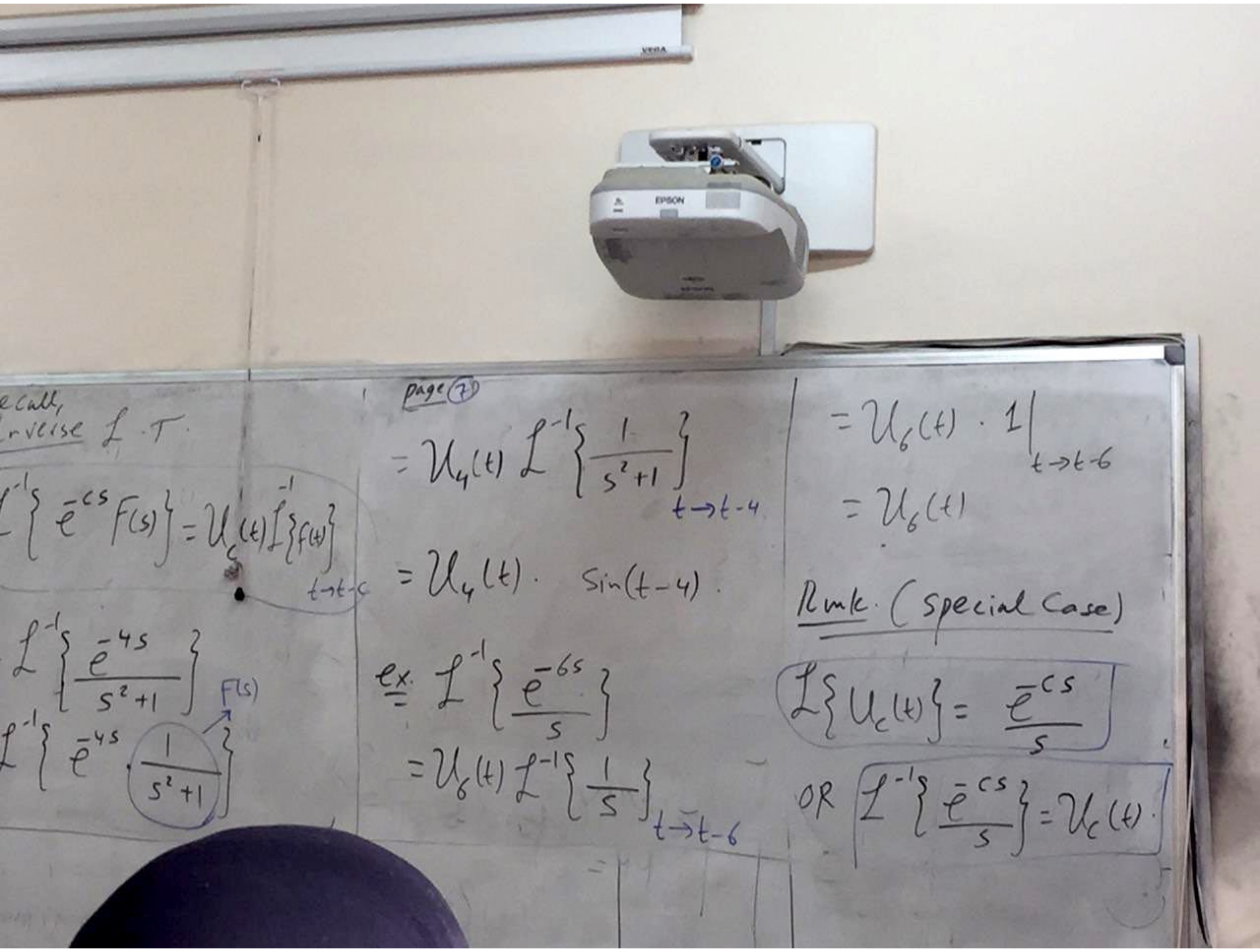
$= \mathcal{U}_4(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$ $\quad t \to$

$= \mathcal{U}_4(t) \cdot \sin(t-4)$

ex. $\mathcal{L}^{-1}\left\{\frac{e^{-6s}}{s}\right\}$

$= \mathcal{U}_6(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$ $\quad t \to$


ecall,
nverse $\mathcal{L}\cdot T$
$\mathcal{L}^{-1}\left\{ e^{-cs} F(s) \right\} = \mathcal{U}_c(t)\, \mathcal{L}^{-1}\left\{ F(s) \right\}\Big|_{t \to t-c}$
$\mathcal{L}^{-1}\left\{ \frac{e^{-4s}}{s^2+1} \right\}$
$\mathcal{L}^{-1}\left\{ e^{-4s} \cdot \frac{1}{s^2+1} \right\}$
F(s)
page (7)
$= \mathcal{U}_4(t)\, \mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\}_{t \to t-4}$
$= \mathcal{U}_4(t) \cdot \sin(t-4)$.
ex. $\mathcal{L}^{-1}\left\{ \frac{e^{-6s}}{s} \right\}$
$= \mathcal{U}_6(t)\, \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\}_{t \to t-6}$
$= \mathcal{U}_6(t) \cdot 1 \Big|_{t \to t-6}$
$= \mathcal{U}_6(t)$
Rmk. (special case)
$\mathcal{L}\left\{ \mathcal{U}_c(t) \right\} = \frac{e^{-cs}}{s}$
OR $\mathcal{L}^{-1}\left\{ \frac{e^{-cs}}{s} \right\} = \mathcal{U}_c(t)$.

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Ex. Solve using $\mathcal{L}.T$

$$\begin{cases} y'' + 4y = \sin t \; \mathcal{U}_{2\pi}(t) \\ y(0) = 1, \; y'(0) = 0. \end{cases}$$

Sol Take $\mathcal{L}.T$:

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t \; \mathcal{U}_{2\pi}(t)\}$$

$$s^2 Y - s y(0) - y'(0) + 4Y = e^{-2\pi s} \mathcal{L}\{\sin(t+2\pi)\}$$

$$(s^2+4)Y = s + e^{-2\pi s} \cdot \mathcal{L}\{\sin t\}$$

$$(s^2+4)Y = s + e^{-2\pi s} \cdot \frac{1}{s^2+1}.$$

$$Y = \frac{s}{s^2+4} + \frac{1}{(s^2+1)(s^2+4)} e^{-2\pi s}.$$

$$y = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)} e^{-2\pi s}\right\}$$

$$y = \cos 2t + \mathcal{U}_{2\pi}(t) \mathcal{L} \ldots$$

$$\cdot \; \mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)(s^2+\ldots} \right.$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} \ldots$$

$+ e^{-2\pi s}\cdot\mathcal{L}\{\sin t\}$

$e^{-2\pi s}\cdot\frac{1}{s^2+1}$.

$\frac{1}{(s^2+1)(s^2+4)}e^{-2\pi s}$

$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}e^{-2\pi s}\right\}$

page (8)

$$y = \cos 2t + \mathcal{U}_{2\pi}(t)\,\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}_{t\to t-2\pi} \quad (*)$$

$$\cdot\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$1 = (As+B)(s^2+4) + (s^2+1)(Cs+D)$$

$A=0$, $B=\frac{1}{3}$, $C=0$

$D=-\frac{1}{3}$ (أقنع نفسك)

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4)}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{1}{3}\sin t - \frac{1}{6}\sin 2t \quad (**)$$

Setting (**) into (*):

$$y = \cos 2t + \mathcal{U}_{2\pi}(t)\left[\frac{1}{3}\sin(t-2\pi) - \frac{1}{6}\sin(2t-4\pi)\right]$$

$$= \cos 2t + \mathcal{U}_{2\pi}(t)\left[\frac{1}{3}\sin t - \frac{1}{6}\sin(2t)\right]$$

$$= \begin{cases} \cos 2t, & 0 \le t < 2\pi \\ \cos 2t + \frac{1}{3}\sin t - \frac{1}{6}\sin 2t, & t \ge 2\pi \end{cases}$$

Ex. (H.W)

Solve $\begin{cases} y'' + y = g(t) \\ y(0) = 0,\ y'(0) = 2 \end{cases}$

where $g(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3, & t \ge 6. \end{cases}$

Sol. $g(t) = \frac{t}{2} + \left(3 - \frac{t}{2}\right)\mathcal{U}_6(t).$

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$$\mathcal{L}\{g(t)\} = \frac{1}{2}\cdot\frac{1}{s^2} + e^{-6s}\mathcal{L}\{\ldots$$

$$= \frac{1}{2s^2} + e^{-6s}\mathcal{L}\{3 - \ldots$$

$$= \frac{1}{2s^2} - e^{-6s}\cdot\frac{1}{2s^2}.$$

Now, Take $\mathcal{L}.T$:

$$s^2 Y - s y(0) - y'(0) + Y = \frac{1}{2s^2} \ldots$$

$\ldots)$

$\ldots + y = g(t)$

$\ldots(0) = 0, \; y'(0) = 2$

$\ldots(t) = \begin{cases} \frac{t}{2}, & 0 \le t < 6 \\ 3, & t \ge 6. \end{cases}$

$\ldots + \left(3 - \frac{t}{2}\right) \mathcal{U}_6(t).$

page

$$\mathcal{L}\{g(t)\} = \frac{1}{2} \cdot \frac{1}{s^2} + e^{-6s} \mathcal{L}\left\{3 - \frac{t}{2}\right\}$$

$t \to t+6$

$$= \frac{1}{2s^2} + e^{-6s} \mathcal{L}\left\{3 - \frac{t+6}{2}\right\}$$

$-\frac{t}{2}$

$$= \frac{1}{2s^2} - e^{-6s} \cdot \frac{1}{2s^2}.$$

Now, Take $\mathcal{L}.T$:

$$s^2 Y - sy(0) - y'(0) + Y = \frac{1}{2s^2} - \frac{1}{2s^2} e^{-6s}$$

$$(s^2+1) Y = 2 + \frac{1}{2s^2} - \frac{1}{2s^2} e^{-6s}$$

$$Y = \frac{2}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{1}{2s^2(s^2+1)} e^{-6s}$$

$\vdots$

Recall, (2nd shifting)

$$\mathcal{L}\{f(t)\,\mathcal{U}_c(t)\} = e^{-cs}\,\mathcal{L}\{f(t+c)\}$$

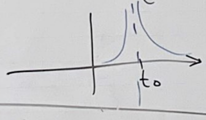
$$\mathcal{L}^{-1}\{e^{-cs}F(s)\}$$

$$= \mathcal{U}_c(t)\,\mathcal{L}^{-1}\{F(s)\}_{t \to t-c}$$

6.5 Impulse Fuctions

Dirac_delta Function

$$\delta(t-t_0) = \begin{cases} \infty, & \text{if } t = t_0 \\ 0, & \text{if } t \neq t_0 \end{cases}$$



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Rmk ① $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$

② $\int_{-\infty}^{\infty} \delta(t-t_0)\,f(t)\,dt = f(t_0)$

ex. $\int_{-\infty}^{\infty} \delta(t-\frac{\pi}{3})\cos t\,dt$

$= \cos\frac{\pi}{3} = \frac{1}{2}$

ex. $\int_{-\infty}^{\infty} t^2\,\delta(t-6)\,dt = 6^2 = 36$

③ $\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}$

④ $\mathcal{L}\{\delta(t-t_0)\,f(t)\}$

$= f(t_0)\,e^{-t_0 s}$

ex. $\mathcal{L}\{\delta(t-\frac{\pi}{3})\cos t\,dt\} = \cos(\frac{\pi}{3})\,e^{-\frac{\pi}{3}s}$

$= \frac{1}{2}e^{-\frac{\pi}{3}s}$

$$\mathcal{L}^{-1}\{ e^{-t_0 s} \} = \delta(t-t_0).$$

ex. $\mathcal{L}^{-1}\{ e^{-6s} \} = \delta(t-6).$

ex. $\mathcal{L}^{-1}\{1\} = \mathcal{L}^{-1}\{ e^{-0\cdot t} \} = \delta(t-0) = \delta(t)$

$\Rightarrow \boxed{\mathcal{L}^{-1}\{1\} = \delta(t)}$

$\mathcal{L}^{-1}\{2020\} = 2020\,\delta(t)$

$\therefore \boxed{\mathcal{L}^{-1}\{k\} = k\,\delta(t)}$

ex. $\begin{cases} y'' + y = 4\delta(t-2\pi). \\ y(0) = 1,\ y'(0) = 0 \end{cases}$

Sol. Take L.T:

$s^2 Y - s y(0) - y'(0) + Y = 4 e^{-2\pi s}.$

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$(s^2+1)Y = s + 4e^{-2\pi s}$

$Y = \frac{s}{s^2+1} + \frac{4}{s^2+1} e^{-2\pi s}$

$y = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} e^{-2\pi s}\right\}$

$= \cos t + 4\,\mathcal{U}_{2\pi}(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}_{t \to t-2\pi}$

$= \cos t + 4\,\mathcal{U}_{2\pi}(t) \sin(t-2\pi)$

$= \cos t + 4\,\mathcal{U}_{2\pi}(t) \sin t$

$= \begin{cases} \cos t, & 0 \le t < 2\pi \\ \cos t + 4\sin t, & t \ge 2\pi \end{cases}$

$$\mathcal{L}^{-1}\{e^{-t_0 s}\} = \delta(t-t_0).$$

ex. $\mathcal{L}^{-1}\{e^{-6s}\} = \delta(t-6).$

ex. $\mathcal{L}^{-1}\{1\} = \mathcal{L}^{-1}\{e^{-0\cdot t}\} = \delta(t-0) = \delta(t)$

$$\Rightarrow \boxed{\mathcal{L}^{-1}\{1\} = \delta(t)}$$

$\mathcal{L}^{-1}\{2020\} = 2020\,\delta(t)$

$$\therefore \boxed{\mathcal{L}^{-1}\{k\} = k\,\delta(t)}$$

ex. $\begin{cases} y'' + y = 4\delta(t-2\pi). \\ y(0)=1,\ y'(0)=0 \end{cases}$

Sol. Take $\mathcal{L}.T$:

$$s^2 Y - s y(0) - y'(0) + Y = 4 e^{-2\pi s}.$$

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$$(s^2+1)Y = s + 4e^{-2\pi s}$$

$$Y = \frac{s}{s^2+1} + \frac{4}{s^2+1} e^{-2\pi s}$$

$$y = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2+1} e^{-2\pi s}\right\}$$

$$= \cos t + 4\,\mathcal{U}_{2\pi}(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}_{t \to t-2\pi}$$

$$= \cos t + 4\,\mathcal{U}_{2\pi}(t)\sin(t-2\pi)$$

$$= \cos t + 4\,\mathcal{U}_{2\pi}(t)\sin t$$

$$= \begin{cases} \cos t, & 0 \le t < 2\pi \\ \cos t + 4\sin t, & t \ge 2\pi \end{cases}$$

Ex. (2)

$$\begin{cases} y'' = t^2\,\delta(t-2) \\ y(0)=0,\ y'(0)=1. \end{cases}$$

Sol. Take $\mathcal{L}.T$:

$$\mathcal{L}\{y''\} = \mathcal{L}\{t^2\,\delta(t-2)\}$$

$$s^2Y - sy(0) - y'(0) = (2)^2 e^{-2s}$$

$$s^2Y = 1 + 4e^{-2s}$$

$$Y = \frac{1}{s^2} + \frac{4}{s^2}e^{-2s}$$

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2}e^{-2s}\right\}$$

$$= t + 4\,\mathcal{U}_2(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}_{t \to t-2}$$

$$= t + 4\,\mathcal{U}_2(t)\,(t-2)$$

$$= \begin{cases} t, & 0 \le t < 2 \\ 5t-8, & t \ge 2 \end{cases}$$

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ex. (3).

$$\begin{cases} 2y'' + y' + 4y = 2\,\delta(t-\frac{\pi}{6})\sin t \\ y(0) = y'(0) = 0 \end{cases}$$

Sol. Take $\mathcal{L}.T$

$$2\mathcal{L}\{y''\} + \mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 2\mathcal{L}\{\delta(t-\tfrac{\pi}{6})\sin t\}$$

$$2[s^2Y - sy(0) - y'(0)] + sY - y(0) + 4Y = 2\sin\tfrac{\pi}{6}\,e^{-\frac{\pi}{6}s}$$

$$(2s^2+s+4)Y = e^{-\frac{\pi}{6}s}$$

$$Y = \frac{e^{-\frac{\pi}{6}s}}{2s^2+s+4} = \frac{1}{2}e^{-\frac{\pi}{6}s}\frac{1}{s^2+\frac{1}{2}s+2}$$

$$y = \frac{1}{2}\,\mathcal{U}_{\frac{\pi}{6}}(t)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+\frac{1}{2}s+2}\right\}_{t \to t-\frac{\pi}{6}}$$

completing square

(4) $\begin{cases} y'' + 3y' + 2y = \delta(t-5) + \mathcal{U}_{10}(t). \\ y(0) = y'(0) = 0 \end{cases}$

(5) $\begin{cases} y'' + y' + y = \delta(t-\pi)\cos t + \mathcal{U}_1(t) \\ y(0) = y'(0) = 0 \end{cases}$

6.6 The Convolution Integrals *

Df. If a function f & g are piece-wise continuous on $[0,\infty)$ then $f*g$ is defined by

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

page (4)

$f*g$ is called the convolution of f & g and it is a function of t.

ex. Find $t * \sin t$

$$t*\sin t = \int_0^t \tau \sin(t-\tau)\, d\tau$$

اشتقاق	تكامل
τ	$\sin(t-\tau)$
1	$\frac{-\cos(t-\tau)}{-1}$
0	$\frac{\sin(t-\tau)}{-1}$

(+), (−)

$$t*\sin t = \tau\cos(t-\tau) + \sin(t-\tau)\Big|_{\tau=0}^{\tau=t}$$

$$= [t\cos(t-t) + \sin(t-t)] - [0 + \sin t]$$

$$= t - \sin t$$

Scanned with CamScanner

ex. $1 * e^t = \int_0^t 1 \cdot e^{t-\tau}\, d\tau$

$= -e^{t-\tau} \Big|_{\tau=0}^{\tau=t}$

$= -e^{t-t} + e^{t-0}$

$= -1 + e^t.$

Rmk. $1 * f \neq f.$

The convolution Thm

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$$

$$= F(s)\, G(s)$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}$$

$$= f * g$$

page (5)

ex. $\mathcal{L}\{t * \sin t\} = \mathcal{L}\{t\} \cdot \mathcal{L}\{\sin t\}$

$= \frac{1}{s^2} \cdot \frac{1}{s^2+1}.$

ex. $\mathcal{L}\{1 * e^t\} = \mathcal{L}\{1\}\mathcal{L}\{e^t\}$

$= \frac{1}{s} \frac{1}{s-1}$

ex. $\mathcal{L}\{\sin t * e^t\} = \mathcal{L}\{\sin t\}\mathcal{L}\{e^t\}$

$= \frac{1}{s^2+1} \cdot \frac{1}{s-1}$

ex. $\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s-1}\right\}$

$F(s)$ $G(s)$

$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$

$= 1 * e^t$

$= \int_0^t 1 \cdot e^{t-\tau}\, d\tau = \cdots = -1 + e^t$

ex. $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$= t * \sin t$

$= \int_0^t \tau \sin(t-\tau)\,d\tau$

$= \cdots = t - \sin t$

ex. Solve the <u>intego-differential eq.</u>

$$\begin{cases} y' - \frac{1}{2}\int_0^t (t-\tau)^2 y(\tau)\,d\tau = -t \\ y(0) = 1 \end{cases}$$

Sol. $y' - \frac{1}{2}\, y * t^2 = -t$

Take $\mathcal{L}$.T

$\mathcal{L}\{y'\} - \frac{1}{2}\mathcal{L}\{y * t^2\} = -\mathcal{L}\{t\}$

page ⑥

$sY - y(0) - \frac{1}{2}\mathcal{L}\{y\}\mathcal{L}\{t^2\} = -\frac{1}{s^2}$

$sY - 1 - \frac{1}{2}Y\cdot\frac{2!}{s^3} = -\frac{1}{s^2}$

$\left(s - \frac{1}{s^3}\right)Y = 1 - \frac{1}{s^2}$

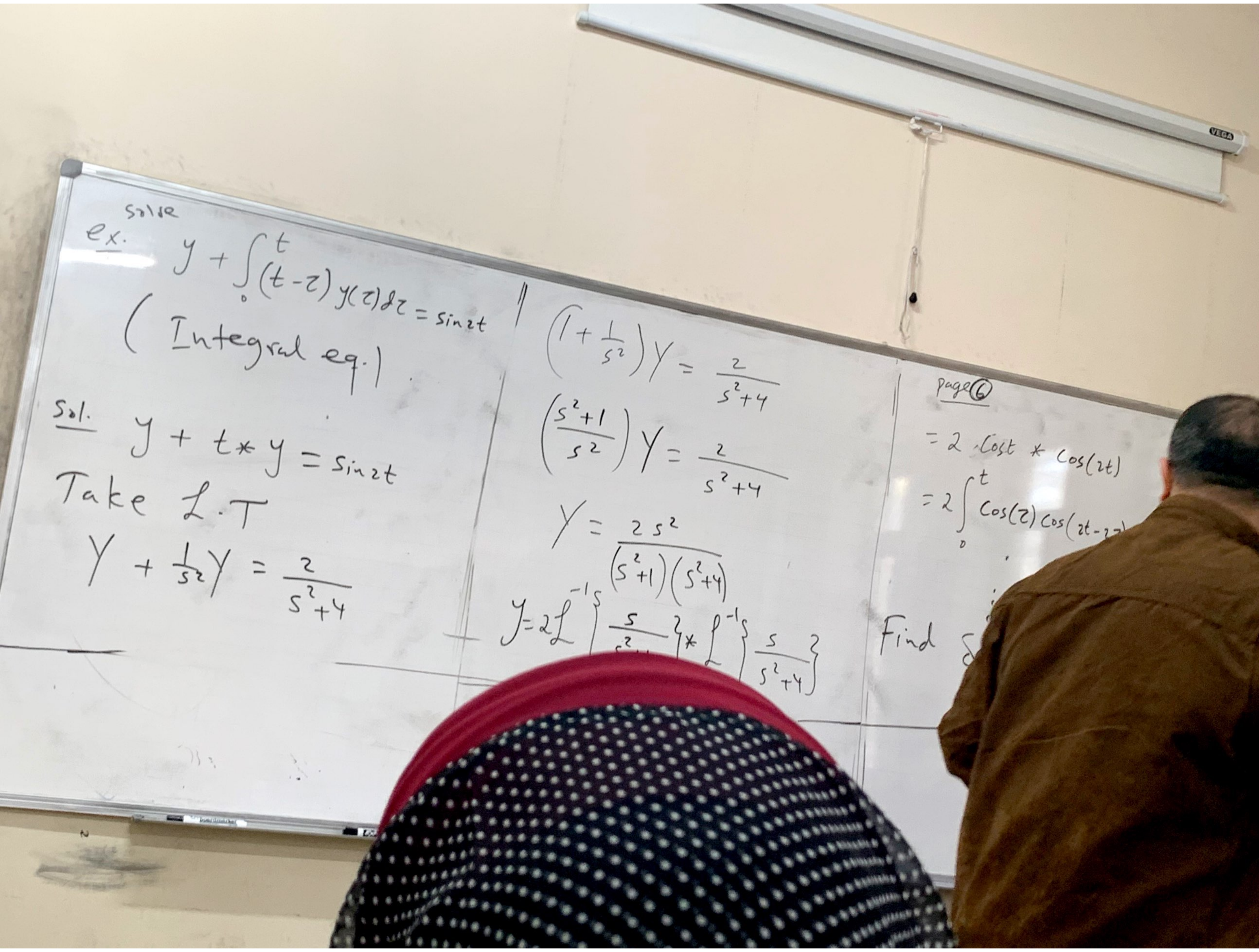
$\frac{s^4-1}{s^3}Y = \frac{s^2-1}{s^2}$

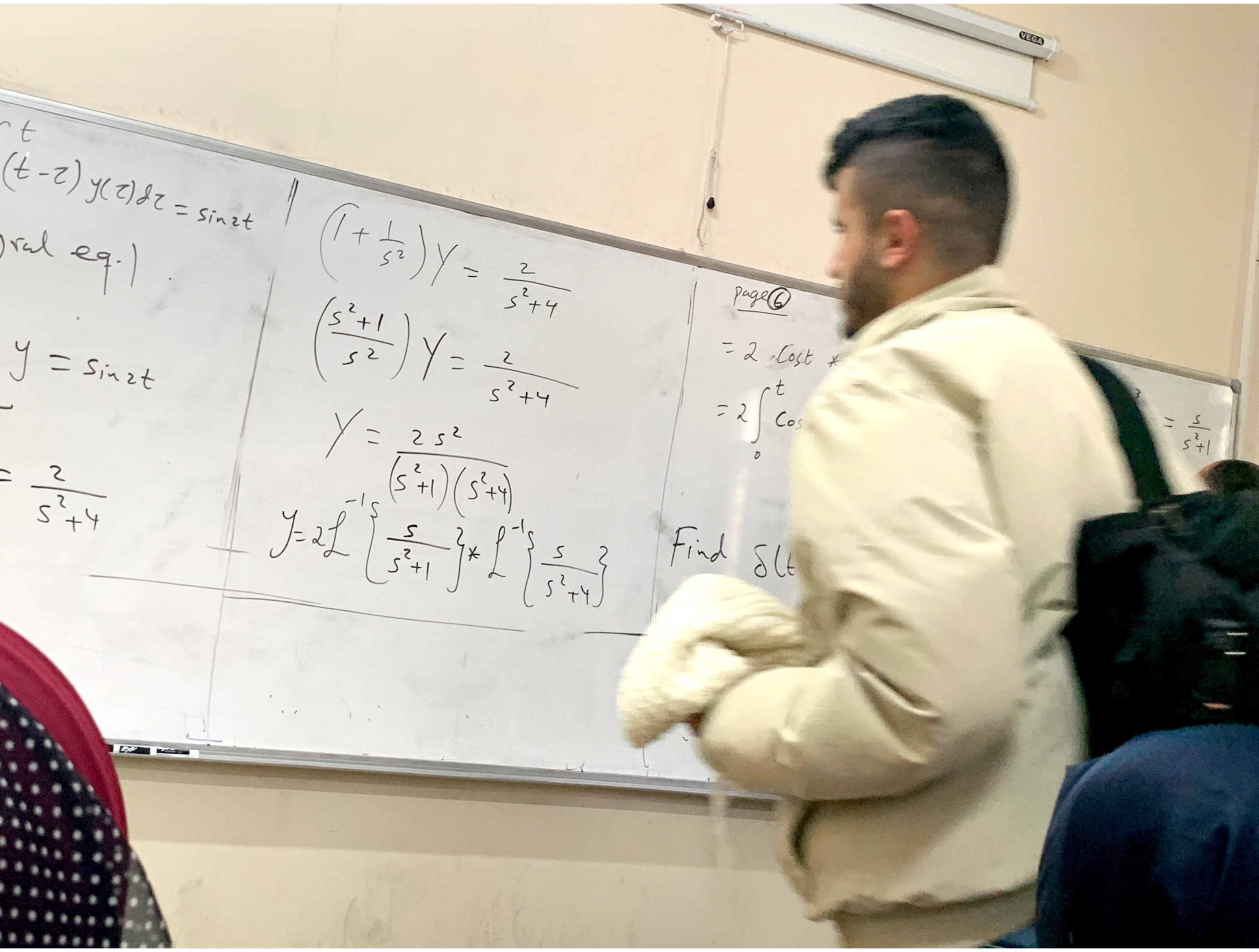
$Y = \frac{s^2-1}{s^2}\cdot\frac{s^3}{s^4-1} = \frac{s}{s^2+1}$

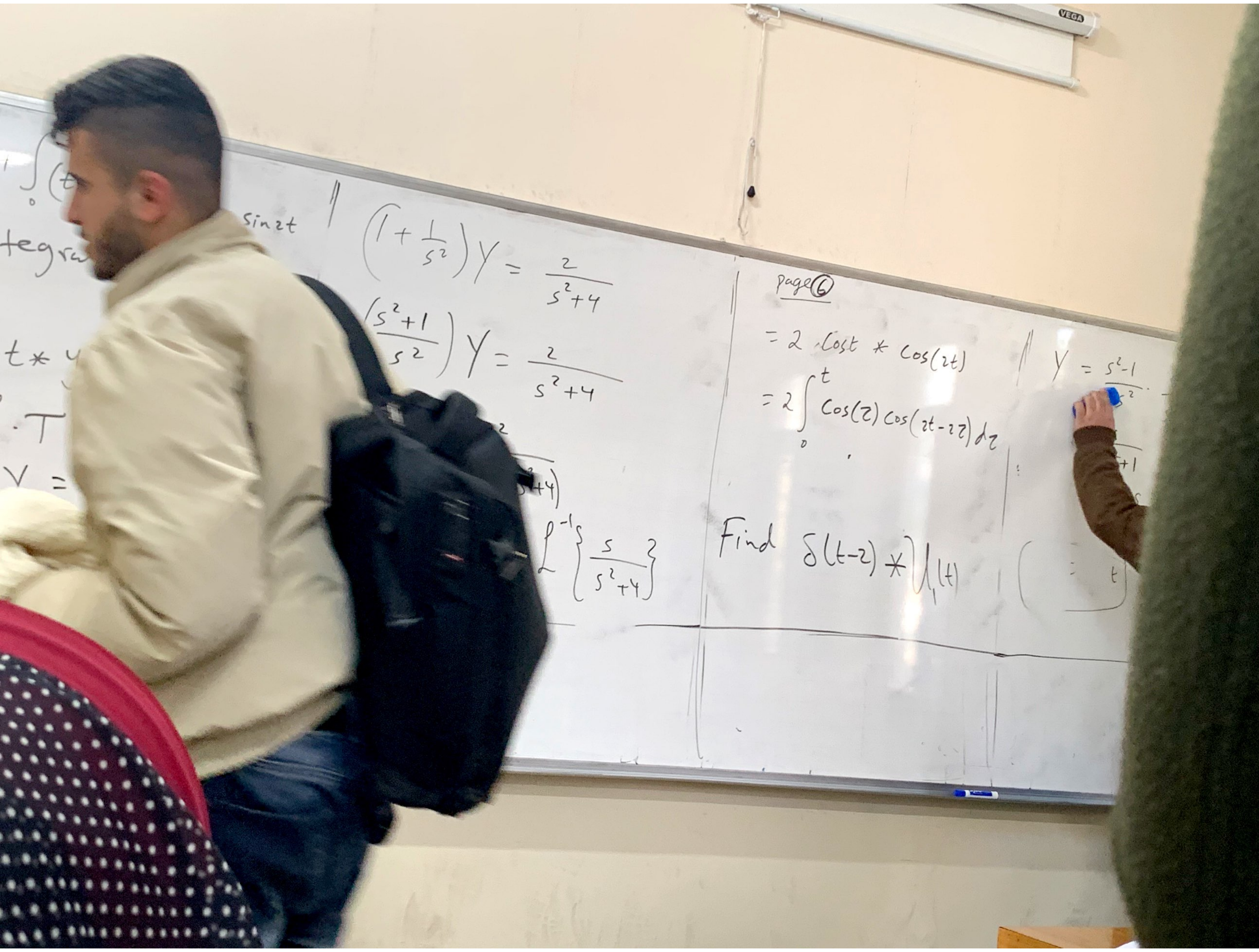
$Y = \frac{s}{s^2+1}$

$y = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$

$\boxed{y = \cos t}$

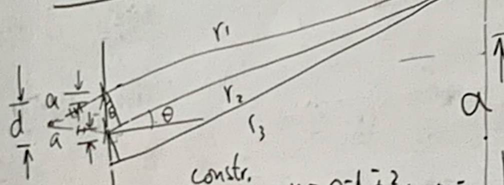

ex. solve
y + ∫₀ᵗ (t - τ) y(τ) dτ = sin2t
(Integral eq.)
Sol. y + t * y = sin2t
Take L.T
Y + 1/s² Y = 2/(s²+4)
(1 + 1/s²) Y = 2/(s²+4)
((s²+1)/s²) Y = 2/(s²+4)
Y = 2s²/((s²+1)(s²+4))
y = 2L⁻¹{s/(s²+1)} * L⁻¹{s/(s²+4)}
page 6
= 2 Cost * cos(2t)
= 2 ∫₀ᵗ cos(τ) cos(2t - 2τ)
Find


y = sin 2t
page 6
= 2 Cost *
Find δ(t


sin2t
page 6
= 2 Cost * Cos(2t)
Find δ(t-2) * U1(t)

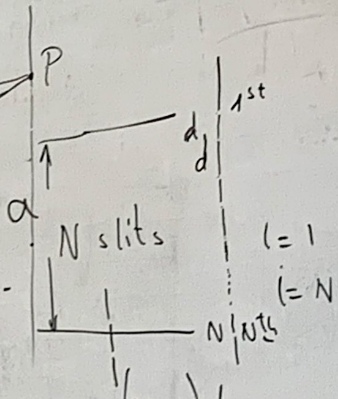
Interferena & Diffraction

Young Experiment



Constr. $r_2 - r_1 = d\sin\theta = n\lambda$ $\quad n = 0, \pm 1, \pm 2, \dots$

$= (n+\frac{1}{2})\lambda$ Destr.



$a = (N-1)d$

Int. from N multiple stils

$$\psi_i = A\cos(kr_i - \omega t)$$

$j = \sqrt{-1}$

$$\psi_i^* = \frac{1}{r_i}\hat{A}e^{j(kr_i - \omega t)}$$

$$I \propto \frac{P}{r^2}$$

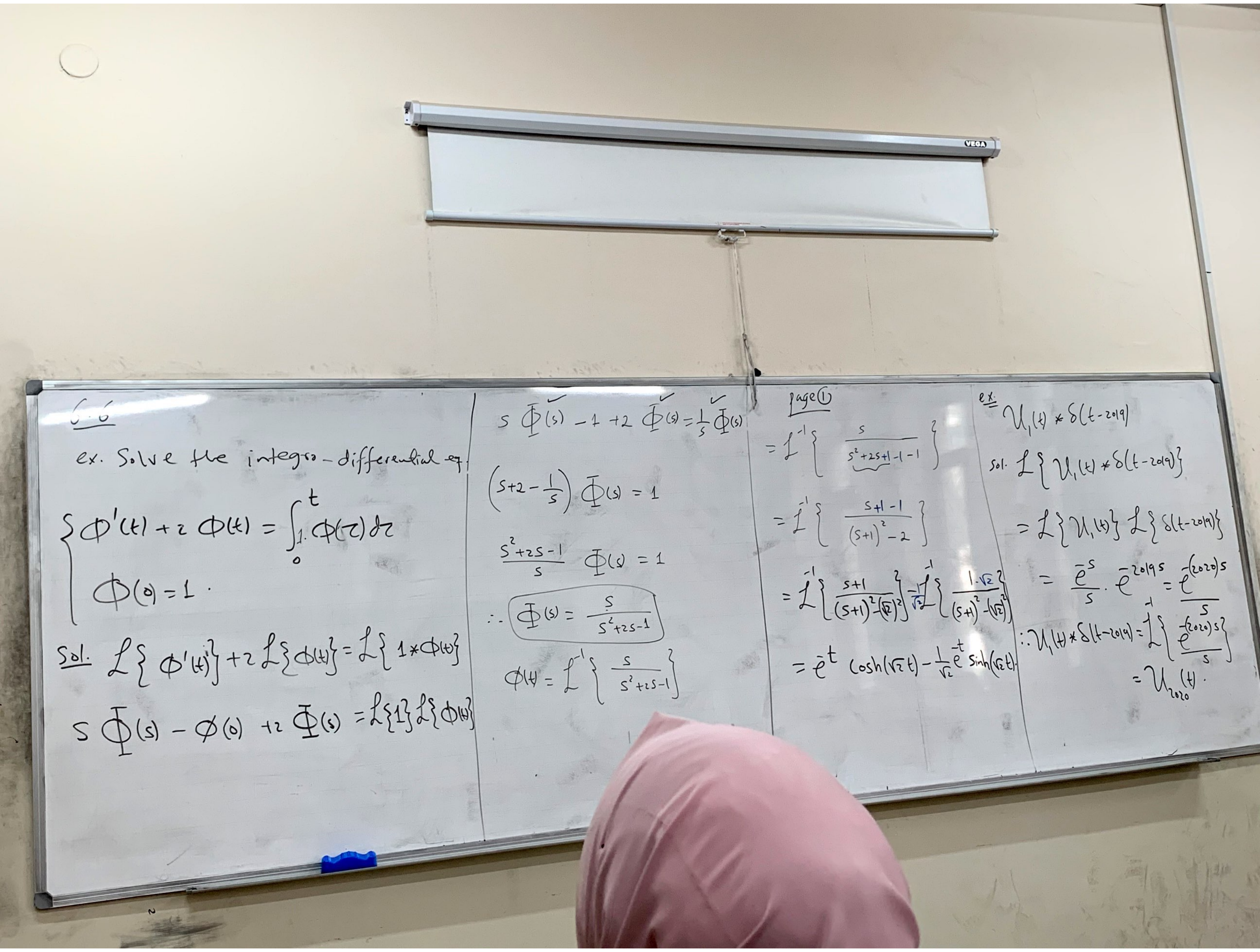
$$A \propto \frac{1}{r}$$

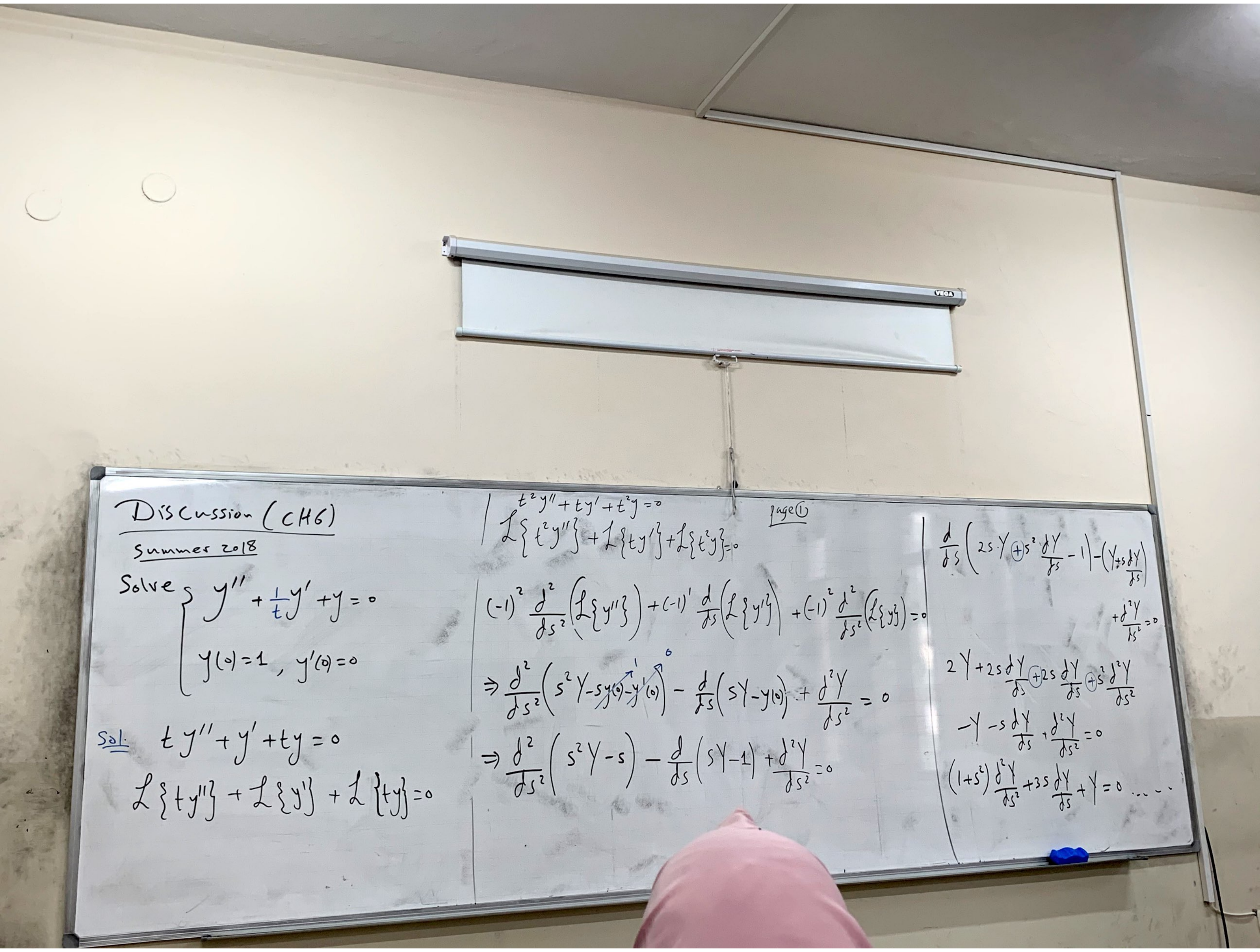
far field approx. $r_i \sim r_{av} = \frac{\sum r_i}{N}$ for all i

$\sum$

$\psi +$

r_i


6.6
ex. Solve the integro-differential eq.
$\begin{cases} \Phi'(t) + 2\Phi(t) = \int_0^t 1 \cdot \Phi(\tau)\, d\tau \\ \Phi(0) = 1. \end{cases}$
Sol. $\mathcal{L}\{\Phi'(t)\} + 2\mathcal{L}\{\Phi(t)\} = \mathcal{L}\{1 * \Phi(t)\}$
$s\Phi(s) - \Phi(0) + 2\Phi(s) = \mathcal{L}\{1\}\mathcal{L}\{\Phi(t)\}$
$s\Phi(s) - 1 + 2\Phi(s) = \frac{1}{s}\Phi(s)$
$\left(s+2-\frac{1}{s}\right)\Phi(s) = 1$
$\frac{s^2+2s-1}{s}\Phi(s) = 1$
$\therefore \Phi(s) = \frac{s}{s^2+2s-1}$
$\Phi(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2s-1}\right\}$
page(1)
$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+1-1-1}\right\}$
$= \mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2-2}\right\}$
$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2-(\sqrt{2})^2}\right\} - \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{1 \cdot \sqrt{2}}{(s+1)^2-(\sqrt{2})^2}\right\}$
$= e^{-t}\cosh(\sqrt{2}t) - \frac{1}{\sqrt{2}}e^{-t}\sinh(\sqrt{2}t)$
ex. $\mathcal{U}_1(t) * \delta(t-2019)$
Sol. $\mathcal{L}\{\mathcal{U}_1(t) * \delta(t-2019)\}$
$= \mathcal{L}\{\mathcal{U}_1(t)\}\mathcal{L}\{\delta(t-2019)\}$
$= \frac{e^{-s}}{s} \cdot e^{-2019s} = \frac{e^{-(2020)s}}{s}$
$\therefore \mathcal{U}_1(t) * \delta(t-2019) = \mathcal{L}^{-1}\left\{\frac{e^{-(2020)s}}{s}\right\}$
$= \mathcal{U}_{2020}(t).$


Discussion (CH6)
Summer 2018
Solve $\begin{cases} y'' + \frac{1}{t}y' + y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{cases}$
Sol. $ty'' + y' + ty = 0$
$\mathcal{L}\{ty''\} + \mathcal{L}\{y'\} + \mathcal{L}\{ty\} = 0$
$t^2y'' + ty' + t^2y = 0$
$\mathcal{L}\{t^2y''\} + \mathcal{L}\{ty'\} + \mathcal{L}\{t^2y\} = 0$
page (1)
$(-1)^2 \frac{d^2}{ds^2}\left(\mathcal{L}\{y''\}\right) + (-1)^1 \frac{d}{ds}\left(\mathcal{L}\{y'\}\right) + (-1)^2 \frac{d^2}{ds^2}\left(\mathcal{L}\{y\}\right) = 0$
$\Rightarrow \frac{d^2}{ds^2}\left(s^2Y - sy(0) - y'(0)\right) - \frac{d}{ds}\left(sY - y(0)\right) + \frac{d^2Y}{ds^2} = 0$
$\Rightarrow \frac{d^2}{ds^2}\left(s^2Y - s\right) - \frac{d}{ds}\left(sY - 1\right) + \frac{d^2Y}{ds^2} = 0$
$\frac{d}{ds}\left(2s \cdot Y + s^2 \frac{dY}{ds} - 1\right) - \left(Y + s\frac{dY}{ds}\right) + \frac{d^2Y}{ds^2} = 0$
$2Y + 2s\frac{dY}{ds} + 2s\frac{dY}{ds} + s^2\frac{d^2Y}{ds^2} - Y - s\frac{dY}{ds} + \frac{d^2Y}{ds^2} = 0$
$(1+s^2)\frac{d^2Y}{ds^2} + 3s\frac{dY}{ds} + Y = 0$ .....
VEGA

$1 * \delta(t-1) = \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot e^{-s}\right\}$

$= \mathcal{U}_1(t)$

ex. $\mathcal{L}\left\{t\,\mathcal{U}_3(t)\,e^{t-3}\right\}$

$= \mathcal{L}\left\{t e^{t-3}\,\mathcal{U}_3(t)\right\}$

$= e^{-3s}\,\mathcal{L}\left\{t e^{t-3}\right\}_{t \to t+3}$

$= e^{-3s}\,\mathcal{L}\left\{(t+3)\,e^{t}\right\}$

$= e^{-3s}\,\mathcal{L}\left\{t e^{t} + 3e^{t}\right\}$

$= e^{-3s}\left[(-1)^1 \frac{d^1}{ds^1}\mathcal{L}\{e^t\} + \frac{3}{s-1}\right]$

$= e^{-3s}\left[-\frac{d}{ds}\left(\frac{1}{s-1}\right) + \frac{3}{s-1}\right]$

$= e^{-3s}\left[\frac{1}{(s-1)^2} + \frac{3}{s-1}\right]$

page (2)