Ch 1

- **Differential Equation (DE)** is an equation (relation) with derivative (changes or rates)

Exp ① $y = 2x + 1$ is not DE but it is an algebraic equation with
x: independent variable
y: dependent variable

② $y' = 2x + 1$ is DE

The solution (or the unknown) is $y(x)$

③ $\frac{d^2y}{dt^2} - e^t = 3$ is DE

The solution (or the unknown) is $y(t)$

④ $\frac{dN}{dr} = N^3 \sin r$ is DE

The solution (or the unknown) is $N(r)$

Remark ① In Exp ①, ②, ③, ④ :

x, t, r are indep. variables
y, N are dep. variables

② If the solution $y(x)$ passes through the point (x_0, y_0) then we write $y(x_0) = y_0$

Def The Initial Value Problem (IVP) is DE with IC where IC is Initial Condition (x_0, y_0) or $y(x_0) = y_0$

$$IVP = DE + IC$$

Exp The following are examples of IVP's:

① $\dfrac{dv}{dt} = 9.8 - 0.2\,v \quad , \quad v(\underbrace{0}_{t_0}) = \underbrace{0}_{v_0}$

The unknown (sol.) is $v(t)$

② $y'' - 2\sin x = \dfrac{7}{y} \quad , \quad y(\underbrace{\pi}_{x_0}) = \underbrace{1}_{y_0} \quad , \quad y'(\pi) = 2$

The unknown (sol.) is $y(x)$

③ $\dfrac{d^3N}{dr^3} = e^{2r} \quad , \quad N(\underbrace{e}_{r_0}) = \underbrace{2}_{N_0} \quad , \quad N'(e) = -1 \quad , \quad N''(e) = 3$

The unknown (sol.) is $N(r)$

Remark The solution of any IVP must satisfy its DE and IC.

Exp show that $y(t) = 0$ is sol. for the IVP:

$$y' = y^{\frac{1}{3}} \quad , \quad y(\underbrace{0}_{t_0}) = \underbrace{0}_{y_0}$$

- $y(t_0) = y(0) = 0 = y_0 \Rightarrow y(t) = 0$ satisfy the IC
- $y(t) = 0 \Rightarrow \left.\begin{array}{l} y' = 0 \\ \Rightarrow y^{\frac{1}{3}} = 0 \end{array}\right\} \Rightarrow y' = 0 = y^{\frac{1}{3}} \Rightarrow y(t)$ satisfy the DE

Exp show that $y(x) = \frac{1}{2} e^{x^2} - \frac{1}{2}$ solves the IVP:

$$y' - 2xy - x = 0 \quad , \quad y(0) = 0$$

The sol. $y(x) = \frac{1}{2} e^{x^2} - \frac{1}{2}$ must satisfy the DE and its IC.

- IC: $y(0) = \frac{1}{2} e^{0^2} - \frac{1}{2}$

$$= \frac{1}{2}(1) - \frac{1}{2}$$
$$= 0 \quad \checkmark$$

- DE: $y' = \frac{1}{2}(2x) e^{x^2} - 0 = x e^{x^2}$

$$y' - 2xy - x = (x e^{x^2}) - 2x\left(\frac{1}{2} e^{x^2} - \frac{1}{2}\right) - x$$
$$= x e^{x^2} - x e^{x^2} + x - x$$
$$= 0 \quad \checkmark$$

Note: ① If $y = f(x)$ has all derivatives, then

Ordinary Derivatives:

$y' = \frac{dy}{dx} = f'(x)$ is y prime

$y'' = \frac{d^2y}{dx^2} = f''(x)$ is y double prime

$y''' = \frac{d^3y}{dx^3} = f'''(x)$ is y tribble prime

$\vdots$

$y^{(n)} = \frac{d^ny}{dx^n} = f^{(n)}(x)$ is y super n or the n^{th} derivative of y

② If $y = f(x,s)$ has all derivatives, then

Partial Derivatives:

$y_x = \frac{\partial y}{\partial x} = f_x$ | $y_{xx} = \frac{\partial^2 y}{\partial x^2} = f_{xx}$

$y_s = \frac{\partial y}{\partial s} = f_s$ | $y_{ss} = \frac{\partial^2 y}{\partial s^2} = f_{ss}$

$y_{xs} = \frac{\partial^2 y}{\partial x \partial s} = f_{xs} = f_{sx} = \frac{\partial^2 y}{\partial s \partial x}$

Exp Given the DE : $y'' + y = 0$

① Show that $y_1 = \sin t$ is sol.

We need to show $y_1'' + y_1 \overset{?}{=} 0$

$y_1' = \cos t$

$y_1'' = -\sin t$

$(-\sin t) + (\sin t) = 0$ ✓

② Verify that $y_2 = \cos t$ is sol.

We need to show that $y_2'' + y_2 \overset{?}{=} 0$

$y_2' = -\sin t$

$y_2'' = -\cos t$

$(-\cos t) + (\cos t) = 0$ ✓

③ Show that $y(t) = c_1 \sin t + c_2 \cos t$ is sol.
where $c_1, c_2 \in \mathbb{R}$

We need to show that $y'' + y \overset{?}{=} 0$

$$\left.\begin{aligned} y' &= c_1 \cos t - c_2 \sin t \\ y'' &= -c_1 \sin t - c_2 \cos t \end{aligned}\right\} \Rightarrow \begin{aligned} &(-c_1 \sin t - c_2 \cos t) + \\ &(c_1 \sin t + c_2 \cos t) = 0 \end{aligned}$$

✓

Exp ① $f(x) = 2x^2 - 4x + 1 \Rightarrow f'(x) = 4x - 4$

② $f(x,y) = 2x^2y^3 - 3e^x y + 5$ then

$f_x = \frac{\partial f}{\partial x} = 4xy^3 - 3ye^x$

$f_y = \frac{\partial f}{\partial y} = 6x^2y^2 - 3e^x$

$f_{xy} = 12xy^2 - 3e^x$
$= f_{yx}$

Def The order of a given DE is the highest derivative appears in the equation

Exp ① The DE $y' = y(y^2 - 3)$ has order 1 (1st order)

② The DE $\frac{d^2y}{dt^2} - e^t = 3$ is of order 2

③ The DE $N''' = N^3 \sin r$ is 3^{rd} order

Question: How to classify the DE's ?

DE

ODE (Ordinary DE)

- The unknown function depends only on a single indep. variable
- Only ordinary derivatives appear in the equation
- Exp ① $\frac{dP}{dt} = 0.5P - 450$

  unknown $P(t)$

  ② $v'' - 2v = t$

  unknown $v(t)$
- Our Course

PDE (Partial DE)

- The unknown function depends on more than one indep. variable
- Partial derivatives appear in the equation

  $\frac{\partial^2 P}{\partial t^2} = \alpha \frac{\partial P}{\partial u}$

  unknown $P(t,u)$

  $V_t - 2V_r = tr$

  unknown $V(t,r)$
- New Course

System of DE's

$\frac{dx}{dt} = 2x - y$

$\frac{dy}{dt} = 3y - x$

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* The general form of ODE of order n is

$$F(t, y, y', y'', \ldots, y^{(n)}) = 0$$

Exp ① $y' - 2ty + 3 = 0 \Rightarrow F(t, y, y') = y' - 2ty + 3$

② $\frac{dN}{dx} = e^{x} - N^{2} \Rightarrow F(x, N, N') = N' - e^{x} + N^{2}$

unknown is $N(x)$ $\Rightarrow N' - e^{x} + N^{2} = 0$

③ $5y'' - 3y' = y \Rightarrow F(t, y, y', y'') = 5y'' - 3y' - y$

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* The ODE is linear if F is linear in $y, y', y'', \ldots, y^{(n)}$. Otherwise, the ODE is nonlinear.

Exp Classify the following DE's:

① $y' - 2y + 5 = 0$ 1^{st} order linear ODE

unknown is $y = y(x)$

② $2y'' - 5y' + 3t = 0$ 2^{nd} order linear ODE

unknown is $y = y(t)$

③ $\frac{d^6R}{dx^6} + \frac{d^3R}{dx^3} - 5 = e^{\sqrt{2} - x}$ 6^{th} order linear ODE

unknown is $R(x)$

④ $u_{xx} - u_{yy} - \cos(xy) = 0$ 2^{nd} order linear PDE

unknown is $u(x, y)$

⑤ $\frac{d^3N}{dt^3} - e^{N} \frac{dN}{dt} = 5t$ 3^{rd} order nonlinear ODE

unknown is $N(t)$