* The general form of ODE of order n is

$$F(t, y, y', y'', \dots, y^{(n)}) = 0$$

Exp ① $y' - 2ty + 3 = 0 \quad \Rightarrow F(t, y, y') = y' - 2ty + 3$

② $\dfrac{dN}{dx} = e^{x} - N^{2} \quad \Rightarrow F(x, N, N') = N' - e^{x} + N^{2}$

unknown is $N(x) \Rightarrow N' - e^{x} + N^{2} = 0$

③ $5y'' - 3y' = y \quad \Rightarrow F(t, y, y', y'') = 5y'' - 3y' - y$

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* The ODE is linear if F is linear in $y, y', y'', \dots, y^{(n)}$. Otherwise, the ODE is nonlinear.

Exp Classify the following DE's:

① $y' - 2y + 5 = 0$ — 1^{st} order linear ODE

unknown is $y = y(x)$

② $2y'' - 5y' + 3t = 0$ — 2^{nd} order linear ODE

unknown is $y = y(t)$

③ $\dfrac{d^6R}{dx^6} + \dfrac{d^3R}{dx^3} - 5 = e^{\sqrt{2} - x}$ — 6^{th} order linear ODE

unknown is $R(x)$

④ $u_{xx} - u_{yy} - \cos(xy) = 0$ — 2^{nd} order linear PDE

unknown is $u(x, y)$

⑤ $\dfrac{d^3N}{dt^3} - e^{N}\dfrac{dN}{dt} = 5t$ — 3^{rd} order nonlinear ODE

unknown is $N(t)$

⑥ $xy' - 2y = \sin x$ $\quad$ 1^{st} order linear ODE

unknown is $y(x)$

⑦ $\frac{1}{t}\frac{dy}{dt} + (\cos t)y = t^2$ $\quad$ 1^{st} order linear ODE

unknown is $y(t)$

⑧ $(\sin t)\frac{d^2y}{dt^2} = t^3$ $\quad$ 2^{nd} order linear ODE

unknown is $y(t)$

⑨ $\left(\frac{dN}{dx}\right)^2 + N = x$ $\quad$ 1^{st} order nonlinear ODE

unknown is $N(x)$

⑩ $ty' + \frac{1}{ty} = 10$ $\quad$ 1^{st} order nonlinear ODE

unknown $y(t)$

⑪ $(x + e^y)\,dy - dx = 0$

$\Rightarrow (x + e^y)\frac{dy}{dx} - 1 = 0$

$$\frac{dy}{dx} = \frac{1}{x + e^y}$$

1^{st} order nonlinear ODE

unknown is $y(x)$

or $\Rightarrow (x + e^y) - \frac{dx}{dy} = 0$

$$\frac{dx}{dy} = x + e^y$$

1^{st} order linear ODE

unknown is $x(y)$

⑫ $\frac{\partial y}{\partial x} - y\frac{\partial^2 y}{\partial x \partial s} = \sin(xs)$ $\quad$ 2^{nd} order nonlinear PDE

unknown is $y(x,s)$

Direction Field (DF)

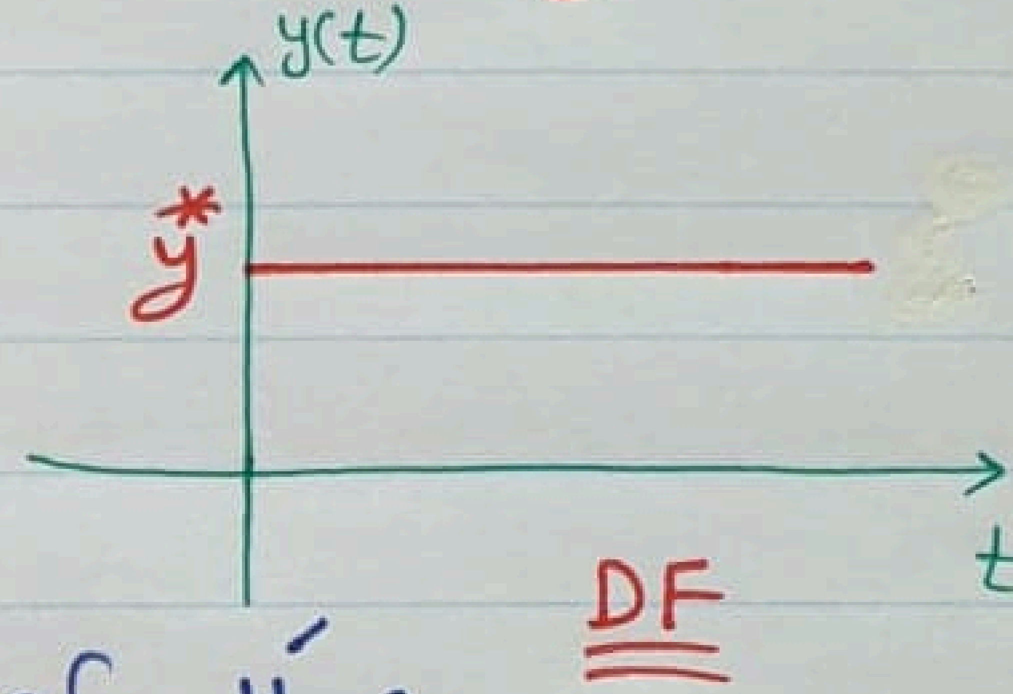
- We use DF to study the behaviour of the solution for a given DE without solving it.

- To draw the DF of a given DE:

$$y' = f(y) \quad \cdots\cdots \ (1)$$

→ First we find the Equilibrium Solution (Eq. Sol.) by setting $y' = 0$ and solve for y^*

→ Draw the Eq. Sol. y^*



→ Substitute values of y_0 above and below y^* in (1) to see the sign of y':

→ if $y' > 0 \Rightarrow y(t) \uparrow$

→ if $y' < 0 \Rightarrow y(t) \downarrow$

→ if $y' = 0 \Rightarrow y(t) = y^*$

Exp 1 Given the DE: $\quad y' - 2y = -4$

① Find Eq. Sol.

② Draw the DF

③ Find $\lim_{t \to \infty} y(t)$ if $y_0 = 3$

① Write the DE in the form (1) $\Rightarrow \boxed{y' = 2y - 4}$ (1)

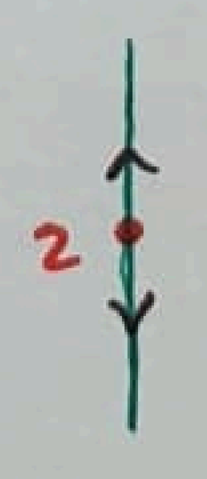
$y' = 0 \quad \Rightarrow \quad 2y - 4 = 0$

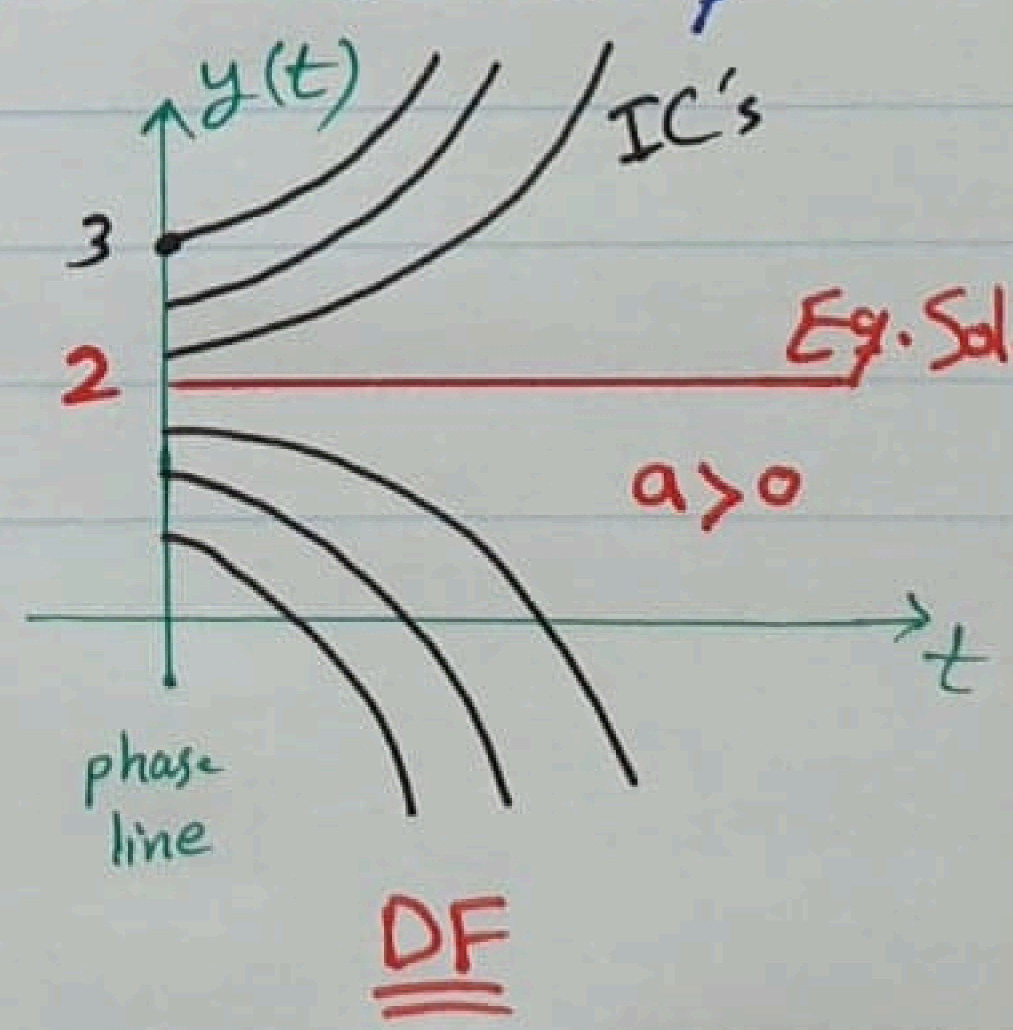
$\Rightarrow \quad 2y = 4 \quad \Rightarrow y^* = 2$ is the Eq. Sol.

② substitute $y_0 = 3 \Rightarrow y' = 2 > 0 \Rightarrow y(t) \uparrow$

$y_0 = 1 \Rightarrow y' = -2 < 0 \Rightarrow y(t) \downarrow$

③ $\lim_{t \to \infty} y(t) = \infty$

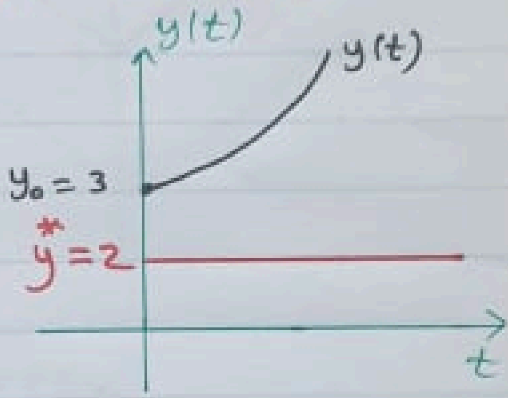




- The DF in Exp' shows the Integral Curves (IC's)
- These curves are all possible solutions for the DE $\Rightarrow$ They depend on the choice of y_0
- In part 3 when $y_0 = 3$ the DF becomes

→ clearly $\lim_{t \to \infty} y(t) = \infty$

→ Here the DF contains only one Integral Curve which is the solution $y(t)$



- Back to Exp' $\Rightarrow$ we can see that the behavior of solution on y_0 as follow:

$$\lim_{t \to \infty} y(t) = \begin{cases} \infty & \text{if } y_0 > 2 \\ 2 & \text{if } y_0 = 2 \\ -\infty & \text{if } y_0 < 2 \end{cases}$$

- If we arrange the DE in Exp' in the form

$$y' = ay - b$$

$$y' = 2y - 4$$

then we can see that $a = 2$, $b = 4$
+

- Next example will be when $a < 0$ to see how the solution behave for different values of the initial condition y_0

$\underline{Exp^2}$ Consider the DE : $y' + 3y = 12$

① Draw the DF

② Find $\lim_{t\to\infty} y(t)$

① First we find the Eq. Sol.

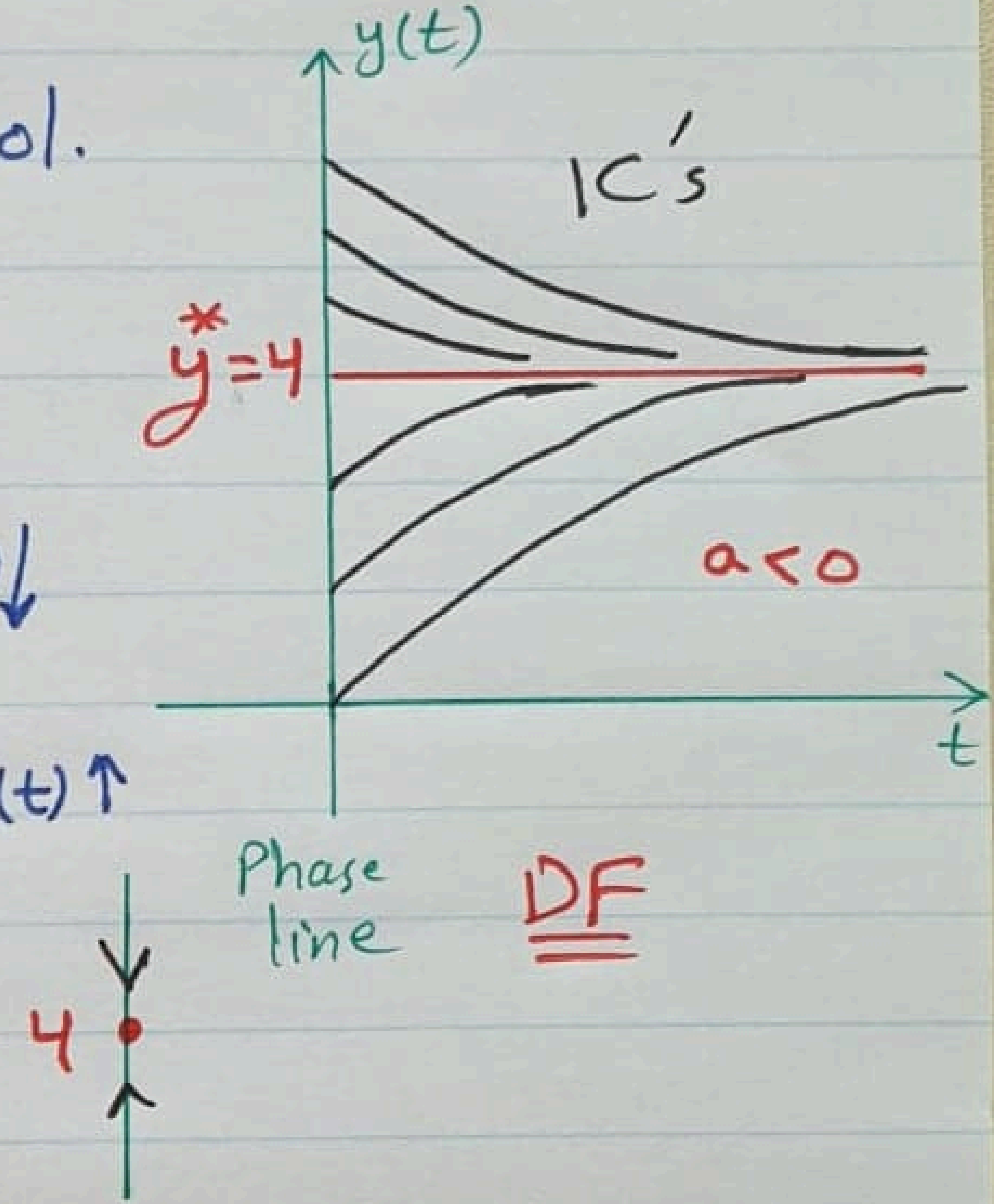
$\Rightarrow y' = 0 \Rightarrow 3y = 12$

$\Rightarrow y^* = 4$

substitue $y_0 = 5 \Rightarrow y' = -3 \Rightarrow y(t)\downarrow$

$y_0 = 0 \Rightarrow y' = 12 > 0 \Rightarrow y(t)\uparrow$

② $\lim_{t\to\infty} y(t) = 4$



• Note that in Exp^2 can be arranged as

$y' = -3y + 12$ and comparing with

$y' = ay - b$ we see $a = -3$, $b = -12$

• In this Exp^2 the behavior of solution :

$$\lim_{t\to\infty} y(t) = \begin{cases} 4 & \text{if } y_0 > 4 \\ 4 & \text{if } y_0 = 4 \\ 4 & \text{if } y_0 < 4 \end{cases}$$

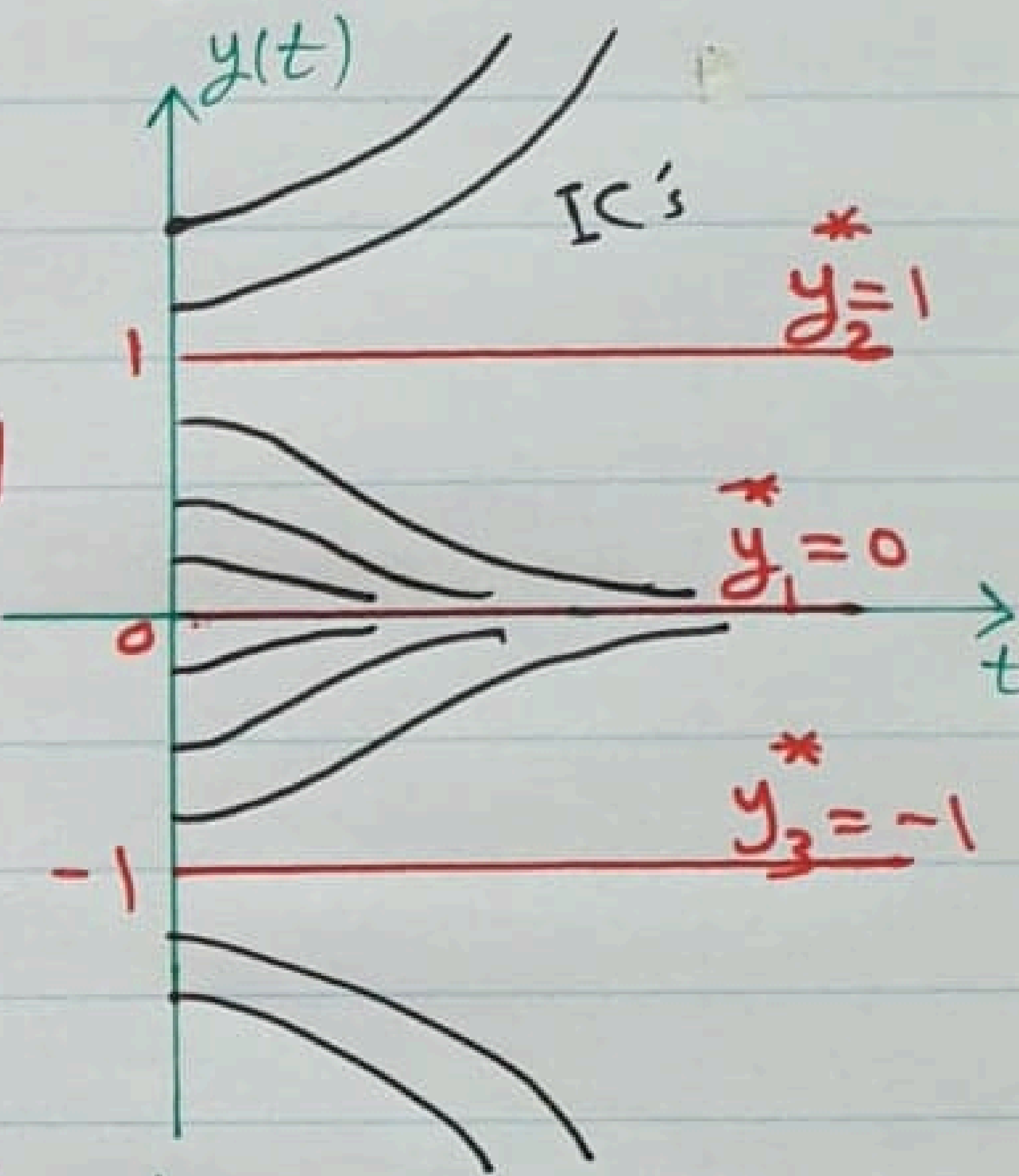
$= 4 \quad \forall\ y_0$

This is because $a < 0$.

- Exp^1 and Exp^2 are examples of linear DE
- However, we can draw DF for some nonlinear DE's

Exp^3 Consider the DE : $y' = y(y^2 - 1)$

① Find Eq. Sol.
② Draw DF
③ Find $\lim_{t \to \infty} y(t)$ if $y(0) = 0.8$



① $y' = 0 \Rightarrow y(y-1)(y+1) = 0$

$y_1^* = 0 , \; y_2^* = 1 , \; y_3^* = -1$

② if $y_0 = 2 \Rightarrow y' = 6 > 0 \Rightarrow y(t) \uparrow$

if $y_0 = \frac{1}{2} \Rightarrow y' < 0 \Rightarrow y(t) \downarrow$

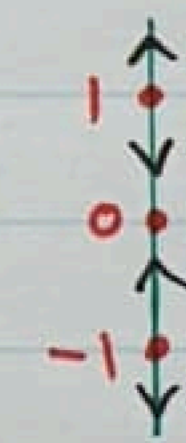
if $y_0 = -\frac{1}{2} \Rightarrow y' > 0 \Rightarrow y(t) \uparrow$

if $y_0 = -2 \Rightarrow y' < 0 \Rightarrow y(t) \downarrow$

Phase line

DF

③ if $y_0 = 0.8 \Rightarrow \lim_{t \to \infty} y(t) = 0$



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* In $Exp^3 \Rightarrow \forall \; y_0 \in (-1, 1) \Rightarrow$ the solution converges to zero
* The behaviour of solution:

$$\lim_{t \to \infty} y(t) = \begin{cases} \infty & \text{if } y_0 > 1 \\ 0 & \text{if } y_0 \in (-1, 1) \\ -\infty & \text{if } y_0 < -1 \\ 1 & \text{if } y_0 = 1 \\ 0 & \text{if } y_0 = 0 \\ -1 & \text{if } y_0 = -1 \end{cases}$$