- Solution of some DE's

Exp How to solve 1^{st} order, linear IVP with constant coefficients and has the form:

$$\boxed{y' = ay - b \ , \quad a \neq 0, \quad y(0) = y_0} \quad (A)$$

- We use the method of calulus :

$$\frac{dy}{dt} = ay - b$$

separable

$$\frac{dy}{dt} = a\left(y - \frac{b}{a}\right) \Rightarrow \int \frac{dy}{y - \frac{b}{a}} = \int a \, dt$$

$$\ln\left|y - \frac{b}{a}\right| = at + c$$

$$\left|y - \frac{b}{a}\right| = e^{at+c}$$

Note that $y^* = \frac{b}{a}$ is the Eq. Sol.

$$y - \frac{b}{a} = \pm e^{c} e^{at}$$

$$y(t) = \frac{b}{a} + D e^{at} \ , \quad D = \pm e^{c}$$

- To find the constant D, we use the IC $y(0) = y_0$

$$y(0) = \frac{b}{a} + D \iff y_0 = \frac{b}{a} + D \iff D = y_0 - \frac{b}{a}$$

- Hence, the solution of the IVP (A) is

$$\boxed{y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}} \longrightarrow A^*$$

- A^* is also called the general sol. because it contains all possible solutions, based on values of y_0.

Exp Solve the IVP:

① $\frac{dv}{dt} = 9.8 - 0.2V \quad , \quad V(\underset{\uparrow}{\overset{t_0}{0}}) = \underset{\uparrow}{\overset{V_0}{0}}$ "Free Fall"

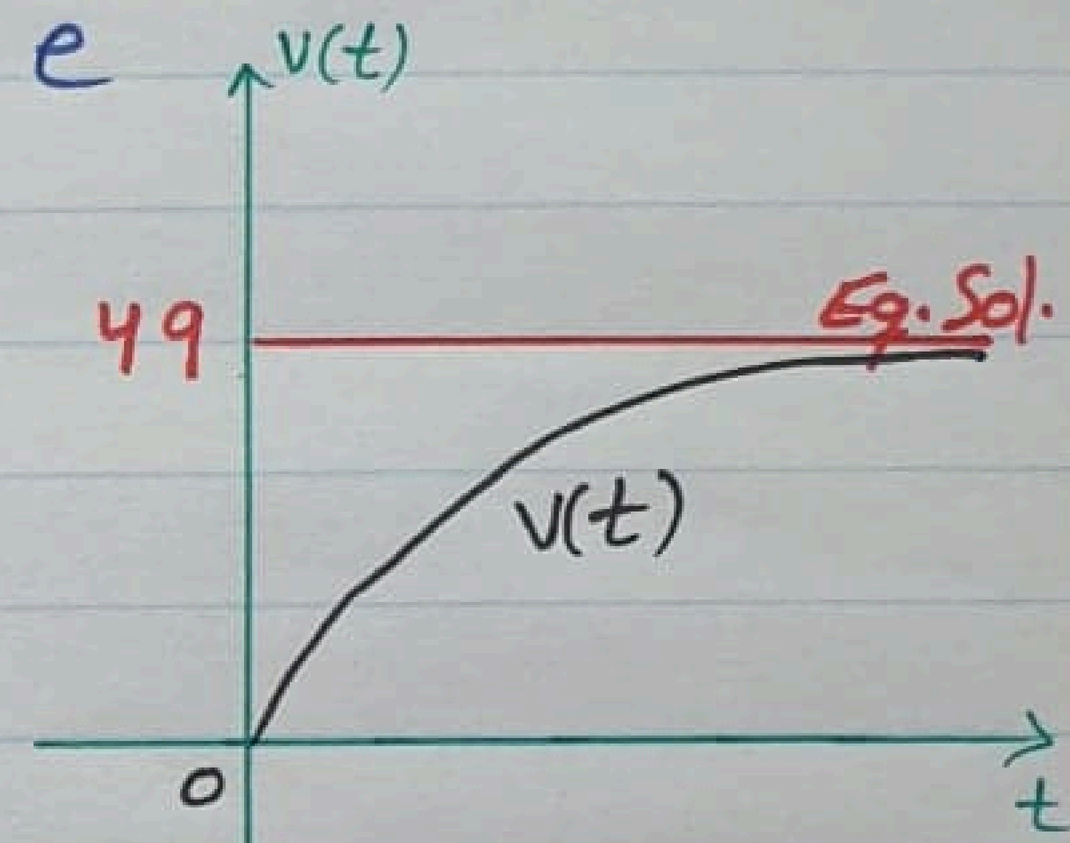
→ Compare with Ⓐ: $y' = ay - b \Rightarrow a = -0.2$, $b = -9.8$

→ Apply A^* since the DE is 1^{st} order linear with constant coefficients

$$v(t) = \frac{b}{a} + \left(V_0 - \frac{b}{a}\right)e^{at}$$

$$= \frac{-9.8}{-0.2} + \left(0 - \frac{-9.8}{-0.2}\right)e^{-0.2t}$$

$$\boxed{V(t) = 49 - 49\,e^{-0.2t}}$$



clearly $\lim_{t\to\infty} v(t) = 49$

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② $\frac{dP}{dt} = 0.5P - 450 \quad , \quad P(0) = 850$ "Mice and Owls"

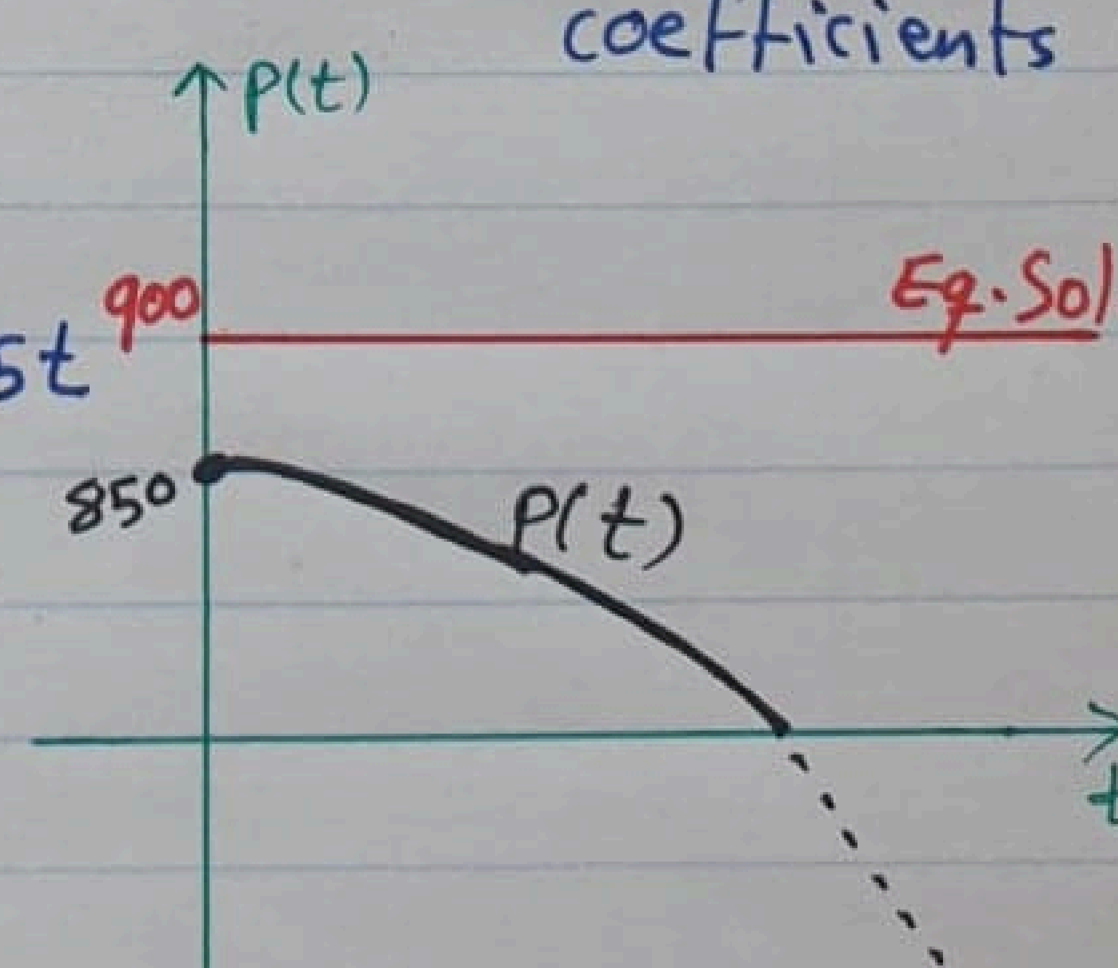
→ compare with Ⓐ: $y' = ay - b \Rightarrow a = 0.5 , b = 450$

→ Apply A^* since the DE is 1^{st} order linear with constant coefficients

$$P(t) = \frac{b}{a} + \left(P_0 - \frac{b}{a}\right)e^{at}$$

$$= \frac{450}{0.5} + \left(850 - \frac{450}{0.5}\right)e^{0.5t}$$

$$\boxed{P(t) = 900 - 50\,e^{0.5t}}$$



$\lim_{t\to\infty} P(t) = 0$ since we talk about population

Exp Given the IVP: $y' - 2y = -4$, $y(0) = 3$. Find $y(\ln 2)$

- $y' = 2y - 4$ This DE satisfy (A) with $a = 2, b = 4$

$\frac{b}{a} = 2$

- Apply A^* $\Rightarrow y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}$

$= 2 + (3 - 2)e^{2t}$

$\boxed{y(t) = 2 + e^{2t}}$ is the sol. of this IVP.

Now $y(\ln 2) = 2 + e^{2\ln 2} = 2 + e^{\ln 4} = 2 + 4 = 6$

Exp Assume a Bacteria population $p(t)$ increases proportionally to its current size. If the population doubles in two days, when it will tribble?

- $\frac{dP}{dt} \propto P \quad \Rightarrow \quad \frac{dP}{dt} = aP$ a: growth rate

- This DE satisfy (A) with $b = 0 \Rightarrow \frac{b}{a} = 0$
- Apply A^* to find the sol. $\Rightarrow$

$P(t) = \frac{b}{a} + \left(P_0 - \frac{b}{a}\right)e^{at} \quad \Rightarrow \quad \boxed{P(t) = P_0 e^{at}}$

- Since the population doubles in two days $\Rightarrow P(2) = 2P_0$

$P(2) = \cancel{P_0} e^{2a} = 2\cancel{P_0} \quad \Leftrightarrow e^{2a} = 2 \quad \Leftrightarrow 2a = \ln 2$

$\Leftrightarrow \boxed{a = \frac{\ln 2}{2}}$

- We need to find the time t^* s.t:

$P(t^*) = 3P_0$

$\cancel{P_0} e^{at^*} = 3\cancel{P_0}$

$e^{at^*} = 3 \quad \Rightarrow at^* = \ln 3 \Rightarrow t^* = \frac{\ln 3}{a} = \frac{\ln 3}{\frac{\ln 2}{2}} = \frac{\ln 9}{\ln 2}$