ch2

$\boxed{2.1}$ 1^{st} order linear DE with variable coefficients (Method of Integrating factors)

- Recall that in ch1 we have solved any 1^{st} order linear DE with constant coefficients of the form

$$y' = ay - b \quad , \quad a \neq 0 \, , \quad y(0) = y_0 \quad \cdots \text{(A)}$$

whose sol. is $y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$ --- A^*

Exp How to solve 1^{st} order linear DE with variable coefficients of the form:

$$y' + p(t)\, y = g(t) \quad \cdots \text{(B)}$$

- Note that the DE (B) is more general than (A).
- This means that (A) is special case of (B).
- If $p(t)$ and $g(t)$ are constants, then (B) becomes (A).
- Hence, the sol. of (B) will solve (A).

- Here the method of calculus does not work, so we look for new method called Integrating factor.

- The idea of this method is to multiply the DE (B) by a positive function $\mu(t)$ so that the resulting equation is easy to integrate:

$$\mu(t) \frac{dy}{dt} + y \boxed{\mu(t)\, p(t)} = \mu(t)\, g(t)$$

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$$\mu'(t) = \mu(t)\, p(t)$$

$$\Downarrow$$

$$\left(\mu(t)\, y(t)\right)' = \mu(t)\, g(t)$$

$$\Downarrow$$

$$\mu(t)\, y(t) = \int \mu(t)\, g(t)\, dt + c$$

Hence, the general sol. of the DE Ⓑ is

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\, dt + c\right]} \rightarrow \overset{*}{B}$$

$$\frac{\mu'(t)}{\mu(t)} = p(t)$$

$$\ln|\mu(t)| = \int p(t)\,dt$$

$$\ln(\mu(t)) = \int p(t)\,dt$$

$$\boxed{\mu(t) = e^{\int p(t)\,dt}}$$

integrating factor

Exp Solve the IVP : $y' + 2y - 4 = 0$, $y(\underset{t_0}{0}) = \underset{y_0}{1}$

Sol. 1 : • This DE has the form of Ⓐ $\Rightarrow$

$$y' = -2y + 4 \quad \text{with} \quad \left.\begin{matrix} a = -2 \\ b = -4 \end{matrix}\right\} \Rightarrow \frac{b}{a} = 2$$

• Apply $\overset{*}{A}$ $\Rightarrow$

$$y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}$$

$$y(t) = 2 + (1-2)\, e^{-2t}$$

✓ $y(t) = 2 - e^{-2t}$ Note $\lim_{t\to\infty} y(t) = 2 = $ Eq. Sol.

Sol. 2 • We can write this DE in the form of Ⓑ $\Rightarrow$

$$y' + 2y = 4 \quad \text{with} \quad p(t) = 2 \quad \text{and} \quad g(t) = 4$$

• $\mu(t) = e^{\int p(t)dt} = e^{\int 2\, dt} = e^{2t}$ is the integrating factor

• The gen. sol. $\overset{*}{B}$ is $y(t) = \frac{1}{\mu}\left[\int \mu g\, dt + c\right]$

$$y(t) = \frac{1}{e^{2t}}\left[\int 4 e^{2t} dt + c\right] = e^{-2t}\left[2e^{2t} + c\right]$$

$y(t) = 2 + c\, e^{-2t}$ To find c we use the IC $\Rightarrow$

$y(0) = 2 + c\, e^{0}$

$1 = 2 + c$

$\boxed{c = -1}$

The sol. becomes

$y(t) = 2 - e^{-2t}$ ✓

Exp Solve the IVP:

$$ty' - 2y = 5t^2, \quad t>0, \quad y(1)=2$$

- Since the DE is 1^{st} order linear with variable coefficients
  $\Rightarrow$ we can only use B^* to solve it

- But first we arrange the DE of the form B to write $p(t)$ and $g(t)$ correctly:

$$y' - \frac{2}{t}y = 5t, \quad p(t) = \frac{-2}{t}, \quad g(t) = 5t$$

- Integrating factor $\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{-2}{t}\,dt} = e^{-2\ln|t|}$

$$= e^{-2\ln t} = e^{\ln t^{-2}} = t^{-2} = \frac{1}{t^2}$$

- Apply $B^* \Rightarrow$

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\,g(t)\,dt + c\right]$$

$$y(t) = \frac{1}{1/t^2}\left[\int \frac{1}{t^2}(5t)\,dt + c\right] = t^2\left[5\ln t + c\right]$$

- To find c we use the IC

$$y(1) = 1^2[5\ln 1 + c]$$
$$2 = [0 + c]$$
$$2 = c$$

- Hence, the gen. sol. is $y(t) = t^2(5\ln t + 2)$

Exp Given the IVP:

$$y' - 2xy - x = 0 \quad , y(0) = \alpha$$

Find α so that the sol. approaches $\frac{-1}{2}$ as $x \to \infty$.

- We need to find α s.t $\lim_{x\to\infty} y(x) = \frac{-1}{2}$
- So first we find $y(x) \Rightarrow$ Apply Ⓑ and Ⓑ*

$$y' - 2xy = x \quad , p(x) = -2x \quad , g(x) = x$$

- $\mu(x) = e^{\int p(x)\,dx} = e^{\int -2x\,dx} = e^{-x^2}$ is the integrating factor
- Apply B^* $\Rightarrow y(x) = \frac{1}{\mu(x)}\left[\int \mu(x) g(x)\,dx + c\right]$

$$= \frac{1}{e^{-x^2}}\left[\int e^{-x^2}(x)\,dx + c\right]$$

$$y(x) = e^{x^2}\left[\frac{-1}{2} e^{-x^2} + c\right]$$

- To find c we use the IC $\Rightarrow y(0) = e^{0}\left[\frac{-1}{2} e^{0} + c\right] = \alpha$

$$\frac{-1}{2} + c = \alpha \Rightarrow \boxed{c = \alpha + \frac{1}{2}}$$

- The sol. becomes $y(x) = e^{x^2}\left[-\frac{1}{2} e^{-x^2} + \alpha + \frac{1}{2}\right]$

$$\boxed{y(x) = \frac{-1}{2} + \left(\alpha + \frac{1}{2}\right) e^{x^2}}$$

- But $\lim_{x\to\infty} y(x) = \cancel{\frac{-1}{2}} = \lim_{x\to\infty}\left(\cancel{\frac{-1}{2}} + \left(\alpha + \frac{1}{2}\right) e^{x^2}\right)$

$$0 = \lim_{x\to\infty}\left(\alpha + \frac{1}{2}\right) e^{x^2} \Leftrightarrow \boxed{\alpha = \frac{-1}{2}}$$