2.2 Separable DE's

<u>Remark.</u> Any 1^{st} order DE can be written as

$$y' = \frac{dy}{dt} = f(t, y)$$

- If we can separate the variables one on each side, then the DE is called separable
- Separable DE's can be solved by integrating each side with respect to its variable.

<u>Exp</u> Solve the DE: $\frac{dy}{dx} = \frac{x^2}{1-y^2}$ Sep. DE

nonlinear ✓

$$\int (1-y^2)\, dy = \int x^2\, dx$$

$$\boxed{y - \frac{y^3}{3} = \frac{x^3}{3} + C} \Rightarrow \text{Implicit solution}$$

<u>Exp</u> Solve the IVP: $\frac{dy}{dx} = \frac{y\cos x}{1+3y^3}$, $y(0) = 1$

nonlinear ✓

$$(1+3y^3)\, dy = y\cos x\, dx$$

$$\int \left(\frac{1}{y} + 3y^2\right) dy = \int \cos x\, dx \quad \text{Sep. DE}$$

$$\ln|y| + y^3 = \sin x + c$$

To find c we use IC:

$$\ln 1 + 1^3 = \sin 0 + c$$

$$\boxed{1 = c}$$

$$\boxed{\ln|y| + y^3 = \sin x + 1} \Rightarrow \text{Implicit solution}$$

<u>Exp</u> Consider the IVP : $\frac{dy}{dx} = \frac{3x^2+4x+2}{2y-2}$, $y(0) = -1$

1) Solve this IVP for implicit sol.

$\boxed{\text{sep. DE}}$
nonlinear ✓

$$\int (2y-2)\,dy = \int (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

⇒ To find c
⇒ $x = 0$, $y = -1$

$$(-1)^2 - 2(-1) = 0 + 0 + 0 + c$$

$$1 + 2 = c \implies \boxed{c = 3}$$

$$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + 3} \Rightarrow \text{Implicit Solution}$$

2) Find Explicit Solution

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 4$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$

$$|y-1| = \sqrt{x^2(x+2) + 2(x+2)}$$

$$y - 1 = \pm\sqrt{(x^2+2)(x+2)}$$

$$y(x) = 1 \pm \sqrt{(x^2+2)(x+2)}$$

$$\boxed{y_1(x) = 1 - \sqrt{(x^2+2)(x+2)}}$$ ✓ only sol. since it satisfies the IC: $y(0) = -1$

$$y_2(x) = 1 + \sqrt{(x^2+2)(x+2)}$$ ✗ not solution since it does not satisfy the IC : $y(0) = -1$

3) Find interval where the sol. is defined

The sol. is defined on

$$I = (-2, \infty)$$

Exp Solve the IVP: $y' = y^{\frac{1}{3}}$, $y(0) = 0$

$$\frac{dy}{dt} = y^{\frac{1}{3}} \Rightarrow \int y^{-\frac{1}{3}}\, dy = \int dt \qquad \boxed{\text{sep. DE}}$$

nonlinear ✓

$$\frac{3}{2} y^{\frac{2}{3}} = t + c$$

To find c we use IC:

$$\frac{3}{2}(0) = 0 + c \Rightarrow \boxed{c = 0}$$

$$\frac{3}{2} y^{\frac{2}{3}} = t$$

$$\left(\frac{3}{2}\right)^3 y^2 = t^3 \Rightarrow y^2 = \frac{8}{27} t^3$$

$$|y| = \sqrt{\frac{8}{27} t^3} \Rightarrow y(t) = \pm\sqrt{\frac{8}{27} t^3}$$

$y_1(t) = \sqrt{\frac{8}{27} t^3}$ is sol. since it also satisfies the IC

$y_2(t) = -\sqrt{\frac{8}{27} t^3}$ is sol. since it also satisfies the IC

$y_3(t) = 0$ is also sol.

Note that if $y(0) = 1$ in the above Exp, then there is a unique sol. whic is

$$y(t) = \sqrt{\left(\frac{2}{3}t + 1\right)^3}$$

Exp solve the DE $\frac{dy}{dx} = -\frac{4x+3y}{2x+y}$

- $(2x+y)\,dy = -(4x+3y)\,dx$ — not sep. DE, nonlinear
- This DE is not separable

Question: Can we change it to sep. DE?

Answer: Yes if the DE is homogenuous.
That is

$$y' = \frac{dy}{dx} = F(x,y)$$

If we can rewrite F as function of $V = \frac{y}{x}$ then the DE is homogenuous and so the DE can be changed to separable DE

$V = \frac{y}{x}$

$y = xV$

$y' = x\frac{dV}{dx} + V$

$$y' = \frac{dy}{dx} = -\frac{4 + 3\frac{y}{x}}{2 + \frac{y}{x}}$$

$$(xV' + V) = -\frac{4+3V}{2+V}$$ Homo.

$$x\frac{dV}{dx} = -V - \frac{4+3V}{2+V}$$

$$-x\frac{dV}{dx} = V + \frac{4+3V}{2+V}$$

$$= \frac{2V + V^2 + 4 + 3V}{2+V}$$

$$= \frac{V^2 + 5V + 4}{2+V}$$

$$-x\frac{dV}{dx} = \frac{(V+1)(V+4)}{2+V}$$

$$\int \frac{2+v}{(v+1)(v+4)}\,dv = \int -\frac{dx}{x}$$ (sep. DE)

$$\int\left(\frac{A}{v+1}+\frac{B}{v+4}\right)dv = -\ln|x| + c$$

$$A = \frac{2+\boxed{-1}}{\boxed{-1}+4} = \frac{1}{3}$$

$$B = \frac{2+\boxed{-4}}{\boxed{-4}+1} = \frac{2}{3}$$

$$\int\left(\frac{\frac{1}{3}}{v+1}+\frac{\frac{2}{3}}{v+4}\right)dv = -\ln|x| + c$$

$$\frac{1}{3}\ln|v+1| + \frac{2}{3}\ln|v+4| = -\ln|x| + c$$

$$\frac{1}{3}\ln\left|\frac{y}{x}+1\right| + \frac{2}{3}\ln\left|\frac{y}{x}+4\right| = -\ln|x| + c$$

$$\frac{1}{3}\ln\left|\frac{y+x}{x}\right| + \frac{2}{3}\ln\left|\frac{y+4x}{x}\right| = -\ln|x| + c$$

$$\frac{1}{3}\ln|y+x| - \cancel{\frac{1}{3}\ln|x|} + \frac{2}{3}\ln|y+4x| - \cancel{\frac{2}{3}\ln|x|} = \cancel{-\ln|x|} + c$$

$$\frac{1}{3}\ln|y+x| + \frac{2}{3}\ln|y+4x| = c$$

Implicit solution

Exp ① $y' = \dfrac{x^3+y^3}{x^3-y^3}$ is homogenous DE

② $\dfrac{dy}{dx} = \dfrac{x^2-y}{x}$ is not homogenous DE