2.3 Modeling with 1^{st} Order DE's

The idea here is to construct MM's that characterize problems in physical, biological and social sciences using DE's.

Newton's law of Cooling: The rate of change of the Temperature $T(t)$ "heat loss" of an object is proportional to the difference between its own Temperature T and the ambient Temperature T_m (the temperature of its surroundings).

That is, the MM (IVP) that describes this phenomena is

$$\frac{dT}{dt} = \alpha (T - T_m) \quad , \; T(0) = T_0 \quad , \; \alpha < 0$$

This DE has the form of (A) ⇒

$$T' = \alpha T - \alpha T_m \quad \text{with } a = \alpha, \; b = \alpha T_m$$

Hence, to find its solution $T(t)$ we apply (A*)

$$T(t) = \frac{b}{a} + \left(T_0 - \frac{b}{a}\right) e^{at}$$

$$\frac{b}{a} = \frac{\alpha T_m}{\alpha} = T_m \quad \text{Eq. Sol.}$$

$$\boxed{T(t) = T_m + (T_0 - T_m) e^{\alpha t}} \; \text{—} *$$

- So any problem obeys to Newton's law of Cooling according to the DE above can be solved directly using $*$.

- If $\alpha > 0$ ⇒ the problem becomes heating instead of cooling.

Exp Suppose that the temperature of a cup of coffee obeys to Newton's law of cooling. If the coffee has temperature of 200 F when freshly poured, and one minute later has cooled to 190 F in a room at 70 F. How long will it take the coffee to reach temperature of 150 F.

- Let $T(t)$ be the temperature of the cup at time t

$$T(t) = T_m + (T_0 - T_m)\, e^{\alpha t}$$

$$= 70 + (200 - 70)\, e^{\alpha t}$$

$T_0 = 200$

$T_m = 70$

$T(1) = 190$

$$\boxed{T(t) = 70 + 130\, e^{\alpha t}}$$

$$T(1) = 70 + 130\, e^{\alpha}$$

$$190 = 70 + 130\, e^{\alpha}$$

$$120 = 130\, e^{\alpha}$$

$$\frac{12}{13} = e^{\alpha}$$

$$\boxed{\alpha = \ln \frac{12}{13}}$$

We need to find the time t^* such that

$$T(t^*) = 150$$

$$70 + 130\, e^{\alpha t^*} = 150$$

$$130\, e^{\alpha t^*} = 80$$

$$e^{\alpha t^*} = \frac{8}{13}$$

$$\alpha t^* = \ln \frac{8}{13}$$

$$t^* = \frac{\ln \frac{8}{13}}{\alpha} = \frac{\ln \frac{8}{13}}{\ln \frac{12}{13}}$$

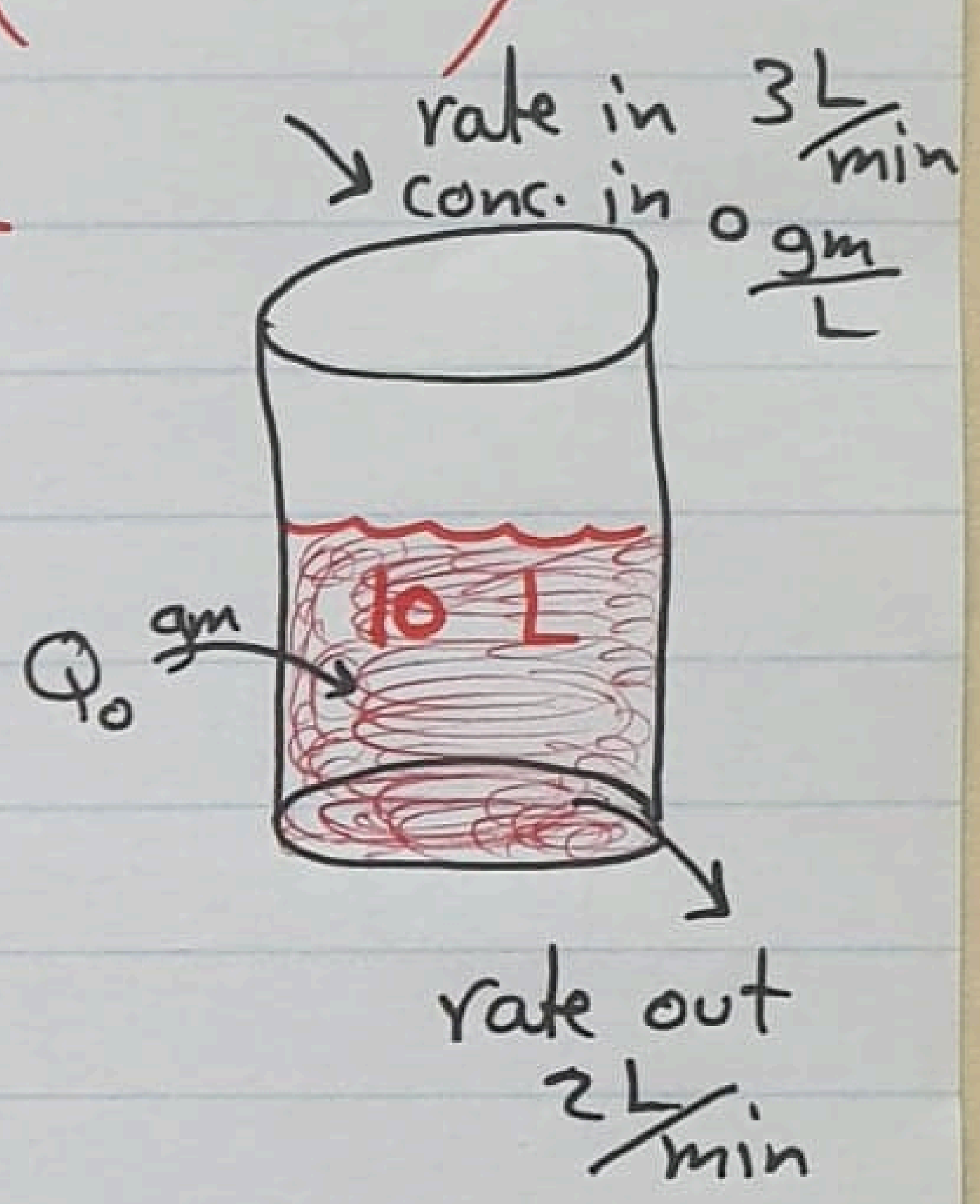
Exp At $t=0$, Q_0 gm of salt are dissolved in 10 L of water. Fresh water flows into the tank at rate 3L/min and the well-stirred mixture flows out at rate 2L/min. Denote the quantity of salt in the tank at time t by $Q(t)$.

1. Set up an IVP that describes this process

$$\frac{dQ}{dt} = (\text{rate in})(\text{Conc. in}) - (\text{rate out})(\text{Conc. out})$$

$$= (3)(0) - (2)\frac{Q(t)}{10+t}$$

$$Q' = -\frac{2Q}{10+t}, \quad Q(0) = Q_0$$



2. Solve the IVP in 1
" Find the quantity of salt in the tank at any time t "

$$Q' + \frac{2}{10+t}Q = 0 \quad \cdots (B) \quad \text{with } p(t) = \frac{2}{10+t}$$

$g(t) = 0$

Apply (B^*) $\Rightarrow$ $Q(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt + c\right]$

$$\mu(t) = e^{\int p(t)\,dt} = e^{2\int \frac{dt}{10+t}} = e^{2\ln|10+t|} = (10+t)^2$$

$$Q(t) = \frac{c}{\mu(t)} = \frac{c}{(10+t)^2}$$

$$Q(0) = \frac{c}{100} = Q_0 \Rightarrow c = 100\,Q_0$$

$$Q(t) = \frac{100\,Q_0}{(10+t)^2}$$

③ Assume at $t = 10$ min the quantity of salt in the tank is 20 gm. Find Q_0.

Since $Q(10) = 20 \Rightarrow$

$$\frac{100\, Q_0}{(10+10)^2} = 20$$

$$100\, Q_0 = (20)(400)$$

$$Q_0 = 80 \text{ gm}$$

④ Find the time where the concentration of salt in the tank is $\frac{1}{8}$ g/L

We need to find time t^* s.t

$$\text{Concentration} = \frac{1}{8}$$

$$\frac{\text{Quantity}}{\text{Volume}} = \frac{1}{8}$$

$$\frac{\frac{100\, Q_0}{(10+t)^2}}{10+t} = \frac{1}{8}$$

$$\frac{100\, Q_0}{(10+t)^3} = \frac{1}{8}$$

$$800\, Q_0 = (10+t)^3$$

$$800\,(80) = (10+t)^3$$

$$(8)(8)(1000) = (10+t)^3$$

$$(2)(2)(10) = (10+t)$$

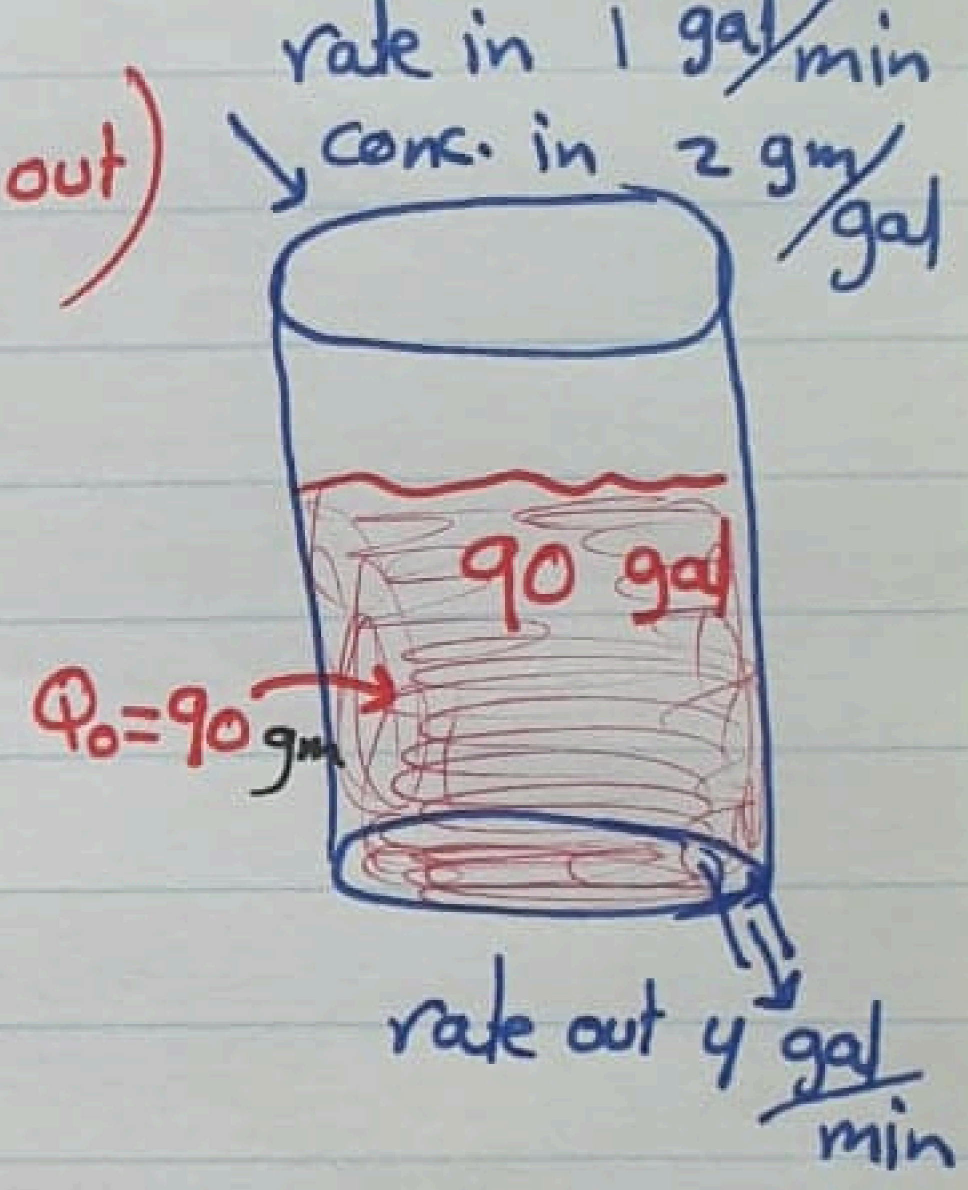
$$(2)(20) = 10+t$$

$$40 = 10 + t$$

$$t = 30 \text{ min}$$

Exp A 120 gallon tank initially contains 90 gm of salt dissolved in 90 gallon of water. Water containing 2 gm/gal of salt enters the tank at rate of 1 gal/min. The well-stirred mixture flows out at rate of 4 gal/min.

[1] Set up the IVP that models the change on $Q(t)$, where $Q(t)$ is the amount of salt in the tank at time t



$$\frac{dQ}{dt} = (\text{rate in})(\text{Conc. in}) - (\text{rate out})(\text{Conc. out})$$

$$= (1)(2) - (4)\frac{Q(t)}{90-3t}$$

$$Q' = 2 - \frac{4Q}{90-3t}, \quad Q(0)=90$$

[2] Solve the IVP

$$Q' + \frac{4}{90-3t} Q = 2 \quad \cdots \text{(B)} \quad \text{with } p(t) = \frac{4}{90-3t}, \; g(t) = 2$$

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{4\,dt}{90-3t}}$$

$$= e^{-\frac{4}{3}\ln|90-3t|} = (90-3t)^{-4/3}$$

$$Q(t) = \frac{1}{\mu(t)}\left[\int \mu(t)\, g(t)\, dt + c\right]$$

$$= \frac{1}{(90-3t)^{-4/3}}\left[\int 2(90-3t)^{-4/3} dt + c\right]$$

$$= \frac{1}{(90-3t)^{-4/3}}\left[2\,\frac{(90-3t)^{\frac{-1}{3}}}{-(\frac{1}{3})}\left(\frac{-1}{3}\right) + c\right]$$

$$Q(t) = \frac{2}{15}(90-3t) + \frac{c}{(90-3t)^{-4/3}}$$

To find c we use the IC : $Q(0) = 90$

$$Q(0) = \left(\frac{2}{15}\right)(90) + \frac{c}{(90)^{-4/3}}$$

$$90 = (180) + \frac{c}{(90)^{-4/3}}$$

$$1 = 2 + \frac{c}{(90)^{-\frac{1}{3}}} \Rightarrow \frac{c}{(90)^{-\frac{1}{3}}} = -1 \Rightarrow c = -\frac{1}{15}(90)^{-\frac{1}{3}}$$

$$Q(t) = \frac{2}{15}(90-3t) - \frac{(90)^{-\frac{1}{3}}}{(90-3t)^{-4/3}}$$

[3] When the tank becomes empty.

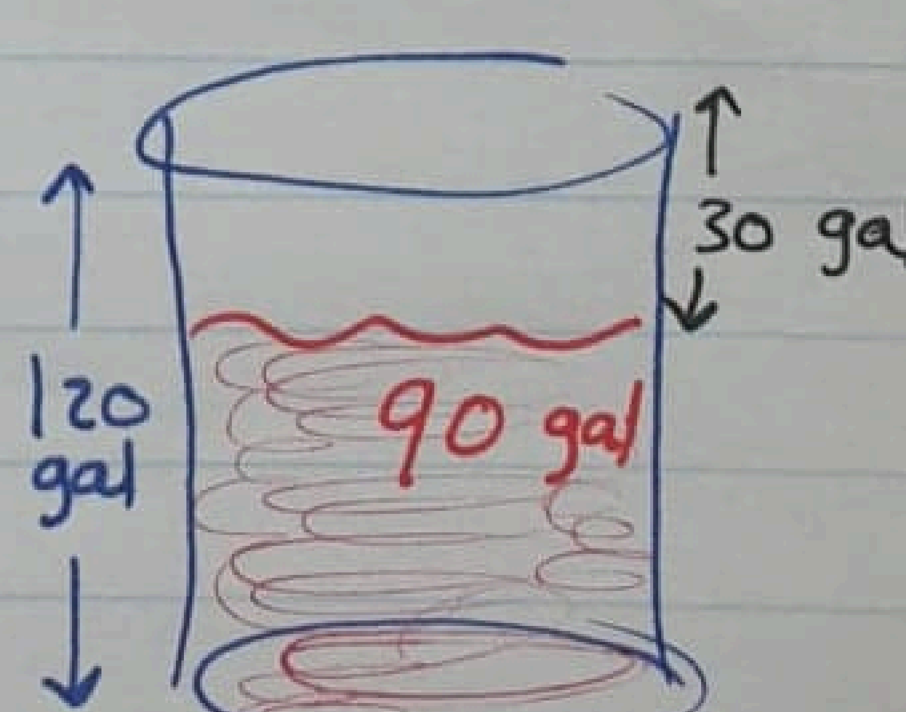
rate in $1\ \text{gal/min}$ and rate out $4\ \frac{\text{gal}}{\text{min}}$ $\Rightarrow$

The tank looses $3\ \frac{\text{gal}}{\text{min}} \Rightarrow \frac{90}{3} = 30\ \underline{\text{min}}$

Hence, after 30 min the tank becomes empty

Exp Consider the same Example above but with rate in $4\ \text{gal/min}$ and rate out $3\ \text{gal/min}$ and answer [1] and [2]

[1] $Q' = (2)(4) - (3)\frac{Q(t)}{90+t}$, $Q(0) = 90$



[2] $Q(t) = 2(90+t) - 90\left(\frac{90}{90+t}\right)^3$

[3] When the tank overflows ?

after $t = 30$ min since the tank increases $1\ \text{gal/min}$

[4] what is the quantity of salt in the tank when it becomes to overflow?

$$Q(30) = 2(90+30) - 90\left(\frac{90}{90+30}\right)^3$$