2.4 Difference Between linear and non linear DE's

- Recall that any 1^{st} order ODE has the general form

$$y' = \frac{dy}{dt} = f(t, y) \quad \cdots \ *$$

- The DE $*$ is linear if f is linear in y.
Otherwise, the DE $*$ is non linear

Question: When the DE $*$ has a unique solution?
How can we find the interval in which the solution is defined?

Th 2.4.1 (linear)
Consider the 1^{st} order linear DE

$$\boxed{y' + p(t)y = g(t) \text{ with } y(t_0) = y_0} \quad \cdots \ (B)$$

If $p(t)$ and $g(t)$ are cont. on an open interval $I = (\alpha, \beta)$ containing t_0, then $\exists$ a unique solution $y(t) = \phi(t)$ satisfies the IVP (B) on I.

Proof Existence is done in section 2.1 pages 18 + 19

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t) g(t)\, dt + c\right], \quad \mu(t) = e^{\int p(t)\, dt}$$

Uniqueness $\mu(t) = e^{\int_{t_0}^{t} p(t)\, dt}$, $y(t_0) = y_0$

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(t) g(t)\, dt + y_0\right]$$

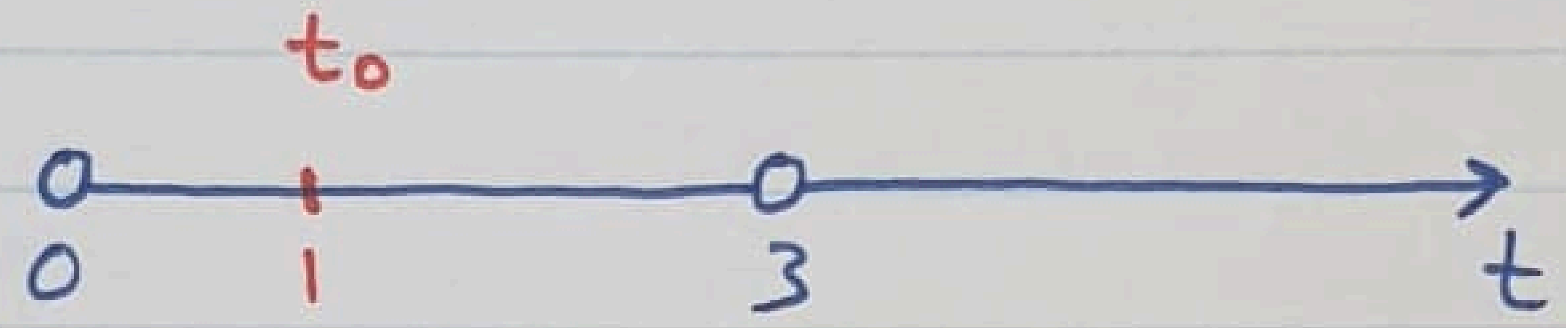
<u>Remark</u> (2.4.1) If the DE is 1^{st} order linear (satisfies (B)), then we can find the interval $I = (\alpha, \beta)$ that contains t_0 without solving the DE

<u>Exp</u> Find the largest interval in which the solution of the following IVP's is valid (defined).

$\boxed{1}$ $(t-3)y' + (\ln t)y = 2t \quad , \quad y(1) = 2$ linear

$$y' + \underbrace{\left(\frac{\ln t}{t-3}\right)}_{p(t) \text{ cont. on } \mathbb{R}^+ \setminus \{3\}} y = \underbrace{\frac{2t}{t-3}}_{g(t) \text{ cont. on } \mathbb{R} \setminus \{3\}} \quad \cdots \text{ (B)}$$

$p(t)$ and $g(t)$ are cont. on $(0,3)$ containing $t_0 = 1$

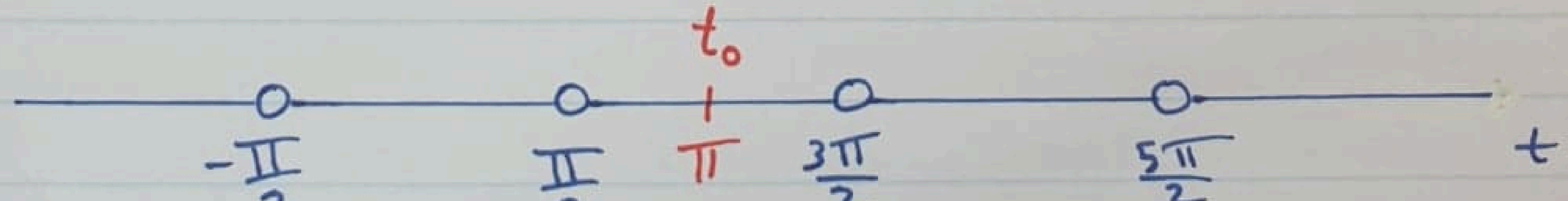


$\boxed{2}$ $(\cos t)\, y' = \sin t\,(\cos t - y) \quad , \quad y(\pi) = 0$

$$y' = \tan t\,(\cos t - y)$$ linear

$$y' + \underbrace{(\tan t)}_{p(t) \text{ cont. on } \mathbb{R} \setminus \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots\}} y = \underbrace{\sin t}_{g(t) \text{ cont. on } \mathbb{R}} \quad \cdots \text{ (B)}$$

$p(t)$ and $g(t)$ are cont. on $I = \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$



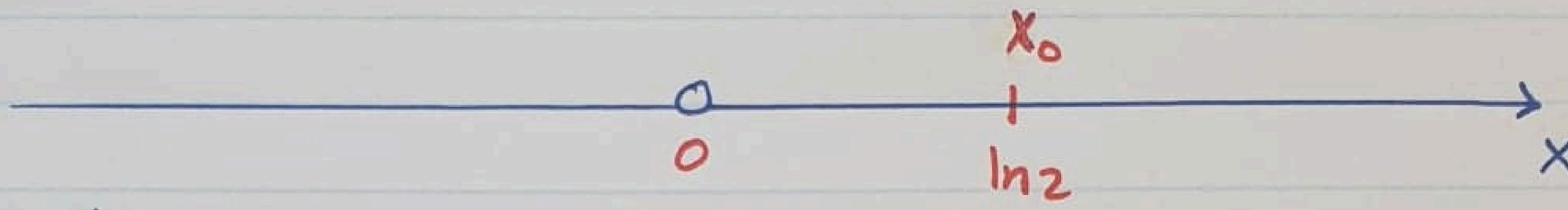
Exp Consider the IVP:

$$xy' = y + x^2 e^{-x}, \quad y(\ln 2) = \ln 2$$

[1] Find an interval in which this IVP has a unique sol.

$$y' - \frac{1}{x} y = x e^{-x} \quad \cdots \ (B) \qquad \text{linear}$$

$p(x)$ cont. on $\mathbb{R} \setminus \{0\}$ $\qquad$ $g(x)$ cont. on $\mathbb{R}$



$p(x)$ and $g(x)$ are cont. on $I = (0, \infty)$ containing x_0

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[2] Find the unique solution on this interval.

Apply B^* $\Rightarrow$ $\mu(x) = e^{\int p(x)dx} = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = e^{\ln x^{-1}} = \frac{1}{x}$

$$y(x) = \frac{1}{\mu(x)} \left[\int \mu(x) g(x) dx + c \right]$$

$$= \frac{1}{\frac{1}{x}} \left[\int \left(\frac{1}{x}\right)(x e^{-x}) dx + c \right]$$

$$= x \left[-e^{-x} + c \right]$$

To find c we use IC: $y(\ln 2) = \ln 2 \left[-e^{-\ln 2} + c \right]$

$$\ln 2 = \ln 2 \left[-\left(\tfrac{1}{2}\right) + c \right]$$

$$1 = -\tfrac{1}{2} + c$$

$$c = \tfrac{3}{2}$$

$$y(x) = x \left(-e^{-x} + \frac{3}{2} \right)$$

$$= \frac{3}{2} x - x e^{-x}$$

Th 2.4.2 (linear or nonlinear)
Consider the IVP $\boxed{\dfrac{dy}{dt} = f(t,y)\ ,\ y(t_0) = y_0}$

If f and f_y are cont. on an open rectangle $R = \{ (t,y) : \alpha < t < \beta \text{ and } \gamma < y < \delta \}$ containing the initial point (t_0, y_0), then $\exists$ a unique solution $y(t) = \phi(t)$ on an open sub-interval $I = (t_0 - h, t_0 + h) \subset (\alpha, \beta)$ s.t $y = \phi(t)$ satisfies the IVP on I, $h >> 0$.

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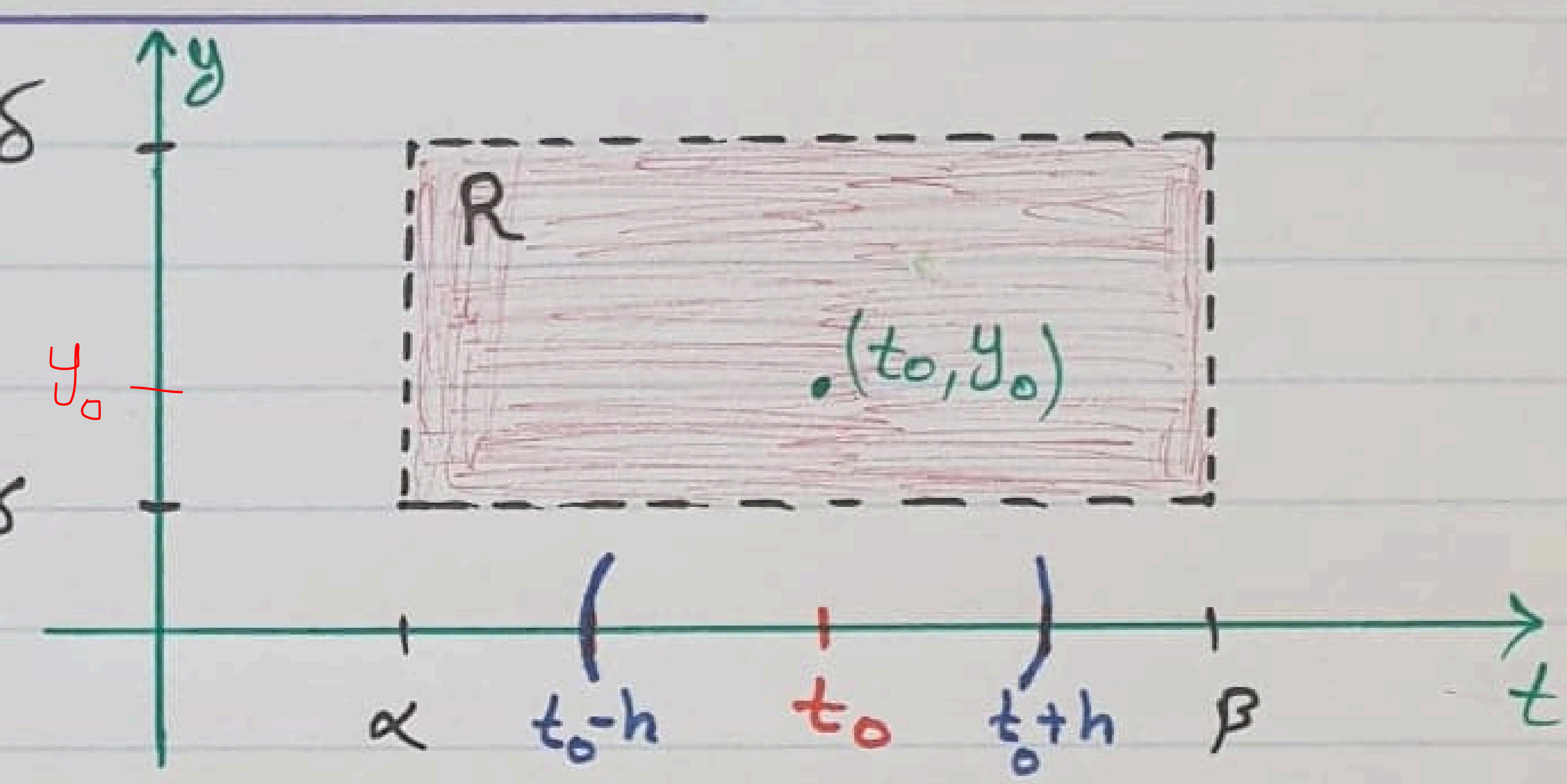
we can draw an open rectangle R containing the initial point (t_0, y_0)



<u>Remark</u> (2.4.2): If the DE is 1^{st} order <u>nonlinear</u>, then to find the interval in which the solution is defined, we need to <u>solve</u> the DE first and find the domain of the solution which contains t_0.

<u>Exp</u> Find the largest interval in which the solution of the following IVP's is valid (defined):

[I] $y' = y^2\ ,\ y(0) = 1$ nonlinear $\Downarrow$ 1^{st} solve to find the interval

$$\frac{dy}{dt} = y^2$$ separable DE

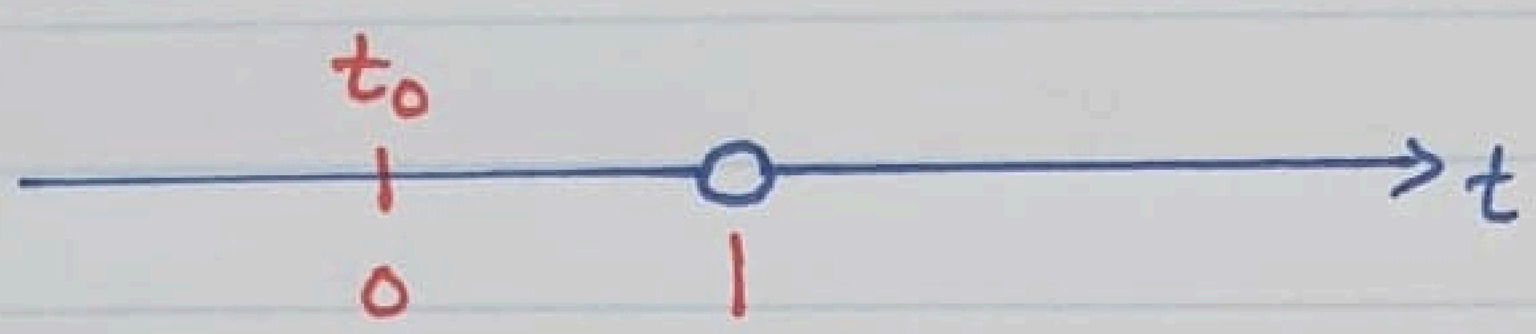
$$\int y^{-2}\, dy = \int dt$$

$$-\frac{1}{y} = t + c \qquad \text{To find } c \text{ we use IC:}$$

$$-\frac{1}{\boxed{1}} = \boxed{0} + c \quad \Rightarrow \quad c = -1$$

$$\frac{-1}{y} = t - 1 \quad \Rightarrow \quad \frac{1}{y} = 1 - t$$

$$y(t) = \frac{1}{1-t}$$



$$I = (-\infty, 1)$$

$\boxed{2}$ $\dfrac{dy}{dx} = \dfrac{3x^2 + 4x + 2}{2(y-1)}$, $y(0) = -1$

nonlinear

$\Downarrow$

1st solve to find the interval

we have solved this Exp in section 2.2 page 23 $\Rightarrow$

$$y(x) = 1 - \sqrt{(x^2+2)(x+2)}$$

with interval $I = (-2, \infty)$

Exp show that the IVP: $y' = y^2$, $y(0) = \underline{\underline{1}}$ has a unique solution

nonlinear

Apply Th 2.4.2

Compare with $y' = f(t, y)$ $\Rightarrow$

$f = y^2$ cont. on $\mathbb{R}$

$f_y = 2y$ cont. on $\mathbb{R}$

By Th 2.4.2 $\exists$ a unique sol. since we can draw an open rectangle containing $(t_0, y_0) = (0, 1)$

Exp show that the IVP:

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0)=-1$$

has a unique sol.

nonlinear
Apply Th 2.4.2

Compare with $y' = f(x,y) \Rightarrow$

$$f = \frac{3x^2+4x+2}{2(y-1)} \text{ is cont. on } \mathbb{R}\setminus\{y=1\}$$

$$f_y = -\frac{3x^2+4x+2}{2(y-1)^2} \text{ is cont. on } \mathbb{R}\setminus\{y=1\}$$

Hence, we can draw an open rectangle R containing the initial point $(x_0, y_0) = (0,-1)$, so $\exists$ a unique sol. by Th 2.4.2

Exp Consider the IVP: $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}, \quad y(0)=1$

[1] Does this IVP have unique sol.?

problem

nonlinear
Apply Th 2.4.2

Compare with $y' = f(x,y) \Rightarrow$

$$f = \frac{3x^2+4x+2}{2(y-1)} \text{ is cont. on } \mathbb{R}\setminus\{\underline{y=1}\}$$

$$f_y = -\frac{3x^2+4x+2}{2(y-1)^2} \text{ is cont. on } \mathbb{R}\setminus\{\underline{y=1}\}$$

We can not draw an open rectangle R containing $(x_0,y_0)=(0,1)$ where f and f_y are cont. Hence, the conditions of Th 2.4.2 do not hold $\Rightarrow$ we can not guarantee the uniqueness.

2 Find the interval in which the sol. is defined.

nonlinear

1^{st} solve to find the interval

$$\int (2y-2)\,dy = \int (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + c$$

To find c $\Rightarrow$ we use IC : $y(0)=1$

$$1^2 - 2(1) = 0 + 0 + 0 + c \Rightarrow c = -1$$

$$y^2 - 2y = x^3 + 2x^2 + 2x - 1 \quad \text{Implicit sol.}$$

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x - 1 + 1$$

$$(y-1)^2 = x(x^2+2x+2)$$

$$|y-1| = \sqrt{x(x^2+2x+2)}$$

The sol. is not unique

$$y(x) = 1 \pm \sqrt{x(x^2+2x+2)}$$

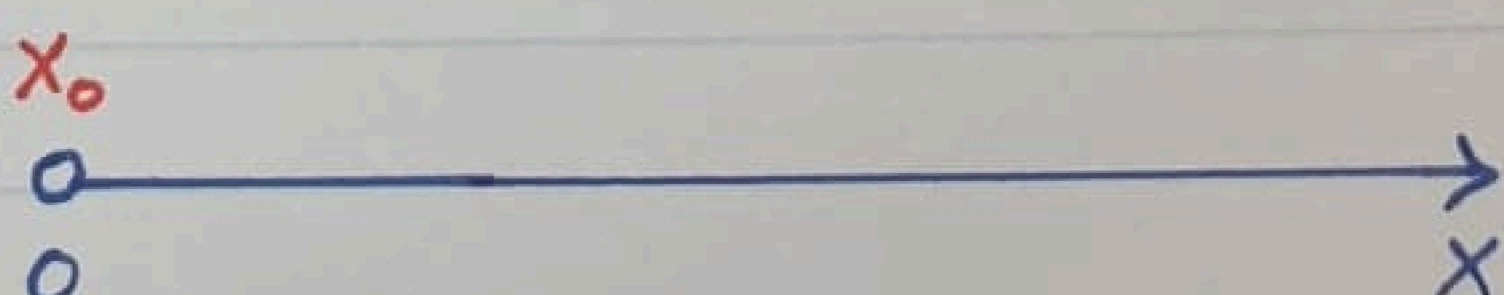
$y_1(x) = 1 + \sqrt{x(x^2+2x+2)}$ ✓ sol. satisfies $y(0)=1$

$y_2(x) = 1 - \sqrt{x(x^2+2x+2)}$ ✓ sol. satisfies $y(0)=1$

$$x^2+2x+2 = (x^2+2x+1)+1 = (x+1)^2+1 \quad \text{is positive}$$

The interval in which the sol. is defined is $I = (0, \infty)$

$y_1(x_0) = y_2(x_0) = 1 = y_0$



$y_0 = 1$ is V. Asy.

Exp Given the IVP: $y' = y^{\frac{1}{3}}$, $y(0) = 0$

1 Does this IVP have unique sol. ?

nonlinear

Apply Th 2.4.2

compare with $y' = f(t,y) \Rightarrow$

$f = y^{\frac{1}{3}}$ cont. on $\mathbb{R}$

$f_y = \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{3} \frac{1}{\sqrt[3]{y^2}}$ cont. on $\mathbb{R} \setminus \{y=0\}$

problem

We can not draw an open rectangle containing $(t_0, y_0) = (0,0)$ in which f and f_y are cont.
Hence, conditions of Th 2.4.2 do not hold $\Rightarrow$
we can not guarantee the uniqueness.

2 Solve this IVP

We solved this IVP in section 2.2 page 24 $\Rightarrow$

$y_1(t) = \sqrt{\frac{8}{27} t^3}$

$y_2(t) = -\sqrt{\frac{8}{27} t^3}$

$y_3(t) = 0$

So there is no unique sol.