Bernoulli DE's (Problem 27 page 73 in book)

Bernoulli DE has the form

$$\boxed{y' + p(t)\,y = q(t)\,y^{n} \quad , n \in \mathbb{R}} \quad \text{—} *$$

- Special case of Bernoulli DE when $n=0 \Rightarrow *$ becomes

$$y' + p(t)\,y = q(t) \quad \cdots \text{(B)} \qquad \text{linear}$$

whose solution is $y(t) = \frac{1}{\mu}\left[\int \mu g\,dt + c\right]$, $\mu = e^{\int p\,dt}$

- Special case of Bernoulli DE when $n=1 \Rightarrow *$ becomes

$$y' + p(t)\,y = q(t)\,y \qquad \text{linear}$$

$$y' + \big(p(t) - q(t)\big)\,y = 0 \quad \cdots \text{(B)}$$

whose solution is $y(t) = \frac{1}{\mu}\left[\int \mu g\,dt + c\right]$, $\mu = e^{\int (p-q)\,dt}$

$\Rightarrow y(t) = \frac{c}{\mu}$

- Special case of Bernoulli DE when $n \neq 0$ and $n \neq 1$

- we use change of variables to solve $*$: nonlinear

$$\text{②} - \; V = y^{1-n} \qquad \Rightarrow V' = (1-n)\,y^{-n}\,y'$$

- Multiply $*$ by $(1-n)\,y^{-n} \Rightarrow$

$$(1-n)\,y^{-n}\,y' + (1-n)\,p(t)\,y^{-n}\,y = (1-n)\,q(t)\,y^{-n}\,y^{n}$$

$$\text{①} - \boxed{V' + (1-n)\,p(t)\,V = (1-n)\,q(t)}$$

we solve ① for V then we solve ② for y

using B^*

Exp Solve this DE: $t^2 y' + 2ty - y^3 = 0, \; t > 0$

. This DE is non linear $\Rightarrow$ think of Bernoulli or Separable

$\Rightarrow$ This DE is not separable

Bernoulli $\Rightarrow$ write the DE in the form of *

$$y' + \frac{2}{t} y = \frac{1}{t^2} y^3 \;, \quad t > 0$$

its Bernoulli with $n = 3$, $p(t) = \frac{2}{t}$, $q(t) = \frac{1}{t^2}$

• First solve ① $\Rightarrow$ $v' + (1-n)\,p(t)\,v = (1-n)\,q(t)$

$$v' + (1-3)\left(\frac{2}{t}\right) v = (1-3)\,\frac{1}{t^2}$$

$$v' - \frac{4}{t} v = \frac{-2}{t^2}$$

$$v(t) = \frac{1}{\mu}\left[\int \mu g \, dt + c\right] \;, \quad \mu(t) = e^{\int \frac{-4}{t} dt} = e^{-4\ln t} = \frac{1}{t^4}$$

$$= \frac{1}{\frac{1}{t^4}}\left[\int \frac{1}{t^4}\left(\frac{-2}{t^2}\right) dt + c\right]$$

$$= t^4 \left[-2 \int t^{-6} \, dt + c\right]$$

$$= t^4 \left(-2 \, \frac{t^{-5}}{-5} + c\right)$$

$$v(t) = \frac{2}{5}\,\frac{1}{t} + c t^4$$

• Now solve ② $\Rightarrow$ $v = y^{1-n}$

$$\frac{2}{5}\,\frac{1}{t} + c t^4 = y^{1-3}$$

$$\frac{2 + 5ct^4 t}{5t} = y^{-2}$$

$$y^2 = \frac{5t}{2 + 5ct^5}$$

$$y(t) = \pm \sqrt{\frac{5t}{2 + 5ct^5}}$$

Exp Consider this IVP:

$$xy' + y = \frac{1}{y^2} \quad , \quad x>0 \quad , \quad y(1) = (2)^{\frac{1}{3}}$$

1. Solve this IVP using Bernoulli

$$y' + \frac{1}{x} y = \frac{1}{x} y^{-2} \qquad p(x) = q(x) = \frac{1}{x} \quad , \quad n=-2$$

First solve $V' + (1-n)p(x)V = (1-n)q(x)$

$$V' + \frac{3}{x} V = \frac{3}{x} \qquad \Rightarrow \mu(x) = e^{\int p(x)dx} = e^{\int \frac{3}{x}dx}$$

$$= e^{3\ln x} = x^3$$

$$V(x) = \frac{1}{\mu}\left[\int \mu g \, dx + c\right]$$

$$= \frac{1}{x^3}\left(\int x^3\left(\frac{3}{x}\right)dx + c\right)$$

$$= \frac{1}{x^3}\left(3\frac{x^3}{3} + c\right)$$

$$V(x) = 1 + \frac{c}{x^3}$$

Second solve $V = y^{1-n} \Leftrightarrow V = y^3$

$$y^3 = 1 + \frac{c}{x^3}$$ To find c we use IC $\Rightarrow$

$$2 = 1 + \frac{c}{1} \Rightarrow c = 2 - 1 \Rightarrow c = 1$$

$$y(x) = \sqrt[3]{1 + \frac{1}{x^3}}$$

[2] Solve this IVP using separable

$$x\frac{dy}{dx} + y = \frac{1}{y^2} \Rightarrow x\frac{dy}{dx} = \frac{1}{y^2} - y = \frac{1-y^3}{y^2}$$

$$\frac{1}{-3}\int \frac{-3y^2}{1-y^3}\,dy = \int \frac{dx}{x}$$

$$\frac{-1}{3}\ln|1-y^3| = \ln x + c$$ To find c we use IC

$$\frac{-1}{3}\ln|1-2| = \ln 1 + c \Rightarrow 0 = 0 + c \Rightarrow c = 0$$

$$\frac{-1}{3}\ln|1-y^3| = \ln x \Rightarrow \ln|1-y^3| = -3\ln x$$

$$|1-y^3| = x^{-3} \Rightarrow 1-y^3 = \pm\frac{1}{x^3}$$

we consider only $1-y^3 = -\frac{1}{x^3}$ since $y(1) = 2^{\frac{1}{3}}$

$$y^3 = 1 + \frac{1}{x^3}$$

$$y(x) = \sqrt[3]{1+\frac{1}{x^3}}$$

[3] Show that this IVP has a unique solution

$$y' = \frac{dy}{dx} = \frac{1-y^3}{xy^2} \qquad y(1) = (2)^{\frac{1}{3}}$$

nonlinear

Apply Th2.4.2

$f = \frac{1-y^3}{xy^2}$ cont. on $\mathbb{R}\setminus\{y=0\}$

$f_y = \frac{(xy^2)(-3y^2)-(1-y^3)(2xy)}{(xy^2)^2}$ cont. on $\mathbb{R}\setminus\{y=0\}$

$\Rightarrow$ by Th 2.4.2 $\exists$ unique sol. since we can draw an open rectangle R contains $(1, \sqrt[3]{2})$ in which f, f_y are cont.