2.6 Exact DE's and Integrating Factors

Question: How to solve this DE :

$$y' = \frac{x - y^2}{2xy + 1}$$

This DE is nonlinear , not separable, not Bernoulli ??

Th. Given the DE : $\boxed{M(x,y) + N(x,y)\, y' = 0}$ — ①

where M, N, M_y, N_x are cont. on an open rectangle $R = \{(x,y) : x \in (\alpha, \beta) \text{ and } y \in (\gamma, \delta)\}$.

- The DE ① is called Exact iff $\boxed{M_y = N_x}$ — ②
- Exact mean $\exists$ a function $\psi(x,y)$ s.t

$$\psi_x = M \text{ and } \psi_y = N \quad \text{iff } M \text{ and } N \text{ satisfy ②.}$$

Exp Solve the DE : $y' = \dfrac{x - y^2}{2xy + 1}$

$$(2xy+1)\, y' = x - y^2 \quad \Rightarrow \quad \underbrace{(y^2 - x)}_{M} + \underbrace{(2xy+1)}_{N} y' = 0$$

$$\left.\begin{array}{ll} M = y^2 - x & \Rightarrow M_y = 2y \\ N = 2xy + 1 & \Rightarrow N_x = 2y \end{array}\right\} \text{Exact} \Rightarrow \exists\ \psi(x,y) \text{ s.t}$$

$$\psi_x = M \text{ and } \psi_y = N$$

$\psi_x = M$

$$\psi = \int \psi_x\, dx = \int M\, dx = \int (y^2 - x)\, dx = y^2 x - \frac{x^2}{2} + h(y)$$

To find $h(y)$ we use $\psi_y = N$

$$\psi_y = \cancel{2xy} - 0 + h'(y) = N = \cancel{2xy} + 1 \quad \Leftrightarrow h'(y) = 1$$

$$\Leftrightarrow h(y) = y$$

$$\psi(x,y) = y^2x - \frac{x^2}{2} + y$$

Since $\psi(x,y) = c \Rightarrow \boxed{y^2x - \frac{x^2}{2} + y = c}$ Implicit Solution

---

Exp Show that $\psi(x,y) = c$ where c is constant.

$$M(x,y) + N(x,y)\, y' = 0$$
$$\psi_x(x,y) + \psi_y(x,y)\frac{dy}{dx} = 0$$
$$\frac{d}{dx}\big(\psi(x,y)\big) = 0$$
$$\psi(x,y) = c$$

Exp Find explicit solution to the IVP:

$$2x + y^2 + 2xy\, y' = 0, \quad y(1) = 1, \ x > 0$$

$$\left.\begin{array}{ll} M = 2x + y^2 & \Rightarrow M_y = 2y \\ N = 2xy & \Rightarrow N_x = 2y \end{array}\right\} \text{The DE is Exact} \Rightarrow$$

$\exists\ \psi(x,y)$ s.t $\psi_x = M$ and $\psi_y = N$

$$\psi_y = N \Rightarrow \psi = \int \psi_y\, dy = \int N\, dy = \int 2xy\, dy = xy^2 + g(x)$$

To find $g(x)$: $\psi_x = \cancel{y^2} + g'(x) = M = 2x + \cancel{y^2}$

$$g'(x) = 2x$$
$$g(x) = x^2$$

$\psi(x,y) = xy^2 + x^2 = c \Rightarrow xy^2 + x^2 = c$

To find c we use $y(1) = 1 \Rightarrow (1)(1)^2 + (1)^2 = c \Rightarrow c = 2$

$xy^2 + x^2 = 2$ is the implicit solution

$$xy^2 = 2 - x^2 \Rightarrow y^2 = \frac{2-x^2}{x} \Rightarrow y = \pm\sqrt{\frac{2-x^2}{x}} \Rightarrow y = \sqrt{\frac{2-x^2}{x}}$$

<u>Exp</u> solve this IVP: $\frac{dy}{dx} = -\frac{x+4y}{4x-y}$, $y(0)=1$

- This IVP is similar to problem solved page 25 in section 2.2
This is homogenuous DE but we can use the method of today

$$(4x-y)y' = -(x+4y) \Rightarrow \underbrace{(x+4y)}_{M} + \underbrace{(4x-y)}_{N}y' = 0$$

$$\left.\begin{aligned} M = x+4y &\Rightarrow M_y = 4 \\ N = 4x-y &\Rightarrow N_x = 4 \end{aligned}\right\} \text{The DE is Exact} \Rightarrow$$

$\exists\ \psi(x,y)$ s.t $\psi_x = M$ and $\psi_y = N$

$$\psi = \int \psi_y\, dy = \int N\, dy = \int (4x-y)\, dy = 4xy - \frac{y^2}{2} + h(x)$$

To find $h(x) \Rightarrow$

$$\psi_x = \cancel{4y} - 0 + h'(x) = M = x + \cancel{4y}$$

$$h'(x) = x$$

$$h(x) = \frac{x^2}{2}$$

$$\psi = 4xy - \frac{y^2}{2} + \frac{x^2}{2} = C$$

To find c we use the IC: $y(0)=1$

$$4(0)(1) - \frac{(1)^2}{2} + \frac{(0)^2}{2} = c \Rightarrow c = \frac{-1}{2}$$

$$4xy - \frac{y^2}{2} + \frac{x^2}{2} = \frac{-1}{2}$$

$$\boxed{8xy - y^2 + x^2 = -1}$$ Implicit Solution

Exp Solve the DE : $\underbrace{(3xy+y^2)}_{M}+\underbrace{(xy+x^2)}_{N}y'=0$, $x>0$

$$\left.\begin{array}{l} M=3xy+y^2 \Rightarrow M_y=3x+2y \\ N=xy+x^2 \Rightarrow N_x=y+2x \end{array}\right\}$$ This DE is not Exact

- In this case, the results of our Theorem do not hold.
- So we need to find a positive function I called integrating factor s.t when we multiply the non exact DE by I, it becomes exact so that we can apply this Theorem.

Question: How to find the positive integrating factor I ?
There are three cases to check:

1. If $\dfrac{M_y-N_x}{N}=f(x)$ then $I(x)=e^{\int f(x)\,dx}$

2. If $\dfrac{M_y-N_x}{M}=g(y)$ then $I(y)=e^{-\int g(y)\,dy}$

3. If $\dfrac{M_y-N_x}{yN-Mx}=h(v)$ then $I(v)=e^{\int h(v)\,dv}$

where $v=xy$

Exp Solve the DE: $(3xy+y^2)+(xy+x^2)y'=0$, $x>0$

$$\left.\begin{array}{l} M=(3xy+y^2) \Rightarrow M_y=3x+2y \\ N=(xy+x^2) \Rightarrow N_x=y+2x \end{array}\right\}$$ Not Exact

- check $\boxed{2}$ $\dfrac{M_y-N_x}{M}=\dfrac{3x+2y-(y+2x)}{3xy+y^2}=\dfrac{x+y}{3xy+y^2}\neq g(y)$

• check (II) $\frac{M_y - N_x}{N} = \frac{x+y}{xy+x^2} = \frac{x+y}{x(x+y)} = \frac{1}{x} = f(x)$

Hence, the integrating factor is

$$I(x) = e^{\int f(x)\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x$$

Now multiply the non exact DE by x to become exact:

$$(3x^2y + xy^2) + (x^2y + x^3)y' = 0 \quad , x > 0$$

$$\left.\begin{aligned} M &= 3x^2y + xy^2 \Rightarrow M_y = 3x^2 + 2xy \\ N &= x^2y + x^3 \Rightarrow N_x = 3x^2 + 2xy \end{aligned}\right\} \text{Exact} \Rightarrow$$

$\exists\ \psi(x,y)$ s.t $\psi_x = M$ and $\psi_y = N$

$$\psi = \int \psi_x\,dx = \int M\,dx = \int (3x^2y + xy^2)\,dx$$

$$= x^3y + \frac{x^2}{2}y^2 + h(y)$$

To find $h(y) \Rightarrow \psi_y = \cancel{x^3} + \cancel{x^2y} + h'(y) = N = \cancel{x^2y} + \cancel{x^3}$

$h'(y) = 0$

$h(y) = c$

$$\psi(x,y) = x^3y + \frac{x^2}{2}y^2 + c$$

$\boxed{x^3y + \frac{x^2}{2}y^2 = c}$ Implicit Solution

Exp (Q31) Solve the DE $\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + \frac{3y}{x}\right)y' = 0$

$M = 3x + \frac{6}{y} \Rightarrow M_y = -\frac{6}{y^2}$

$N = \frac{x^2}{y} + \frac{3y}{x} \Rightarrow N_x = \frac{2x}{y} - \frac{3y}{x^2}$

→ not Exact

- Now we need to find integrating factor $I \Rightarrow$

Case 1

$$\frac{M_y - N_x}{N} = \frac{-\frac{6}{y^2} - \left(\frac{2x}{y} - \frac{3y}{x^2}\right)}{\frac{x^2}{y} + \frac{3y}{x}} \neq f(x)$$

Case 2

$$\frac{M_y - N_x}{M} = \frac{-\frac{6}{y^2} - \left(\frac{2x}{y} - \frac{3y}{x^2}\right)}{3x + \frac{6}{y}} \neq g(y)$$

Case 3

$$\frac{M_y - N_x}{yN - xM} = \frac{-\frac{6}{y^2} - \frac{2x}{y} + \frac{3y}{x^2}}{x^2 + \frac{3y^2}{x} - 3x^2 - \frac{6x}{y}}$$

$$= \frac{\frac{3y}{x^2} - \left(\frac{6 + 2xy}{y^2}\right)}{\frac{3y^3 - 6x^2}{xy} - 2x^2} = \frac{\frac{3y^3 - 6x^2 - 2x^3y}{x^2y^2}}{\frac{3y^3 - 6x^2 - 2x^3y}{\cancel{xy}}}$$

$$= \frac{1}{xy} = \frac{1}{v} = h(v) \checkmark$$

Hence, the integrating factor is $I(v) = e^{\int h(v)\,dv} = e^{\int \frac{dv}{v}}$

$$= e^{\ln v} = v = xy$$

- Now multiply our DE by xy to become exact

$$\left(3x^2y + 6x\right) + \left(x^3 + 3y^2\right)y' = 0$$

$M = 3x^2y + 6x \Rightarrow M_y = 3x^2$

$N = x^3 + 3y^2 \Rightarrow N_x = 3x^2$

$\rightarrow$ Exact $\Rightarrow$

$\exists$ a function $\psi(x,y)$ s.t $\psi_x = M$ and $\psi_y = N$

- To find $\psi \Rightarrow$ use

$$\psi_x = M \Rightarrow \psi(x,y) = \int \psi_x \, dx = \int M \, dx = \int (3x^2y + 6x) dx$$

$$= x^3y + 3x^2 + g(y)$$

- To find $g(y) \Rightarrow$ use $\psi_y = N$

$$\psi_y = x^3 + 0 + g'(y)$$

$$N = x^3 + g'(y)$$

$$\cancel{x^3} + 3y^2 = \cancel{x^3} + g'(y)$$

$$g'(y) = 3y^2 \Rightarrow g(y) = y^3$$

- Hence, ψ becomes

$$\psi = x^3y + 3x^2 + y^3 \Rightarrow \psi = c$$

$$x^3y + 3x^2 + y^3 = c$$ Implicit solution

not Bernoulli

not separable

nonlinear

52

Exp Solve the DE : $\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3}$, $x>0$, $y>0$

$(x^3 + y^3) y' = x^2 y \Rightarrow$

$$x^2 y - (x^3 + y^3) y' = 0 \quad *$$

$M = x^2 y \Rightarrow M_y = x^2$

$N = -(x^3 + y^3) \Rightarrow N_x = -3x^2$

$\rightarrow$ not exact DE

case 1: $\frac{M_y - N_x}{N} = \frac{x^2 - -3x^2}{-(x^3+y^3)} = \frac{4x^2}{-(x^3+y^3)} \neq f(x)$

case 2: $\frac{M_y - N_x}{M} = \frac{4x^2}{x^2 y} = \frac{4}{y} = g(y)$ ✓

Hence, the integrating factor is

$$I = e^{-\int g(y)\,dy} = e^{-\int \frac{4}{y}} = e^{-4\ln y} = e^{\ln y^{-4}} = \frac{1}{y^4}$$

now multiply the nonexact* DE by $\frac{1}{y^4}$ to become exact $\Rightarrow$

$\frac{x^2}{y^3} - \left(\frac{x^3 + y^3}{y^4}\right) y' = 0 \Rightarrow$

$$x^2 y^{-3} - (x^3 y^{-4} + y^{-1}) y' = 0$$

$M = x^2 y^{-3} \Rightarrow M_y = -3x^2 y^{-4}$

$N = -(x^3 y^{-4} + y^{-1}) \Rightarrow N_x = -3x^2 y^{-4}$

$\rightarrow$ exact DE

Hence, $\exists$ a function $\psi(x,y)$ s.t $\psi_x = M$ and $\psi_y = N$

$$\psi = \int \psi_x\,dx = \int M\,dx = \int x^2 y^{-3}\,dx = \frac{x^3}{3} y^{-3} + h(y)$$

To find $h(y)$ we use $\psi_y = N$

$$\psi_y = -\cancel{x^3 y^{-4}} + h'(y) = N = -\cancel{x^3 y^{-4}} - y^{-1}$$

$$h'(y) = -\frac{1}{y}$$

$$h(y) = -\ln y$$

Hence,

$$\psi = \frac{x^3}{3} y^{-3} - \ln y = c$$

$$\frac{x^3}{3y^3} - \ln y = c$$ Implicit solution