2.8 The Existance and Uniqueness Theorem

Th 2.8.1 Consider the IVP:

$$\frac{dy}{dt} = f(t, y) \quad , \quad y(0) = 0 \qquad \text{---- } *$$

If f and f_y are cont. on a rectangle
$R = \{(t, y) : -a \leq t \leq a \text{ and } -b \leq y \leq b\}$,
then $\exists$ a unique solution $y(t) = \phi(t)$ defined on a sub-interval $|t| \leq h \leq a$ that satisfies the IVP $*$.

Note that Th 2.8.1 differes from Th 2.4.2 only in the initial condition. That is, Th 2.8.1 has IC starts at origin.

Remark: In general we can transform any IVP starts at (t_0, y_0) to an equivalent one starts at origin.

Exp Transform the following IVP's to an equivalent ones starting at origin:

① $y' - y^3 = t^2 \quad , \quad y(2) = -6$

Let $s = t - 2 \quad \Rightarrow t = 2 + s$
$\quad\; z = y + 6 \quad \Rightarrow y = z - 6$
$\qquad\qquad\qquad\quad y' = z'$

The equivalent IVP is

$$z' - (z - 6)^3 = (2 + s)^2 \quad , \quad z(0) = 0$$

- Now we will learn a method used to prove the existance of solution for Th2.8.1

- This method is called Picard's Iteration or it is also called The Method of Successive Approximation

$$\frac{dy}{dt} = f(t,y) \quad , \quad y(\underset{t_0}{0}) = \underset{y_0}{0} \quad \text{---} \; *$$

$$\int_{t_0}^{t} dy = \int_{t_0}^{t} f(t,y)\, dt$$

$$y(t)\Big|_{0}^{t} = \int_{0}^{t} f(t,y)\, dt$$

$$y(t) - y_0 = \int_{0}^{t} f(t,y)\, dt$$

$$\boxed{y(t) = \phi(t) = \int_{0}^{t} f(t, \phi(t))\, dt \quad \text{---}(T)}$$

where $\phi(t)$ is the solution of the IVP $*$

- (T) is called integral equation

- The solution of $*$ is the solution of (T).

- Now we will construct a sequence of functions
$\phi_1, \phi_2, \phi_3, \cdots, \phi_n, \cdots$
all satisfy the IC $y(0)=0$ but in general non of them satisfies the DE in $*$

- If the sequence $\phi_n(t)$ converges to $y=\phi(t)$, then $y=\phi(t)$ will be the solution for the IVP *

Here how to construct the sequence (iteration) ϕ_n:

Determine $f(t,y)$ from *

$\phi_0 = y_0 = 0$

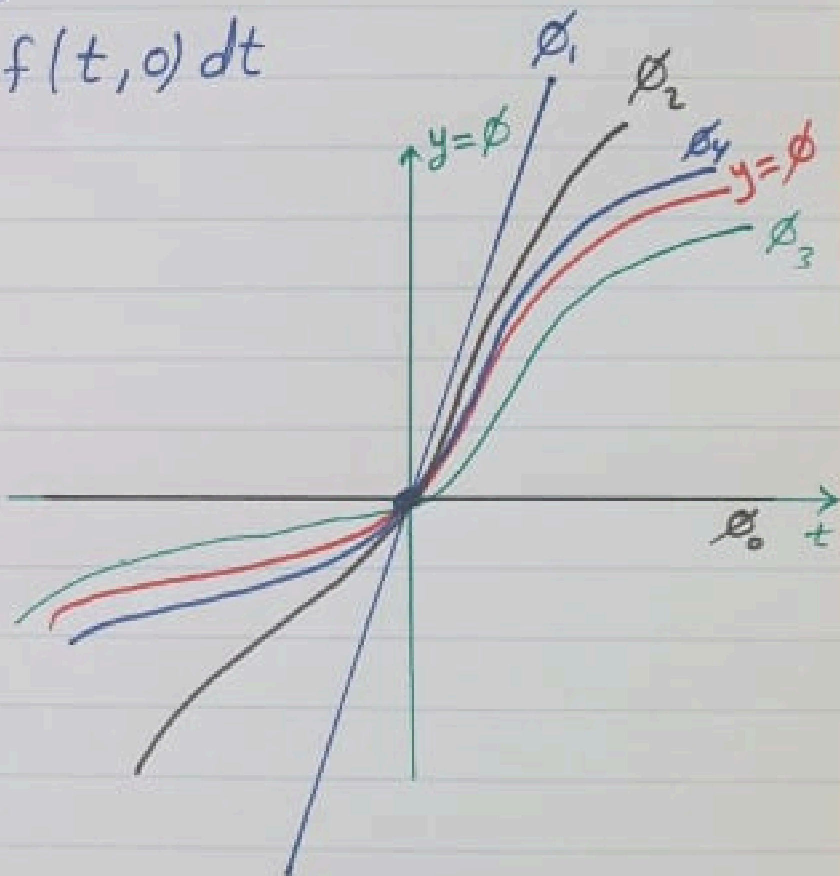
$$\phi_1 = \int_0^t f(t,\phi_0)\,dt = \int_0^t f(t,0)\,dt$$

$$\phi_2 = \int_0^t f(t,\phi_1)\,dt$$

$$\phi_3 = \int_0^t f(t,\phi_2)\,dt$$

$\vdots$

$$\phi_n(t) = \int_0^t f(t,\phi_{n-1}(t))\,dt$$



- If $\lim_{n\to\infty} \phi_n(t) = \phi(t)$, then $\phi(t)$ is the solution of the IVP *.

Remark
- If the iteration diverges, then this method will not be able to find the solution
- We may apply Ratio Test to prove an infinite series converges

Exp Use Picard's iteration to solve the IVP

$$y' = 2t(1+y) , \quad y(0) = 0$$

Compare with $y' = f(t,y) \Rightarrow \boxed{f(t,y) = 2t(1+y)}$

$\phi_0 = y_0 = 0$

$$\phi_1 = \int_0^t f(t,\phi_0)\,dt = \int_0^t f(t,0)\,dt = \int_0^t 2t\,dt = t^2 \Big|_0^t = t^2$$

$$\phi_2 = \int_0^t f(t,\phi_1)\,dt = \int_0^t f(t,t^2)\,dt = \int_0^t 2t(1+t^2)\,dt$$

$$= \int_0^t (2t + 2t^3)\,dt = t^2 + \frac{t^4}{2}$$

$$\phi_3 = \int_0^t f(t,\phi_2)\,dt = \int_0^t f(t, t^2 + \frac{t^4}{2})\,dt = \int_0^t 2t(1 + t^2 + \frac{t^4}{2})\,dt$$

$$= \int_0^t (2t + 2t^3 + t^5)\,dt = t^2 + \frac{t^4}{2} + \frac{t^6}{6}$$

$$\phi_4 = t^2 + \frac{t^4}{2} + \frac{t^6}{6} + \frac{t^8}{24}$$

$\vdots$

$$\phi_n(t) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \frac{t^8}{4!} + \cdots + \frac{t^{2n}}{n!}$$

$$\lim_{n\to\infty} \phi_n(t) = \sum_{n=1}^{\infty} \frac{t^{2n}}{n!} = e^{t^2} - 1 = \phi(t) \quad \text{since} \Downarrow$$

The Maclurine Series of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{t^2} = 1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} = 1 + \sum_{n=1}^{\infty} \frac{t^{2n}}{n!}$$

$$e^{t^2} - 1 = \sum_{n=1}^{\infty} \frac{t^{2n}}{n!} \checkmark$$

Exp Consider the IVP : $y' - 2y - 2 = 0 \ , \ y(0) = 0$

[I] Use the Method of Successive Approximation to find ϕ_3

compare with $y' = f(t,y) \Rightarrow y' = 2y + 2$
$= 2(y+1)$

$$\boxed{f(t,y) = f(y) = 2(y+1)}$$

$\phi_0 = y_0 = 0$

$$\phi_1 = \int_0^t f(t,\phi_0)\,dt = \int_0^t f(t,0)\,dt = \int_0^t 2\,dt = 2t \Big|_0^t = 2t$$

$$\phi_2 = \int_0^t f(t,\phi_1)\,dt = \int_0^t f(t,2t)\,dt = \int_0^t 2(2t+1)\,dt = \int_0^t (4t+2)\,dt$$
$$= 2t + \frac{4t^2}{2}$$
$$= 2t + 2t^2$$

$$\phi_3 = \int_0^t f(t,\phi_2)\,dt = \int_0^t f(t, 2t + 2t^2)\,dt = \int_0^t 2(2t + 2t^2 + 1)\,dt$$

$$= \int_0^t (4t + 4t^2 + 2)\,dt = 2t + 2t^2 + \frac{4t^3}{3}$$

② Find the solution of this IVP using this method

$$\phi_4(t) = 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4$$

$$\vdots$$

$$\boxed{\phi_n(t) = 2t + \frac{2^2 t^2}{2!} + \frac{2^3 t^3}{3!} + \frac{2^4 t^4}{4!} + \cdots + \frac{2^n t^n}{n!}}$$

$$\lim_{n\to\infty} \phi_n(t) = \sum_{n=1}^{\infty} \frac{2^n t^n}{n!} = e^{2t} - 1 = \phi(t) \quad \text{since} \Downarrow$$

The Maclurine series of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{2t} = 1 + 2t + \frac{2^2 t^2}{2!} + \frac{2^3 t^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{2^n t^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{2^n t^n}{n!}$$

$$e^{2t} - 1 = \sum_{n=1}^{\infty} \frac{2^n t^n}{n!}$$